

A note on spacelike surfaces in Minkowski 3-space

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Abstract. We characterize and classify spacelike surfaces endowed with a canonical principal direction in Minkowski 3-space \mathbb{E}_1^3 . Under the maximality condition, a new characterization for the catenoid of the 1st kind is obtained.

1. Introduction

A recent problem in the field of classical differential geometry consists in the study of constant angle surfaces. By definition, a surface for which the unit normal ξ in every point makes a constant angle θ with a fixed direction k is called a constant angle surface. Full classification results were obtained in different homogeneous 3-spaces and a survey on this topic is given in [9] (see also References therein). In particular, constant angle surfaces in Euclidean 3-space \mathbb{E}^3 are assumed to be known in literature, but a new approach on their classification is presented in [10]. Passing now from the Euclidean 3-space to the Minkowski 3-space \mathbb{E}_1^3 , similar techniques are used in [7] in order to classify constant angle surfaces in \mathbb{E}_1^3 . Another class of spacelike surfaces in \mathbb{E}_1^3 , namely the constant slope ones, i.e. their unit normal makes constant angle with the position vector, were studied in [3].

An important property of constant angle surfaces in Euclidean 3-space \mathbb{E}^3 and product spaces $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$ is the following. If we denote by U the projection of the fixed direction k on the tangent plane of the surface, then U is a principal direction of the surface with the corresponding principal curvature 0. A new problem that appears in the context of constant angle surfaces is to study those surfaces for which U remains a principal direction but the corresponding principal curvature is different from zero. First results were given for surfaces isometrically immersed in $\mathbb{S}^2 \times \mathbb{R}$ [1], $\mathbb{H}^2 \times \mathbb{R}$ [2] and \mathbb{E}^3 [11], where U was called a *canonical principal direction*.

In the present note we continue this study for spacelike surfaces in the Minkowski 3-space \mathbb{E}_1^3 . Recalling the general theory of surfaces in \mathbb{E}_1^3 , see for example [4], [6], the angle function θ is well defined between future-directed timelike vectors ξ and k and it is called *hyperbolic angle function*. In the next section we formulate the main results, namely characterization and classification theorems for spacelike surfaces endowed with a canonical principal direction. Notice that an extensively studied property of spacelike surfaces is the maximality, i.e. spacelike surfaces with vanishing mean curvature, see e.g. [5], [8], [12] a.s.o. With this motivation in mind, we study the maximal spacelike surfaces endowed with a canonical principal direction, and we obtain a new characterization for the catenoid of the 1st kind. Finally, in the last section we prove all these classification results.

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2. Background and main results

Let us denote by $\langle , \rangle = dx_1^2 + dx_2^2 - dx_3^2$ the Minkowski metric on \mathbb{E}_1^3 . We briefly recall (see e.g. [6]) that a vector $v \in \mathbb{E}_1^3$ is called *spacelike* if $\langle v, v \rangle > 0$ or $v = 0$, *timelike* if $\langle v, v \rangle < 0$ and *lightlike* if $\langle v, v \rangle = 0$, $v \neq 0$. In a similar manner, a surface M in \mathbb{E}_1^3 endowed with the metric g given by the restriction of \langle , \rangle to M is called *spacelike* if the metric g is positive definite, *timelike* if g is indefinite and *lightlike* or *isotropic* if the matrix associated to g has rank 1. In other words, if $\xi \neq 0$ is the normal of the surface, then the surface is *spacelike* (*timelike*, respectively *lightlike*) if and only if, at every point $p \in M$, ξ is a *timelike* (*spacelike*, respectively *lightlike*) vector.

Consider $r : M \rightarrow \mathbb{E}_1^3$ a *spacelike* immersion, i.e. the induced metric on M is a Riemannian metric. Hence, any normal vector field ξ on M is *timelike* in each point. In particular, if r is spacelike, then the surface M is orientable.

According to [7], a constant angle spacelike surface in \mathbb{E}_1^3 is a spacelike surface whose unit normal vector ξ makes a constant hyperbolic angle θ with the timelike vector $k = (0, 0, 1)$. Notice that the concept of *angle* between two vectors is well defined for timelike vectors; more precisely, when the two vectors are future-directed. In our case, ξ is future-directed if $\langle \xi(p), k \rangle < 0$, in any point of the surface $p \in M$, and the hyperbolic angle θ between ξ and k is defined by $\cosh \theta = -\langle \xi, k \rangle$.

Projecting the fixed direction k on the tangent plane to the surface, we get

$$k = U + \cosh \theta \xi, \quad (1)$$

where U is the tangent component of k .

If M is a constant angle spacelike surface in \mathbb{E}_1^3 , namely $\theta \geq 0$ is constant, then a property of this surface tells us that U is a principal direction with the corresponding principal curvature identically zero. (Check [7] for details.)

At this point, the following problem may be formulated: *Study spacelike surfaces in \mathbb{E}_1^3 endowed with a canonical principal direction, i.e. those surfaces for which U is principal direction.*

In the present note we solve this problem and we provide characterization and classification results.

Theorem 2.1 (Characterization theorem). *Let M be a spacelike surface in \mathbb{E}_1^3 , and $\theta \neq 0$ be the hyperbolic angle function. Let (u, v) be local orthogonal coordinates on M such that ∂_u is in the direction of U . Then, U is a principal direction for M if and only if $\theta_v = 0$.*

Remark that a similar result is obtained in [11] to characterize surfaces with a canonical principal direction in Euclidean 3-space.

In the sequel we give the classification of spacelike surfaces with a canonical principal direction.

Theorem 2.2 (Classification theorem). *Let $r : M \rightarrow \mathbb{E}_1^3$ be a spacelike surface isometrically immersed in \mathbb{E}_1^3 and let $\theta \neq 0$ be the hyperbolic angle function. Then, M has a canonical principal direction if and only if M is parametrized by one of the following:*

$$(a) \quad r(u, v) = (\cos v, \sin v, 0)\phi(u) - (0, 0, 1)\chi(u) + \gamma(v),$$

$$\text{where } \gamma(v) = \left(\int \psi(v) \sin v \, dv, -\int \psi(v) \cos v \, dv, 0 \right), \psi \in C^\infty(M),$$

$$(b) \quad r(u, v) = (\cos v_0, \sin v_0, 0)\phi(u) - (0, 0, 1)\chi(u) + \gamma_0(v),$$

$$\text{where } \gamma_0(v) = (-\sin v_0 v, \cos v_0 v, 0), \text{ and } v_0 \text{ is a real constant.}$$

In both cases $\phi(u) = \int^u \cosh \theta(\tau) d\tau$ and $\chi(u) = \int^u \sinh \theta(\tau) d\tau$.

Under additional assumptions of maximality, respectively flatness, we may formulate the following two results.

Theorem 2.3. *The only maximal spacelike surfaces in \mathbb{E}_1^3 with a canonical principal direction are the catenoids of the 1st kind, parameterized in local coordinates (u, v) as:*

$$(u, v) \mapsto (\sqrt{u^2 - c^2} \cos v, \sqrt{u^2 - c^2} \sin v, c \ln(u + \sqrt{u^2 - c^2})), \tag{2}$$

where $c \in \mathbb{R} \setminus \{0\}$.

Theorem 2.4. *The only flat spacelike surfaces in \mathbb{E}_1^3 with a canonical principal direction are generalized cylinders, parameterized in local coordinates (u, v) as:*

$$(u, v) \mapsto \sigma(u) + \mathbf{v}_0 v, \tag{3}$$

where $\sigma(u) = (\cos v_0 \int^u \cosh \theta(\tau) d\tau, \sin v_0 \int^u \cosh \theta(\tau) d\tau, -\int^u \sinh \theta(\tau) d\tau)$, $\mathbf{v}_0 = (-\sin v_0, \cos v_0, 0)$, $v_0 \in \mathbb{R}$, and $\theta \neq 0$ denotes the hyperbolic angle function.

Remark 2.5. We may regard Theorem 2.3 as a new characterization for the catenoid of the 1st kind, namely the only maximal spacelike surface endowed with a canonical principal direction.

Remark 2.6. The catenoid of the 1st kind which we obtained as a maximal spacelike surface with a canonical principal direction U may be generated by rotating the curve $(c \sinh(\frac{t}{c} - \ln c), 0, t)$ around the Oz axis. These rotations are Lorentz transformations of the Minkowski 3-space \mathbb{E}_1^3 . See [12].

Remark 2.7. The flat spacelike surfaces endowed with a canonical principal direction classified in Theorem 2.4 are given by the generalized cylinders from case (b) of Theorem 2.2. More precisely, these surfaces are cylinders over spacelike curves with spacelike rulings orthogonal to $k = (0, 0, 1)$.

3. Proof of theorems

Let $r : M \rightarrow \mathbb{E}_1^3$ be a spacelike immersion endowed with the Riemannian metric g given by the restriction of the Minkowski metric $\langle \cdot, \cdot \rangle$ from the ambient space in the points of M .

Denote by ∇ and $\tilde{\nabla}$ the Levi Civita connections on M and \mathbb{E}_1^3 respectively, and by R the curvature tensor on M . Recall the structural equations of the surface M , consisting of the classical Gauss and Weingarten formulas:

$$(G) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$(W) \quad \tilde{\nabla}_X \xi = -A_\xi X,$$

together with the equations of Gauss and Codazzi:

$$(E.G.) \quad \langle R(X, Y)Z, W \rangle = g(AY, Z)g(AX, W) - g(AX, Z)g(AY, W),$$

$$(E.C.) \quad \nabla_X AY - \nabla_Y AX + A[X, Y] = 0,$$

where $X, Y, Z, W \in \mathfrak{X}(M)$ are tangent vector fields to M , h is a symmetric $(1, 2)$ -tensor field called the second fundamental form of the surface, and A is a symmetric $(1, 1)$ -tensor field called the shape operator associated to the normal ξ .

Proposition 3.1. *For any vector X tangent to the surface M , we have:*

$$\nabla_X U = (\cosh \theta) AX, \tag{4}$$

$$X(\cosh \theta) = g(AU, X). \tag{5}$$

Proof. On one hand $\tilde{\nabla}_X k = 0$, for any $X \in \mathfrak{X}(M)$, and on the other hand we may compute $\tilde{\nabla}_X k$ using Gauss formula (G). Identifying the tangent and normal parts, we get the expressions (4) and (5). □

Define an orthonormal basis $\{e_1, e_2\}$ in each tangent plane T_pM , such that $e_1 = \frac{U}{|U|}$ and e_2 is orthogonal to e_1 . Taking into account that the fixed direction is unitary and future directed, $\langle k, k \rangle = -1$, together with expression (1), we get that $|U| = \sinh \theta$ and the decomposition (1) of k becomes:

$$k = \sinh \theta e_1 + \cosh \theta \xi. \tag{6}$$

Proof. [of Theorem 2.1] It is a known fact that one can always choose orthogonal coordinates (u, v) on M such that the metric is given by

$$g = \alpha(u, v)^2 du^2 + \beta(u, v)^2 dv^2, \quad \alpha, \beta \in C^\infty(M). \tag{7}$$

Then, from the choice of the principal direction U parallel to ∂_u , we have $U = \frac{\sinh \theta}{\alpha} \partial_u$.

By straightforward reasoning and using Proposition 3.1, we write $AU = \frac{\theta_u \sinh \theta}{\alpha^2} \partial_u + \frac{\theta_v \sinh \theta}{\beta^2} \partial_v$.

Hence, if U is a principal direction, then $\theta_v = 0$.

Conversely, from (5) we get $g(AU, \partial_v) = 0$ which means that U is parallel to ∂_u , and thus U is a principal direction, concluding the proof. □

Concerning the geometry of a spacelike surface in Minkowski 3-space endowed with the principal direction given by U , we formulate the following result.

Proposition 3.2. *Let M be a spacelike surface in \mathbb{E}_1^3 , and the hyperbolic angle function $\theta \neq 0$. If U is a principal direction of M , then we may choose local coordinates (u, v) on the surface such that ∂_u is in the direction of U , the metric is given by*

$$g = du^2 + \beta(u, v)^2 dv^2, \tag{8}$$

and the shape operator may be expressed in the basis $\{\partial_u, \partial_v\}$ as:

$$A = \begin{pmatrix} \theta_u & 0 \\ 0 & \tanh \theta \frac{\beta_u}{\beta} \end{pmatrix}. \tag{9}$$

Moreover, the hyperbolic angle function θ satisfies $\theta_v = 0$, and β has one of the following expressions:

$$\beta(u, v) = - \int^u \cosh \theta(\tau) d\tau + \psi(v), \quad \text{where } \psi \in C^\infty(M), \tag{10.a}$$

$$\beta(u, v) = \beta(v). \tag{10.b}$$

Proof. Using standard computation techniques, starting from the metric (7) one obtains the Levi Civita connection ∇ and the shape operator A in orthogonal coordinates (u, v) . Furthermore, due to Proposition 3.1, the symmetry property of A yields

$$\alpha_v \tanh \theta + \alpha \theta_v = 0,$$

which implies $\partial_v(\alpha \sinh \theta) = 0$ and hence $\alpha = \frac{\psi(u)}{\sinh \theta}$, for a certain function ψ . Since U is a principal direction, from Theorem 2.1 we have that $\theta_v = 0$, i.e. θ depends only on u . After a change of coordinates such that $\psi(u) = \sinh \theta$, we get $\alpha \equiv 1$. Then, formulas (8) and (9) are proved. From the Codazzi equation (E.C.) we obtain that θ and β satisfy the following partial differential equation:

$$\beta_u \theta_u \tanh \theta - \beta_{uu} = 0.$$

Solving this equation we find the two solutions for β given by (10). □

We call the coordinates (u, v) from Proposition 3.1 *canonical coordinates* on the spacelike surface M endowed with the principal direction U . At this point we are ready to prove the classification theorem of spacelike surfaces with a canonical principal direction in \mathbb{E}_1^3 .

Proof. [of Theorem 2.2] From Proposition 3.2 we choose local coordinates (u, v) on the spacelike surface M . Then, the metric on M has expression (8), with β given by (10).

Case (a). Let us consider first $\beta(u, v) = -\int^u \cosh \theta(\tau) d\tau + \psi(v)$.

The Levi Civita connection ∇ and the shape operator A may be expressed as:

$$\nabla_{\partial_u} \partial_u = 0, \quad \nabla_{\partial_u} \partial_v = -\frac{\cosh \theta}{\beta} \partial_v = \nabla_{\partial_v} \partial_u, \quad \nabla_{\partial_v} \partial_v = \beta \cosh \theta \partial_u + \frac{\psi'(v)}{\beta} \partial_v, \tag{11}$$

$$A = \begin{pmatrix} \theta_u & 0 \\ 0 & -\frac{\sinh \theta}{\beta} \end{pmatrix}. \tag{12}$$

The Gauss formula **(G)** and $h(X, Y) = -g(AX, Y)\xi$, together with (11) and (12) yield that the immersion $r(u, v) = (r_1(u, v), r_2(u, v), r_3(u, v))$ is the solution of the following system of partial differential equations:

$$r_{uu} = -\theta_u \xi, \tag{13}$$

$$r_{uv} = -\frac{\cosh \theta}{\beta} r_v, \tag{14}$$

$$r_{vv} = \beta \cosh \theta r_u + \frac{\psi'(v)}{\beta} r_v + \beta \sinh \theta \xi. \tag{15}$$

Using (6), one computes $\langle k, r_u \rangle = \sinh \theta$ and $\langle k, r_v \rangle = 0$. So, the third component of the parametrization may be immediately obtained,

$$r_3(u, v) = -\int^u \sinh \theta(\tau) d\tau. \tag{16}$$

Moreover, the same decomposition (6) yields the expression of the normal

$$\xi = \left(-(r_j)_u \tanh \theta, \cosh \theta \right), \quad j = 1, 2. \tag{17}$$

From the Weingarten formula **(W)** we get that $\xi_u = -\theta_u r_u$ and comparing it with the expression of ξ_u obtained from (17), we get that

$$(r_j)_u = f(v) \cosh \theta, \text{ where } f(v) = (f_1(v), f_2(v)). \tag{18}$$

Since $\langle r_u, r_u \rangle = 1$, there exists a real function $\varphi(v)$ such that $f(v) = (\cos \varphi(v), \sin \varphi(v))$. After a first integration in (18), and taking into account (16), it follows

$$r(u, v) = \left(\cos \varphi(v), \sin \varphi(v), 0 \right) \int^u \cosh \theta(\tau) d\tau - (0, 0, 1) \int^u \sinh \theta(\tau) d\tau + \gamma(v), \tag{19}$$

where $\gamma(v) = (\gamma_1(v), \gamma_2(v), 0)$.

Computing r_{uv} , taking successive derivatives in (19) with respect to u and v and combining the obtained expression with (14), we have

$$r_v = -\beta \varphi'(v) \left(\cos \varphi(v), \sin \varphi(v), 0 \right). \tag{20}$$

Since β has expression (10.a) and $\langle r_v, r_v \rangle = \beta^2(u, v)$, we get that $(\varphi'(v))^2 = 1$ and hence we may take $\varphi(v) = v$.

Comparing now the expression of r_v obtained from (19) with the previous expression (20), we find the curve γ , namely

$$\gamma(v) = \left(\int \psi(v) \sin v \, dv, - \int \psi(v) \cos v \, dv, 0 \right). \tag{21}$$

Hence, case (a) of Theorem 2.2 is proved by plugging (21) in (19).

Case (b). Suppose $\beta(u, v) = \beta(v)$.

After a change of v -coordinate, we may assume $\beta(u, v) \equiv 1$, and thus, the metric becomes $g = du^2 + dv^2$. Now, the Levi Civita connection is identically zero, $\nabla_{\partial_u} \partial_u = \nabla_{\partial_u} \partial_v = \nabla_{\partial_v} \partial_u = \nabla_{\partial_v} \partial_v = 0$, and the shape operator is $A = \begin{pmatrix} \theta_u & 0 \\ 0 & 0 \end{pmatrix}$. In the same way as in the previous case, we get item (b) from Theorem 2.2.

In order to conclude the proof, the converse part follows immediately, checking that the surfaces parameterized by the two cases admit a canonical principal direction. □

Proof. [of Theorem 2.3]

As in the previous proof, let us study the two cases corresponding to each expression of β .

If β is given by (10.a), then, under the maximality assumption, we get from the expression of the shape operator (12) that β and θ fulfill:

$$\theta_u - \frac{\sinh \theta}{\beta} = 0. \tag{22}$$

Replacing the expression of β by (10.a), and solving the obtained partial differential equation, we get that the hyperbolic angle function is:

$$\theta = -\operatorname{arctanh} \frac{c}{u}, \quad c \in \mathbb{R}^*. \tag{23}$$

From (22) we notice that β depends only on u , and comparing with (10.a) we get that $\psi(v) = 0$. Then, applying Theorem 2.2, case (a), we find that $\gamma(v) = 0$.

Using (23), we immediately compute

$$- \int^u \sinh \theta(\tau) d\tau = c \ln(u + \sqrt{u^2 - c^2}), \quad \int^u \cosh \theta(\tau) d\tau = \sqrt{u^2 - c^2},$$

and substituting these expressions in case (a) of Theorem 2.2 we obtain (2), namely the parametrization of the catenoid of 1st kind.

On the other hand, if β is given by (10.b), namely $\beta(u, v) = \beta(v)$, a change of the v -coordinate allows us to take $\beta(u, v) \equiv 1$ and the shape operator has expression (3). The maximality assumption yields that θ satisfies $\theta_u = 0$. Then, combining it with $\theta_v = 0$, it follows that θ is constant, case excluded in our study. □

Proof. [of Theorem 2.4] As before, let us discuss separately the two cases corresponding to the expressions of the function β .

If β is given by (10.a), then the shape operator may be written as (12). From the flatness condition, the hyperbolic angle function θ satisfies the following partial differential equation:

$$\theta_u \sinh \theta = 0.$$

But this case cannot occur since the hyperbolic angle function θ cannot be constant.

If β has expression (10.b), then a change of v -coordinate furnishes $\beta \equiv 1$. The flatness assumption yields precisely case (b) from the classification Theorem 2.2, and hence, parametrization (3) of generalized cylinders is obtained. □

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