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Extreme values of an infinite mixture of normally distributed variables

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Abstract. We study distribution of extreme values of a mixture of an infinite sequence of independent normally distributed variables with the same mean and an increasing sequence of standard deviations, and prove that the common distribution function belongs to the domain of attraction of Gumbel extreme value distribution. The norming constants for the maximum also are given.

1. Introduction

Let (X_n) be a sequence of independent random variables with common distribution function F. Suppose that there exist sequences $a_n > 0$ and b_n of real numbers such that

$$P\Big(\max_{1\le j\le n} X_j \le \frac{x}{a_n} + b_n\Big) \xrightarrow{d} G(x),$$

where G is a non degenerate distribution function; then G belongs to one of the three classes of *maximum stable* distributions, which, possibly after linear transformation of the argument, have the following form:

Type I. (Gumbel) $G_1(x) = \exp(-e^{-x})$, $-\infty < x < +\infty$;
Type II. (Fréchet) $G_2(x) = 0$, for $x \le 0$ and $G_2(x) = \exp(-x^{-\alpha})$, for $x > 0$ and some $\alpha > 0$;
Type III. (Weibull) $G_3(x) = \exp(-(-x)^{\alpha})$, for $x < 0$ and some $\alpha > 0$, and $G_3(x) = 1$, for $x \ge 0$.

These three types of distributions are called *the extreme values distributions*, and we say that the common distribution function F of the corresponding i.i.d. variables $X_1, X_2, \dots$ belongs to *the domain of attraction* of G, with *normalizing constants* $a_n > 0$ and b_n. Note that the normalizing constants are not unique.

Throughout this paper we will use the notation $M_n = \max\{X_1, \dots X_n\}$.

In this paper, we study distribution of extreme values of a mixture of an infinite sequence of independent normally distributed variables with the same mean and an increasing sequence of standard deviations. This paper extends the result of Mladenović [2], where extreme values of mixture of two independent normally distributed variables were studied, to the case of a mixture of an infinite sequence of such variables.

We will show that the common distribution function of a mixture of an infinite sequence of independent normally distributed variables belongs to the domain of attraction of Type I. Thus, limiting distribution of the maximum of the mixture is given by $P\left(a_n(M_n - b_n) \le x\right) \to e^{-e^{-x}}$, with the normalizing constants a_n and b_n also computed in the paper.

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Our result will make use of two Theorems from [1] (for general reference, see also [3, 4]). The first Theorem enables us to determine the domain of attraction and its type for our common distribution function.

Theorem 1.1 (Theorem 1.6.2 [1]). *Let (X_n) be a sequence of independent random variables with the common distribution function $F(x)$, $x \in \mathbb{R}$, and $x_F = \sup\{x|F(x) < 1\}$. Necessary and sufficient conditions for the function F to belong to the domain of attraction of possible types are*

Type I. There exist a strictly positive function $g(t)$ defined on the set $(-\infty, x_F)$, such that for every real number x the equality $\lim_{t\to x_F-} \frac{1-F(t+xg(t))}{1-F(t)} = e^{-x}$ holds true.

Type II. $x_F = +\infty$ and $\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = x^{-\alpha}$, for some $\alpha > 0$ and all $x > 0$.

Type III. $x_F < +\infty$ and $lim_{h\to 0+} \frac{1-F(x_F-hx)}{1-F(x_F-h)} = x^{\alpha}$, for some $\alpha > 0$ and all $x > 0$.

The second Theorem enables us to find the normalizing constants.

Theorem 1.2 (Theorem, 1.5.1 [1]). *Let (X_n) be a sequence of independent random variables with the common distribution function $F(x)$, $x \in \mathbb{R}$. Let (u_n) be a sequence of real numbers, $0 \le \tau \le +\infty$, and $M_n = \max\{X_1, X_2, \dots X_n\}$. Then the equality*

$$\lim_{n\to\infty} P\left(M_n \le u_n\right) = e^{-\tau}$$

holds true if and only if $\lim_{n\to\infty} n(1 - F(u_n)) = \tau$.

We say that a random variable X is a *mixture* of an infinite sequence of random variables (Z_k) with probabilities p_k, $\sum_{k=1}^{\infty} p_k = 1$, if $X = Z_K$, where $K \in \mathbb{N}$ is a random variable independent of random variables Z_k with $P(K = k) = p_k$, i.e. X is equal to Z_k with probability p_k.

In this paper we consider an infinite sequence of normally distibuted random variables, $Z_k \sim N(\mu_k, \sigma_k^2)$, such that $\mu_k = \mu_0$, $k \ge 1$ and $0 < \sigma_1 < \sigma_2 < \dots < \sigma_k \to \sigma_0$ as $k \to \infty$.

The distribution function of variable Z_k is

$$F_k(x) = P\left(Z_k \le x\right) = \Phi\Big(\frac{x-\mu_k}{\sigma_k}\Big),$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt.$$

Then a random variable X which is a mixture of an infinite sequence of variables $Z_1, Z_2, \dots Z_k, \dots$ with probabilities $p_1, p_2, \dots p_k$, $\sum_{k=1}^{\infty} p_k = 1$, has the distribution function given by

$$F(x) = \sum_{k=1}^{\infty} p_k \Phi\Big(\frac{x-\mu_k}{\sigma_k}\Big). \tag{1}$$

2. Main results

We are now ready to determine the type of the domain of attraction and corresponding normalizing constants for our mixture.

Theorem 2.1. *Let (X_n) be a sequence of independent random variables with common distribution $F(x)$ defined by* (1), *then F belongs to the domain of attraction of Gumbel extreme value distribution, i.e., there exist norming constants $a_n > 0$ and b_n such that*

$$\lim_{n\to\infty} P\Big(M_n \le \frac{x}{a_n} + b_n\Big) = \lim_{n\to\infty} F\Big(\frac{x}{a_n} + b_n\Big) = \exp(-e^{-x}),$$

where $M_n = \max\{X_1, X_2 \dots X_n\}$.

Proof. Let

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

and

$$\Phi(x) = \int_{-\infty}^{x} \varphi(t)dt.$$

We shall use the following asymptotic relation

$$1 - \Phi(x) = \frac{1}{x}\varphi(x)(1 + R(x)),$$

where $R(x) \to 0$ as $x \to \infty$.

Suppose $X_k \in N(\mu_k, \sigma_k^2)$ for $k \in \mathbb{N}$. Then

$$1 - F(t) = \sum_{k=1}^{\infty} p_k \left(\varphi\left(\frac{t-\mu_k}{\sigma_k}\right) \frac{\sigma_k}{t-\mu_k} \right) \left(1 + R\left(\frac{t-\mu_k}{\sigma_k}\right)\right).$$

Let

$$R_1(t) = \max_{1 \le k < \infty} \left| R\left(\frac{t-\mu_k}{\sigma_k}\right) \right|.$$

Since $\mu_k = \mu_0$ and $\sigma_0 > \sigma_k > 0$, for $t > \mu_0$, we have $\frac{t-\mu_k}{\sigma_k} \ge \frac{t-\mu_0}{\sigma_0}$. Therefore, since $R(x) \to 0$ as $x \to \infty$, $R_1(t) \to 0$ as $t \to \infty$. Hence

$$\left| 1 - F(t) - \sum_{k=1}^{\infty} p_k \left(\varphi\left(\frac{t-\mu_k}{\sigma_k}\right) \frac{\sigma_k}{t-\mu_k} \right) \right| < R_1(t) \sum_{k=1}^{\infty} p_k \varphi\left(\frac{t-\mu_k}{\sigma_k}\right) \frac{\sigma_k}{t-\mu_k} \to 0, \text{ as } t \to \infty.$$

Therefore

$$1 - F(t) = o(1) + \sum_{k=1}^{\infty} p_k \left(\varphi\left(\frac{t-\mu_k}{\sigma_k}\right) \frac{\sigma_k}{t-\mu_k} \right),$$

$$1 - F(t) + o(1) = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{\infty} p_k \frac{\sigma_k}{t-\mu_k} e^{-\frac{1}{2}\left(\frac{t-\mu_k}{\sigma_k}\right)^2},$$

$$1 - F(t) + o(1) = \frac{1}{\sqrt{2\pi}} \frac{\sigma_0}{t-\mu_0} e^{-\frac{1}{2}\left(\frac{t-\mu_0}{\sigma_0}\right)^2} \sum_{k=1}^{\infty} p_k \frac{\sigma_k}{\sigma_0} \frac{t-\mu_0}{t-\mu_k} e^{-\frac{1}{2}\Delta_k(t)},$$

where

$$\Delta_k(t) = \left(\frac{t-\mu_k}{\sigma_k}\right)^2 - \left(\frac{t-\mu_0}{\sigma_0}\right)^2,$$

$$1 - F(t) + o(1) = \frac{1}{\sqrt{2\pi}} \frac{\sigma_0}{t-\mu_0} e^{-\frac{1}{2}\left(\frac{t-\mu_0}{\sigma_0}\right)^2} p(t),$$

where

$$p(t) = \sum_{k=1}^{\infty} p_k \frac{\sigma_k}{\sigma_0} \frac{t-\mu_0}{t-\mu_k} e^{-\frac{1}{2}\Delta_k(t)}.$$

Let

$$g(t) = \frac{\sigma_0^2}{t-\mu_0}.$$

We have to prove that

$$\frac{p(t+xg(t))}{p(t)} \to 1, \text{ as } t \to \infty.$$

Note that p is an analytic function. Using the Taylor expansion with the Lagrange form of the reminder, we get $p(t+h) = p(t) + p'(\xi)h$, where $t \le \xi \le t+h$, and

$$\frac{p(t+h)}{p(t)} = \frac{p(t)+p'(\xi)h}{p(t)} = 1 + \frac{p'(\xi)}{p(t)}h + o(h).$$

Put

$$h = xg(t) = \frac{x\sigma_0^2}{t-\mu_0}.$$

Note that $h \to 0$ as $t \to \infty$. We have $ht_1 = x\sigma_0^2$ where $t_1 = t - \mu_0$.

Since or assumption is that $\mu_k = \mu_0$, we have

$$\Delta_k(t) = \left(\frac{t-\mu_0}{\sigma_k}\right)^2 - \left(\frac{t-\mu_0}{\sigma_0}\right)^2 = t_1^2\left(\frac{1}{\sigma_k^2} - \frac{1}{\sigma_0^2}\right) = \delta_k t_1^2,$$

where $\delta_k = \frac{1}{\sigma_k^2} - \frac{1}{\sigma_0^2}$.

Hence,

$$p(t) = \sum_{k=1}^{\infty} p_k \frac{\sigma_k}{\sigma_0} e^{-\frac{1}{2}\delta_k t_1^2}.$$

We now compute, using for instance the Dominated convergence theorem to justify term by term differentiation:

$$p'(t) = -\sum_{k=1}^{\infty} p_k \frac{\sigma_k}{\sigma_0} \delta_k t_1 e^{-\frac{1}{2}\delta_k t_1^2},$$

$$-\frac{p'(t)}{t_1} = \sum_{k=1}^{\infty} p_k \frac{\sigma_k}{\sigma_0} \delta_k e^{-\frac{1}{2}\delta_k t_1^2}.$$

Note that $\delta_k \to 0$ as $k \to \infty$.

Note also that $|p'(t)|$ is a decreasing function of t, and hence $\frac{p'(\xi)}{p(t)}h \to 0$ if $\frac{p'(t)}{p(t)}h \to 0$. Since $h = xg(t) = x\frac{\sigma_0^2}{t_1}$, if x is kept constant, the condition $\frac{p'(t)}{t_1 p(t)} \to 0$ as $t \to \infty$ will imply $\frac{p'(t)}{p(t)}h \to 0$ as $t \to 0$.

We have:

$$\frac{p'(t)}{p(t)t_1} = -\frac{\sum_{k=1}^{\infty} p_k \frac{\sigma_k}{\sigma_0} \delta_k e^{-\frac{1}{2}\delta_k t_1^2}}{\sum_{k=1}^{\infty} p_k \frac{\sigma_k}{\sigma_0} e^{-\frac{1}{2}\delta_k t_1^2}} \to 0,$$

as $t_1 \to \infty$, and therefore

$$\frac{p(t+xg(t))}{p(t)} = 1 + o(1), \text{ as } t \to \infty.$$

Note also that if x is kept constant, $\frac{t_1}{t_1+xg(t)} = 1 + o(1)$, as $t \to \infty$ too.

We have in fact also proven that

$$\frac{p(t + O(1/t))}{p(t)} = 1 + o(1), \tag{2}$$

as $t \to \infty$.

We now consider asymptotic behavior as $x \to \infty$. For $x > 0$ we have, as $t \to \infty$

$$\frac{1 - F(t + xg(t))}{1 - F(t)} = e^{\frac{t_1^2 - (t_1 + xg(t))^2}{2\sigma_0^2}} \frac{t_1}{t_1 + xg(t)} \frac{p(t + xg(t))}{p(t)}(1 + o(1)) = e^{-\frac{x^2 g(t)^2}{2\sigma_0^2}} e^{-\frac{xg(t)t_1}{\sigma_0^2}} (1 + o(1)).$$

Recall that $g(t) = \frac{\sigma_0^2}{t_1}$, and hence

$$\frac{1 - F(t + xg(t))}{1 - F(t)} = e^{-\frac{x^2 g(t)^2}{2\sigma_0^2}} e^{-x}(1 + o(1)) \to e^{-x}, \text{ as } t \to \infty.$$

We conclude that the distribution function $F(x)$ belongs to the domain of attraction of the function $G_1(x)$, and we have the type I of extreme value distribution. □

Now we proceed to find the normalizing constants a_n^* and b_n^*.

Theorem 2.2. *Limiting distribution of the maximum of the mixture, in notation of the previous theorem, is given by*

$$P\{a_n^*(M_n - b_n^*) \le x\} \to e^{-e^{-x}},$$

where

$$a_n^* = \frac{\sqrt{2\ln(n)}}{\sigma_0},$$

$$b_n^* = \mu_0 + \sigma_0 \sqrt{2\ln(n)} - \frac{\sigma_0}{2\sqrt{2\ln(n)}}(\ln(\ln(n)) + \ln(4\pi)) - \tau_n,$$

with $\tau_n > 0$ *the smallest positive solution of an equation*

$$\tau = \frac{\sigma_0}{\sqrt{2\ln(n)}} \left| \ln p\Big(\mu_0 + \sigma_0 \sqrt{2\ln(n)} - \frac{\sigma_0}{2\sqrt{2\ln(n)}}(\ln(\ln(n)) + \ln(4\pi)) - \tau\Big) \right|,$$

when $2\ln(n) - \frac{1}{2}(\ln(\ln(n)) + \ln(4\pi)) > |\ln p(\mu_0)|$, *and* 0 *otherwise.*

Proof. Let $v_n^{(0)} = \frac{u_n - \mu_0}{\sigma_0}$ and $v_n^{(k)} = \frac{u_n - \mu_k}{\sigma_k}$, so that $F_k(u_n) = \Phi(v_n^{(k)})$. We have (see proof of the previous theorem)

$$1 - \sum_{k=1}^{\infty} p_k \Phi(v_n^{(k)}) \sim \sum_{k=1}^{\infty} p_k \varphi(v_n^{(k)})/v_n^{(k)} = \frac{1}{\sqrt{2\pi} v_n^{(0)}} e^{-\frac{1}{2}(v_n^{(0)})^2} p(u_n)(1 + o(1)),$$

where

$$p(u_n) = \sum_{k=1}^{\infty} p_k \frac{\sigma_k}{\sigma_0} \frac{u_n - \mu_0}{u_n - \mu_k} e^{-\frac{1}{2}\Delta_k(u_n)},$$

$$\Delta_k(u_n) = \left(\frac{u_n - \mu_k}{\sigma_k}\right)^2 - \left(\frac{u_n - \mu_0}{\sigma_0}\right)^2 = \left(\frac{1}{\sigma_k^2} - \frac{1}{\sigma_0^2}\right) u_n^2 + Au_n + B \to \infty, \text{ as } n \to \infty.$$

Hence, the constant u_n should be determined from the conditions $v_n^{(0)} = \frac{u_n-\mu_0}{\sigma_0}$, i.e. $u_n = \mu_0 + \sigma_0 v_n^{(0)}$, and

$$p(u_n)\frac{\varphi(v_n^{(0)})}{v_n^{(0)}} \sim \frac{1}{n}e^{-x}, \tag{3}$$

as $n \to \infty$.

This asymptotic relation can be rewritten as

$$\frac{1}{np(u_n)}e^{-x}\frac{v_n^{(0)}}{\varphi(v_n^{(0)})} \to 1,$$

or, by taking logarithms,

$$-\ln(n) - \ln(p(u_n)) - x + \ln(v_n^{(0)}) - \ln(\varphi_n^{(0)}) \to 0,$$

which is equivalent to

$$-\ln(n) - \ln(p(u_n)) - x + \ln(v_n^{(0)}) - \left(-\frac{1}{2}\ln(2\pi) - \frac{1}{2}(v_n^{(0)})^2\right) \to 0,$$

i.e.

$$-\ln(n) - \ln(p(u_n)) - x + \ln(v_n^{(0)}) + \frac{1}{2}\ln(2\pi) + \frac{1}{2}(v_n^{(0)})^2 \to 0.$$

Provided that $\frac{\ln(p(u_n))}{\ln(n)} = o(1)$, we will have $\frac{(v_n^{(0)})^2}{2\ln(n)} \to 1$ as $n \to \infty$ and hence, by taking logarithms again

$$2\ln(v_n^{(0)}) - \ln(2) - \ln(\ln(n)) = o(1).$$

Hence

$$\ln(v_n^{(0)}) = \frac{1}{2}(\ln(2) + \ln(\ln(n))) + o(1). \tag{4}$$

Substituting back this expression for $\ln(v_n^{(0)})$, we find

$$\frac{1}{2}(v_n^{(0)})^2 = x + \ln(n) + \ln(p(u_n)) - \frac{1}{2}\ln(4\pi) - \frac{1}{2}\ln(\ln(n)) + o(1). \tag{5}$$

Conversely, if (5) holds and $\frac{\ln(p(u_n))}{\ln(n)} = o(1)$, the right hand side of (5) will be equal to $\ln(n)(1 + o(1))$ and hence, taking logarithm of both sides, we see that (4) will also hold, and hence (3) holds as well.

We will choose a sequence u_n so that it satisfies $\frac{\ln(p(u_n))}{\ln(n)} = o(1)$. For this, it is sufficient that $u_n = O(\sqrt{\ln(n)})$ as $n \to \infty$. To check that this is indeed enough, we use the assumption that $\mu_k = \mu_0$. Hence the formula

$$p(u_n) = \sum_{k=1}^{\infty} p_k\frac{\sigma_k}{\sigma_0}e^{-\frac{1}{2}\delta_k(u_n-\mu_0)^2}$$

holds, and

$$p(u_n) > \frac{p_k\sigma_k}{\sigma_0}e^{-\frac{1}{2}\delta_k(u_n-\mu_0)^2},$$

i.e.

$$|\ln(p(u_n))| < \frac{1}{2}\delta_k(u_n-\mu_0)^2 - \ln\left(\frac{p_k\sigma_k}{\sigma_0}\right).$$

Now using $u_n = O(\sqrt{\ln(n)})$, we get

$$\frac{|\ln(p(u_n))|}{\ln(n)} < \delta_k O(1)$$

for every k, and thus $\frac{\ln(p(u_n))}{\ln(n)} = o(1)$, since $\delta_k \to 0$ as $k \to \infty$.

Now let us find $v_n^{(0)}$ so that (5) holds. Let

$$w_n = \mu_0 + \sigma_0 \sqrt{2\ln(n)} - \frac{\sigma_0}{2\sqrt{2\ln(n)}}(\ln(4\pi) + \ln(\ln(n))).$$

Note that $p(\mu_0) = \sum_{k=1}^{\infty} p_k \frac{\sigma_k}{\sigma_0}$ is the maximum of function p, and for $t > \mu_0$ the function $p(t)$ is decreasing. Define τ_n to be 0 when $w_n - \mu_0 \leq \frac{\sigma_0}{\sqrt{2\ln(n)}}|\ln p(\mu_0)|$, and τ_n to be a solution of an equation

$$\tau = \frac{\sigma_0}{\sqrt{2\ln(n)}}|\ln p(w_n - \tau)| \tag{6}$$

otherwise.

The solution to (6) exists and is unique when $w_n - \mu_0 > \frac{\sigma_0}{\sqrt{2\ln(n)}}|\ln p(\mu_0)|$, since the right hand side of (6) is a decreasing function of τ for $0 < \tau < w_n - \mu_0$, and becomes smaller than the left hand side for $\tau = w_n - \mu_0$. The solution will satisfy $0 < \tau_n < w_n - \mu_0$. But since $w_n = O(\sqrt{\ln(n)})$ we will have that $p(w_n - \tau_n) = o(\ln(n))$ and hence $\tau = o(\sqrt{\ln(n)})$, because of (6).

Note that since $w_n \to \infty$ as $n \to \infty$, for n sufficiently large the condition $w_n - \mu_0 > \frac{\sigma_0}{\sqrt{2\ln(n)}}|\ln p(\mu_0)|$ will be satisfied.

We will use the sequence $u_n = w_n - \tau_n + \frac{\sigma_0 x}{\sqrt{2\ln(n)}}$. Note that since $\tau_n = o(\sqrt{\ln(n)})$ we have $u_n \sim \sigma_0 \sqrt{2\ln(n)}$.

Taking square roots in (5) and using Taylor expansion for $\sqrt{(1+\varepsilon)}$, we see that

$$v_n^{(0)} = \sqrt{2\ln(n)}\left(1 + \frac{1}{2\ln(n)}\left(x - \frac{1}{2}\ln(4\pi) - \frac{1}{2}\ln(\ln(n)) + \ln(p(u_n))\right) + o\left(\frac{1}{\ln(n)}\right)\right) \tag{7}$$

is needed in order to have (3), in addition to $u_n = O(\sqrt{\ln(n)})$.

For n sufficiently large, we will have $\tau_n = \frac{\sigma_0}{\sqrt{2\ln(n)}}\left|\ln p(u_n - \frac{\sigma_0 x}{\sqrt{2\ln(n)}})\right|$. Using $u_n \sim \sigma_0\sqrt{2\ln(n)}$ and (2), we get that $\left|\ln p(u_n - \frac{\sigma_0 x}{\sqrt{2\ln(n)}})\right| / |ln(u_n)| = 1 + o(1)$ and hence $\tau_n = \frac{\sigma_0}{\sqrt{2\ln(n)}}|\ln p(u_n)| + o(1/\sqrt{\ln(n)})$. Therefore (7) is equivalent to

$$u_n = \mu_0 + \sigma_0\sqrt{2\ln(n)}\left(1 + \frac{1}{2\ln(n)}\left(x - \frac{1}{2}\ln(4\pi) - \frac{1}{2}\ln(\ln(n)) + \ln(p(u_n))\right) + o\left(\frac{1}{\ln(n)}\right)\right),$$

$$u_n = \mu_0 + \frac{\sigma_0 x}{\sqrt{2\ln(n)}} + \sigma_0\sqrt{2\ln(n)} - \frac{\sigma_0}{2\sqrt{2\ln(n)}}(\ln(\ln(n)) + \ln(4\pi)) - \tau_n + o\left(\frac{1}{\sqrt{\ln(n)}}\right),$$

$$u_n = w_n + \frac{\sigma_0 x}{\sqrt{2\ln(n)}} - \tau_n + o\left(\frac{1}{\sqrt{\ln(n)}}\right).$$

The last equation is obviously satisfied for n large enough, since $u_n = w_n - \tau_n + \frac{\sigma_0 x}{\sqrt{2\ln(n)}}$.

Note that for u_n the relation $u_n = O(\sqrt{\ln(n)})$ also holds, since $u_n \sim \sigma_0\sqrt{2\ln(n)}$, and so the corresponding $v_n^{(0)}$ will satisfy all the equations used in the above calculation.

This gives us the required constants in the expression $u_n = \frac{x}{a_n^*} + b_n^*$:

$$a_n^* = \frac{\sqrt{2\ln(n)}}{\sigma_0},$$

$$b_n^* = \mu_0 + \sigma_0\sqrt{2\ln(n)} - \frac{\sigma_0}{2\sqrt{2\ln(n)}}(\ln(\ln(n)) + \ln(4\pi)) - \tau_n,$$

where τ_n is the solution of equation (6) when $w_n - \mu_0 > \frac{\sigma_0}{\sqrt{2\ln(n)}}|\ln p(\mu_0)|$, and 0 otherwise. $\square$

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