The mixed-type reverse order laws for generalized inverses of the product of two matrices

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Abstract. The relationship between generalized inverses of AB and the product of generalized inverses of A and B have been studied in this paper. The necessary and sufficient conditions for a number of mixed-type reverse order laws of generalized inverses of two matrix products are derived by using the maximal ranks of the generalized Schur complements.

1. Introduction

Let $\mathbb{C}^{m \times n}$ denote the set of $m \times n$ matrices with complex entries and $\mathbb{C}^m$ denote the set of m-dimensional vectors. I_k denotes the identity matrix of order k and $O_{m \times n}$ be the $m \times n$ matrix of all zero entries (if no confusion occurs, we will drop the subscript). For a matrix $A \in \mathbb{C}^{m \times n}$, A^* and $r(A)$ denote the conjugate transpose and the rank of the matrix A, respectively.

The concept of generalized inverses of a matrix $A \in \mathbb{C}^{m \times n}$ has a long history. The most commonly used definition of generalized inverses was introduced by Penrose in [11], now is known as the Moore-Penrose conditions, which is a matrix $X \in \mathbb{C}^{n \times m}$ satisfying some of the following four equations:

$$(1)\ AXA = A, \quad (2)\ XAX = X, \quad (3)\ (AX)^* = AX, \quad (4)\ (XA)^* = XA. \tag{1.1}$$

For a subset $\{i, j, \cdots, k\}$ of the set $\{1, 2, 3, 4\}$, the set of $n \times m$ matrices satisfying the equations (i), (j), $\cdots$, (k) from among equations $(1) - (4)$ is denoted by $A\{i, j, \cdots, k\}$. A matrix in $A\{i, j, \cdots, k\}$ is called an $\{i, j, \cdots, k\}$-inverse of A and is denoted by $A^{(i,j,\cdots,k)}$. For example, a matrix X of the set $A\{1\}$ is called a g-inverse of A and is denoted by $X = A^{(1)}$. One usually denotes any $\{1, 3\}$-inverse of the set $A\{1, 3\}$ as $A^{(1,3)}$ which is also called a least squares g-inverse of A. Any $\{1, 4\}$-inverse of the set $A\{1, 4\}$ is denoted by $A^{(1,4)}$ which is also called a minimum norm g-inverse of A. Similarly, any $\{1, 3, 4\}$-inverse of the set $A\{1, 3, 4\}$ is denoted by $A^{(1,3,4)}$. The unique $\{1, 2, 3, 4\}$-inverse of A is denoted by $A^\dagger$, which is called the Moore-Penrose inverse of A. We refer the reader to [1, 12, 20] for basic results on generalized inverses.

Let A_i, $i = 1, 2, \cdots, n$, be n matrices such that the product $A_1 A_2 \cdots A_n$ exists. If each of the n matrices is nonsingular, then the product $A_1 A_2 \cdots A_n$ is nonsingular too, and the inverse of $A_1 A_2 \cdots A_n$ satisfies the reverse order law $(A_1 A_2 \cdots A_n)^{-1} = A_n^{-1} A_{n-1}^{-1} \cdots A_1^{-1}$. However, this reverse order law does not hold for

generalized inverses. Naturally, the reverse order laws for the generalized inverses of products of multiple matrices yield a class of interesting problems that are fundamental in the theory of generalized inverses of matrices see [1, 14, 20]. The reverse order law for the Moore-Penrose inverse seems first to have been studied by Greville [7], in the 60's, giving a necessary and sufficient condition for the reverse order law $(AB)^\dagger = B^\dagger A^\dagger$, for matrices A and B. This was followed (see [6]) by further equivalent conditions for the same thing. Sun and Wei [15] extended the reverse order law conditions to the weighted Moore-Penrose inverse, and Hartwig [8] and Tian [16, 17] to the product of three and more matrices, respectively. The next step was to consider the reverse order law for $\{i, j, \cdots, k\}$-inverses, where $\{i, j, \cdots, k\} \subseteq \{1, 2, 3, 4\}$. Werner [21] presented conditions for the reverse order law $B\{1\}A\{1\} \subseteq (AB)\{1\}$ to hold. Wei [4, 22], Wei and Guo [23] studied reverse order laws for $\{1\}$-inverses, $\{1, 2\}$-inverses, $\{1, 3\}$-inverses and $\{1, 4\}$-inverses of matrix products. In [3, 5, 24–26], the reverse order laws for $\{1, 3\}$, $\{1, 4\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$-inverses were considered. For other interesting results on this subject see [2, 9, 10, 13, 18].

In this paper, by applying the maximal ranks of generalized Schur complement [18, 19], we obtain necessary and sufficient conditions for the following mixed-type reverse order laws:

$$B\{1, 3, 4\}A\{1, 3, 4\} \subseteq (AB)\{1\}, \tag{1.2}$$

$$B\{1\}A\{1\} \subseteq (AB)\{1, 3, 4\}, \tag{1.3}$$

$$B\{1, 3\}A\{1, 3\} \subseteq (AB)\{1, 3, 4\}, \tag{1.4}$$

$$B\{1, 4\}A\{1, 4\} \subseteq (AB)\{1, 3, 4\}, \tag{1.5}$$

$$B\{1, 3, 4\}A\{1, 3, 4\} \subseteq (AB)\{1, 3\}, \tag{1.6}$$

$$B\{1, 3, 4\}A\{1, 3, 4\} \subseteq (AB)\{1, 4\}, \tag{1.7}$$

$$B\{1, 3, 4\}A\{1, 3, 4\} \subseteq (AB)\{1, 3, 4\}. \tag{1.8}$$

The significance of our results lies in the fact that the conditions given in this paper are only related to the ranks of the known matrices.

The main tools in the later discussion are the following three lemmas. The first lemma gives the formulas of the maximal ranks of the generalized Schur complements related to the generalized inverses, and the second shows the characterizations of some generalized inverses of a matrix.

Lemma 1.1 *[18, 19] Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times k}$, $C \in \mathbb{C}^{l \times n}$ and $D \in \mathbb{C}^{l \times k}$. Then for any $A^{(i, j, \cdots, k)} \in A\{i, j, \cdots, k\}$,*

$$\max_{A^{(1)}} r(D - CA^{(1)}B) = \min \left\{ r\begin{pmatrix} C & D \end{pmatrix}, \ r\begin{pmatrix} B \\ D \end{pmatrix}, \ r\begin{pmatrix} A & B \\ C & D \end{pmatrix} - r(A) \right\}, \tag{1.9}$$

$$\max_{A^{(1,3)}} r(D - CA^{(1,3)}B) = \min \left\{ \ r\begin{pmatrix} A^*A & A^*B \\ C & D \end{pmatrix} - r(A), \ \ r\begin{pmatrix} B \\ D \end{pmatrix} \right\}, \tag{1.10}$$

$$\max_{A^{(1,3,4)}} r(D - CA^{(1,3,4)}B) = \min \left\{ r\begin{pmatrix} A^*A & A^*B \\ C & D \end{pmatrix} - r(A), \ \ r\begin{pmatrix} AA^* & B \\ CA^* & D \end{pmatrix} - r(A) \right\} \tag{1.11}$$

and

$$r(D - CA^\dagger B) = r\begin{pmatrix} A^*AA^* & A^*B \\ CA^* & D \end{pmatrix} - r(A). \tag{1.12}$$

Lemma 1.2 [17] *Let* $A \in \mathbb{C}^{m \times n}$ *and* $G \in \mathbb{C}^{n \times m}$. *Then*

$$G \in A\{1\} \Leftrightarrow AGA = A, \tag{1.13}$$

$$G \in A\{1,3\} \Leftrightarrow A^*AG = A^*, \tag{1.14}$$

$$G \in A\{1,4\} \Leftrightarrow GAA^* = A^*, \tag{1.15}$$

$$G \in A\{1,3,4\} \Leftrightarrow A^*AG = A^* \text{ and } GAA^* = A^*. \tag{1.16}$$

Lemma 1.3 [1] *Let* $A \in \mathbb{C}^{m \times n}$, *then the general expressions of the following types of g-inverses of A can be written as:*

$$A\{1\} = \{A^{(1)} : \ A^{(1)} = A^\dagger + F_A V + W E_A\}, \tag{1.17}$$

$$A\{1,3\} = \{A^{(1,3)} : \ A^{(1,3)} = A^\dagger + F_A V\}, \tag{1.18}$$

$$A\{1,4\} = \{A^{(1,4)} : \ A^{(1,4)} = A^\dagger + W E_A\}, \tag{1.19}$$

$$A\{1,3,4\} = \{A^{(1,3,4)} : \ A^{(1,3,4)} = A^\dagger + F_A V E_A\}, \tag{1.20}$$

where $F_A = I_n - A^\dagger A$, $E_A = I_m - AA^\dagger$ *and* $V \in \mathbb{C}^{n \times m}$ *and* $W \in \mathbb{C}^{n \times m}$ *are two arbitrary matrices.*

2. The necessary and sufficient conditions for the mixed-type reverse order laws (1.2) and (1.3).

In this section we will present the necessary and sufficient conditions for the mixed-type reverse order laws $B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1\}$ and $B\{1\}A\{1\} \subseteq (AB)\{1,3,4\}$. The relative results are included in the following two theorems.

Theorem 2.1 *Let* $A \in \mathbb{C}^{m \times n}$ *and* $B \in \mathbb{C}^{n \times m}$. *Then* $B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1\}$ *if and only if*

$$r\begin{pmatrix} A^* & B \end{pmatrix} + r(AB) = r(A) + r(B). \tag{2.21}$$

Proof. From the formula (1.13) in Lemma 1.2, we know that the mixed-type reverse order law $B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1\}$ holds if and only if the following equation

$$AB = ABB^{(1,3,4)}A^{(1,3,4)}AB, \tag{2.22}$$

holds for any $A^{(1,3,4)} \in A\{1,3,4\}$ and $B^{(1,3,4)} \in B\{1,3,4\}$, which is equivalent to the following rank identity

$$\max_{B^{(1,3,4)}, \ A^{(1,3,4)}} r(AB - ABB^{(1,3,4)}A^{(1,3,4)}AB) = 0. \tag{2.23}$$

Using the formula (1.11) in Lemma 1.1, we have

$$
\begin{aligned}
\max_{A^{(1,3,4)}} r(AB - ABB^{(1,3,4)}A^{(1,3,4)}AB) &= \min\left\{ r\begin{pmatrix} A^*A & A^*AB \\ ABB^{(1,3,4)} & AB \end{pmatrix} - r(A), \ r\begin{pmatrix} AA^* & AB \\ ABB^{(1,3,4)}A^* & AB \end{pmatrix} - r(A) \right\} \\
&= \min\left\{ r\begin{pmatrix} A & AB \\ ABB^{(1,3,4)} & AB \end{pmatrix} - r(A), \ r\begin{pmatrix} AA^* & AB \\ ABB^{(1,3,4)}A^* & AB \end{pmatrix} - r(A) \right\} \\
&= r\begin{pmatrix} AA^* & AB \\ ABB^{(1,3,4)}A^* & AB \end{pmatrix} - r(A) \\
&= r\begin{pmatrix} AA^* & O \\ ABB^{(1,3,4)}A^* & AB - ABB^{(1,3,4)}A^*(A^*)^\dagger B \end{pmatrix} - r(A) \\
&= r(AB - ABB^{(1,3,4)}A^\dagger AB). \tag{2.24}
\end{aligned}
$$

Again by Lemma 1.1 (1.11), we have

$$
\begin{aligned}
\max_{B^{(1,3,4)},\, A^{(1,3,4)}} r(AB - ABB^{(1,3,4)}A^{(1,3,4)}AB) &= \max_{B^{(1,3,4)}} r(AB - ABB^{(1,3,4)}A^\dagger AB) \\
&= \min\left\{ r\begin{pmatrix} B^*B & B^*A^\dagger AB \\ AB & AB \end{pmatrix} - r(B),\ r\begin{pmatrix} BB^* & A^\dagger AB \\ ABB^* & AB \end{pmatrix} - r(B) \right\} \\
&= \min\left\{ r\begin{pmatrix} B^*B & B^*A^\dagger AB \\ AB & AB \end{pmatrix} - r(B),\ r\begin{pmatrix} B & A^\dagger AB \\ AB & AB \end{pmatrix} - r(B) \right\} \\
&= r\begin{pmatrix} B^*B & B^*A^\dagger AB \\ AB & AB \end{pmatrix} - r(B) \\
&= r(B^*B - B^*A^\dagger AB) + r(AB) - r(B). \tag{2.25}
\end{aligned}
$$

By the formula (1.12) in Lemma 1.1, we have

$$
r(B^*B - B^*A^\dagger AB) = r\begin{pmatrix} A^*AA^* & A^*AB \\ B^*A^* & B^*B \end{pmatrix} - r(A) = r\left(\begin{pmatrix} A \\ B^* \end{pmatrix}\begin{pmatrix} A^* & B \end{pmatrix}\right) - r(A) = r\begin{pmatrix} A^* & B \end{pmatrix} - r(A). \tag{2.26}
$$

Combining (2.24), (2.25) with (2.26), we have

$$
\max_{B^{(1,3,4)},\, A^{(1,3,4)}} r(AB - ABB^{(1,3,4)}A^{(1,3,4)}AB) = r\begin{pmatrix} A^* & B \end{pmatrix} + r(AB) - r(A) - r(B). \tag{2.27}
$$

Let the right hand side of (2.27) be zero, then we obtain the result in Theorem 2.1.

Example 1. Let

$$
A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.
$$

Then, it is easy to obtain that $r(A) = 2$, $r(B) = 2$, $r\begin{pmatrix} A^* & B \end{pmatrix} = 2$, $r(AB) = 2$ and

$$
r\begin{pmatrix} A^* & B \end{pmatrix} + r(AB) = r(A) + r(B).
$$

From Theorem 2.1, we can conclude that the following mixed-type reverse order law holds

$$
B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1\}.
$$

Now we verify this statement. By the definition of $\{1,3,4\}$-inverse in Lemma 1.3, we have

$$
A\{1,3,4\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}-a & a \\ 0 & a & \frac{1}{2}-a \end{pmatrix} \,\middle|\, a \in \mathbb{C} \right\}
$$

and

$$
B\{1,3,4\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & b & -b \end{pmatrix} \,\middle|\, b \in \mathbb{C} \right\}.
$$

Hence the matrix set $B\{1,3,4\}A\{1,3,4\}$ can be expressed as

$$
B\{1,3,4\}A\{1,3,4\} = \left\{ M : M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}-a & a \\ 0 & a & \frac{1}{2}-a \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & b & -b \end{pmatrix} \,\middle|\, a,\, b \in \mathbb{C} \right\}.
$$

It is easy to verify that the identities $(AB)M(AB) = AB$ hold for any matrix $M \in B\{1,3,4\}A\{1,3,4\}$, that is

$$
B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1\}.
$$

Theorem 2.2 *Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times m}$. Then $B\{1\}A\{1\} \subseteq (AB)\{1,3,4\}$ if and only if $AB = O$ or $m + n = r(A) + r(B)$.*

Proof. From the formula (1.16) in Lemma 1.2, we know that the mixed-type reverse order law $B\{1\}A\{1\} \subseteq (AB)\{1,3,4\}$ holds if and only if the following two equations:

$$B^*A^* = B^*A^*ABB^{(1)}A^{(1)}$$

and

$$B^*A^* = B^{(1)}A^{(1)}ABB^*A^*$$

hold for any $A^{(1)} \in A\{1\}$ and $B^{(1)} \in B\{1\}$, which are respectively equivalent to the following two rank identities:

$$\max_{B^{(1)},\, A^{(1)}} r(B^*A^* - B^*A^*ABB^{(1)}A^{(1)}) = 0 \tag{2.28}$$

and

$$\max_{A^{(1)},\, B^{(1)}} r(B^*A^* - B^{(1)}A^{(1)}ABB^*A^*) = 0. \tag{2.29}$$

Using the formula (1.9) in Lemma 1.1, we have

$$\max_{A^{(1)}} r(B^*A^* - B^*A^*ABB^{(1)}A^{(1)}) = \min\left\{ r\begin{pmatrix} B^*A^*ABB^{(1)} & B^*A^* \end{pmatrix},\ r\begin{pmatrix} I_m \\ B^*A^* \end{pmatrix},\ r\begin{pmatrix} A & I_m \\ B^*A^*ABB^{(1)} & B^*A^* \end{pmatrix} - r(A) \right\}$$
$$= \min\left\{ r(AB),\ r(B^*A^*ABB^{(1)} - B^*A^*A) + m - r(A) \right\}. \tag{2.30}$$

Again by Lemma 1.1 (1.9), we have

$$\max_{B^{(1)}} r(B^*A^*ABB^{(1)} - B^*A^*A) = \min\left\{ r\begin{pmatrix} B^*A^*AB & B^*A^*A \end{pmatrix},\ r\begin{pmatrix} I_n \\ B^*A^*A \end{pmatrix},\ r\begin{pmatrix} B & I_n \\ B^*A^*AB & B^*A^*A \end{pmatrix} - r(B) \right\}$$
$$= \min\left\{ r(AB),\ n - r(B) \right\}. \tag{2.31}$$

Combining (2.30) with (2.31), we have

$$\max_{B^{(1)},\, A^{(1)}} r(B^*A^* - B^*A^*ABB^{(1)}A^{(1)}) = \min\left\{ r(AB),\ \max_{B^{(1)}} r(B^*A^*ABB^{(1)} - B^*A^*A) + m - r(A) \right\}$$
$$= \min\left\{ r(AB),\ m + n - r(A) - r(B) \right\}. \tag{2.32}$$

On the other hand, using the formula (1.9) in Lemma 1.1, we have

$$\max_{B^{(1)}} r(B^*A^* - B^{(1)}A^{(1)}ABB^*A^*) = \min\left\{ r\begin{pmatrix} I_m & B^*A^* \end{pmatrix},\ r\begin{pmatrix} A^{(1)}ABB^*A^* \\ B^*A^* \end{pmatrix},\ r\begin{pmatrix} B & A^{(1)}ABB^*A^* \\ I_m & B^*A^* \end{pmatrix} - r(B) \right\}$$
$$= \min\left\{ r(AB),\ r(BB^*A^* - A^{(1)}ABB^*A^*) + m - r(B) \right\}. \tag{2.33}$$

Again by Lemma 1.1 (1.9), we have

$$\max_{A^{(1)}} r(BB^*A^* - A^{(1)}ABB^*A^*) = \min\left\{ r\begin{pmatrix} I_n & BB^*A^* \end{pmatrix},\ r\begin{pmatrix} ABB^*A^* \\ BB^*A^*A \end{pmatrix},\ r\begin{pmatrix} A & ABB^*A^* \\ I_n & BB^*A^* \end{pmatrix} - r(A) \right\}$$
$$= \min\left\{ r(AB),\ n - r(A) \right\}. \tag{2.34}$$

Combining (2.33) with (2.34), we have

$$\max_{A^{(1)},\, B^{(1)}} r(B^*A^* - B^{(1)}A^{(1)}ABB^*A^*) = \min\left\{ r(AB),\ \max_{A^{(1)}} r(BB^*A^* - A^{(1)}ABB^*A^*) + m - r(B) \right\}$$
$$= \min\left\{ r(AB),\ m + n - r(A) - r(B) \right\}. \tag{2.35}$$

Finally, from the formulas (2.28), (2.29), (2.32) and (2.35), we obtain that the mixed-type reverse order law $B\{1\}A\{1\} \subseteq (AB)\{1,3,4\}$ holds if and only if

$$\min\{r(AB), \ m + n - r(A) - r(B)\} = 0.$$

Similar to Example 1, we can easily verify the following three matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

satisfy the mixed-type reverse order law

$$B\{1\}A\{1\} \subseteq (AB)\{1,3,4\}.$$

3. The necessary and sufficient conditions for the mixed-type reverse order laws (1.4) and (1.5).

In this section, by applying the maximal ranks of some generalized Schur complement, we will present the necessary and sufficient conditions for the mixed-type reverse order laws $B\{1,3\}A\{1,3\} \subseteq (AB)\{1,3,4\}$ and $B\{1,4\}A\{1,4\} \subseteq (AB)\{1,3,4\}$.

Theorem 3.1 *Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times m}$. Then $B\{1,3\}A\{1,3\} \subseteq (AB)\{1,3,4\}$ if and only if*

$$r\begin{pmatrix} B^* \\ B^*A^*A \end{pmatrix} = r(B) \quad and \quad \min\left\{ r\begin{pmatrix} A \\ B^* \end{pmatrix} + m - r(A) - r(B), \ r(AB) \right\} = 0.$$

Proof. From the formula (1.16) in Lemma 1.2, we know that the mixed-type reverse order law $B\{1,3\}A\{1,3\} \subseteq (AB)\{1,3,4\}$ holds if and only if the following two equations:

$$B^*A^* = B^*A^*ABB^{(1,3)}A^{(1,3)}$$

and

$$B^*A^* = B^{(1,3)}A^{(1,3)}ABB^*A^*$$

hold for any $A^{(1,3)} \in A\{1,3\}$ and $B^{(1,3)} \in B\{1,3\}$, which are respectively equivalent to the following two rank identities:

$$\max_{B^{(1,3)}, \ A^{(1,3)}} r(B^*A^* - B^*A^*ABB^{(1,3)}A^{(1,3)}) = 0 \tag{3.36}$$

and

$$\max_{A^{(1,3)}, \ B^{(1,3)}} r(B^*A^* - B^{(1,3)}A^{(1,3)}ABB^*A^*) = 0. \tag{3.37}$$

Using the formula (1.10) in Lemma 1.1, we have

$$
\begin{aligned}
\max_{A^{(1,3)}} r(B^*A^* - B^*A^*ABB^{(1,3)}A^{(1,3)}) &= \min\left\{ r\begin{pmatrix} A^*A & A^* \\ B^*A^*ABB^{(1,3)} & B^*A^* \end{pmatrix} - r(A), \ r\begin{pmatrix} I_m \\ B^*A^* \end{pmatrix} \right\} \\
&= \min\left\{ m, \ r(B^*A^*ABB^{(1,3)} - B^*A^*A) \right\} \\
&= r(B^*A^*ABB^{(1,3)} - B^*A^*A). \tag{3.38}
\end{aligned}
$$

Again by Lemma 1.1 (1.10), we have

$$
\begin{aligned}
\max_{B^{(1,3)}} r(B^*A^*ABB^{(1,3)} - B^*A^*A) &= \min\left\{ r\begin{pmatrix} B^*B & B^* \\ B^*A^*AB & B^*A^*A \end{pmatrix} - r(B), \ r\begin{pmatrix} I_n \\ B^*A^*A \end{pmatrix} \right\} \\
&= \min\left\{ n, \ r\begin{pmatrix} B^* \\ B^*A^*A \end{pmatrix} - r(B) \right\} \\
&= r\begin{pmatrix} B^* \\ B^*A^*A \end{pmatrix} - r(B). \tag{3.39}
\end{aligned}
$$

Combining (3.38) with (3.39), we have

$$\max_{B^{(1,3)},\, A^{(1,3)}} r(B^*A^* - B^*A^*ABB^{(1,3)}A^{(1,3)}) = \max_{B^{(1,3)}} r(B^*A^*ABB^{(1,3)} - B^*A^*A) = r\begin{pmatrix} B^* \\ B^*A^*A \end{pmatrix} - r(B). \tag{3.40}$$

On the other hand, using the formula (1.10) in Lemma 1.1, we have

$$\max_{B^{(1,3)}} r(B^*A^* - B^{(1,3)}A^{(1,3)}ABB^*A^*) = \min\left\{ r\begin{pmatrix} B^*B & B^*A^{(1,3)}ABB^*A^* \\ I_m & B^*A^* \end{pmatrix} - r(B),\ r\begin{pmatrix} A^{(1,3)}ABB^*A^* \\ B^*A^* \end{pmatrix} \right\}$$

$$= \min\left\{ r(B^*BB^*A^* - B^*A^{(1,3)}ABB^*A^*) + m - r(B),\ r(AB) \right\}. \tag{3.41}$$

Again by Lemma 1.1 (1.10), we have

$$\max_{A^{(1,3)}} r(B^*BB^*A^* - B^*A^{(1)}ABB^*A^*) = \min\left\{ r\begin{pmatrix} A^*A & A^*ABB^*A^* \\ B^* & B^*BB^*A^* \end{pmatrix} - r(A),\ r\begin{pmatrix} ABB^*A^* \\ B^*BB^*A^* \end{pmatrix} \right\}$$

$$= \min\left\{ r\begin{pmatrix} A^*A \\ B^* \end{pmatrix} - r(A),\ r(AB) \right\}$$

$$= \min\left\{ r\begin{pmatrix} A \\ B^* \end{pmatrix} - r(A),\ r(AB) \right\}. \tag{3.42}$$

Combining (3.41) with (3.42), we have

$$\max_{A^{(1,3)},\, B^{(1,3)}} r(B^*A^* - B^{(1,3)}A^{(1,3)}ABB^*A^*) = \min\left\{ r(AB),\ \max_{A^{(1,3)}} r(B^*BB^*A^* - B^*A^{(1,3)}ABB^*A^*) + m - r(B) \right\}$$

$$= \min\left\{ r(AB),\ r\begin{pmatrix} A \\ B^* \end{pmatrix} + m - r(A) - r(B) \right\}. \tag{3.43}$$

Finally, from the formulas (3.36), (3.37), (3.40) and (3.43), we obtain that the mixed-type reverse order law $B\{1,3\}A\{1,3\} \subseteq (AB)\{1,3,4\}$ holds if and only if

$$r\begin{pmatrix} B^* \\ B^*A^*A \end{pmatrix} = r(B) \quad \text{and} \quad \min\left\{ r\begin{pmatrix} A \\ B^* \end{pmatrix} + m - r(A) - r(B),\ r(AB) \right\} = 0.$$

Example 2. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

It is easy to obtain that $r(A) = 2$, $r(B) = 2$, $r\begin{pmatrix} A \\ B^* \end{pmatrix} = 2$, $r(AB) = 2$, $r\begin{pmatrix} B^* \\ B^*A^*A \end{pmatrix} = 2$. Then, we have

$$r\begin{pmatrix} B^* \\ B^*A^*A \end{pmatrix} = r(B) \quad \text{and} \quad \min\left\{ r\begin{pmatrix} A \\ B^* \end{pmatrix} + m - r(A) - r(B),\ r(AB) \right\} = 0.$$

From Theorem 3.1, we can conclude that the following mixed-type reverse order law holds

$$B\{1,3\}A\{1,3\} \subseteq (AB)\{1,3,4\}.$$

Now we verify this statement. By the definition of $\{1,3\}$-inverse in Lemma 1.3, we have

$$A\{1,3\} = \{A^\dagger + (I_3 - A^\dagger A)W :\ W \in \mathbb{C}^{3\times 2}\} = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ a & b \end{pmatrix} \mid a,\ b \in \mathbb{C} \right\}$$

and

$$B\{1,3\} = \{B^\dagger + (I_2 - B^\dagger B)V : V \in \mathbb{C}^{2\times 3}\} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

Hence the matrix set $B\{1,3\}A\{1,3\}$ can be expressed as

$$B\{1,3\}A\{1,3\} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \\ a & b \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}.$$

It is easy to verify that

$$B\{1,3\}A\{1,3\} \subseteq (AB)\{1,3,4\}.$$

In the remainder of this section, we will present the necessary and sufficient conditions for the mixed-type reverse order law (1.5). Notice that $GAA^* = A^*$ is equivalent to the equation $AA^*G^* = A$. This implies that, by the formulas (1.14) and (1.15) in Lemma 1.2, $G \in A\{1,4\}$ if and only if $G^* \in A^*\{1,3\}$. So from the results obtained in Theorem 3.1, we can immediately get the necessary and sufficient conditions for the mixed-type reverse order law (1.5), which are stated below without proofs.

Theorem 3.2 *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$. Then $B\{1,4\}A\{1,4\} \subseteq (AB)\{1,3,4\}$ if and only if*

$$r\begin{pmatrix} A \\ ABB^* \end{pmatrix} = r(A) \ \text{ and } \ \min\left\{r\begin{pmatrix} B & A^* \end{pmatrix} + m - r(A) - r(B), \ r(AB)\right\} = 0.$$

4. The necessary and sufficient conditions for the mixed-type reverse order laws (1.6), (1.7) and (1.8).

In this section we will present the necessary and sufficient condition for the the reverse order laws $B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1,3\}$ and $B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1,4\}$ and $B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1,3,4\}$. The relative results are included in the following two lemmas.

Lemma 4.1 *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$. Then $B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1,3\}$ if and only if the following condition holds:*

$$\min\left\{r\begin{pmatrix} B^* \\ B^*A^*A \end{pmatrix} - r(B), \ r\begin{pmatrix} B^*B & B^*A^* \\ B^*A^*AB & B^*A^*AA^* \end{pmatrix} + m - r(A) - r(B)\right\} = 0.$$

Proof. From Lemma 1.2 (1.14), we know that the mixed-type reverse order law

$$B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1,3\}$$

holds if and only if the equation

$$B^*A^* = B^*A^*ABB^{(1,3,4)}A^{(1,3,4)} \tag{4.44}$$

holds for any $A^{(1,3,4)} \in A\{1,3,4\}$ and $B^{(1,3,4)} \in B\{1,3,4\}$, which is equivalent to the following rank identity:

$$\max_{A^{(1,3,4)}, \ B^{(1,3,4)}} r(B^*A^* - B^*A^*ABB^{(1,3,4)}A^{(1,3,4)}) = 0. \tag{4.45}$$

Then using the formula (1.11) in Lemma 1.1, we have

$$
\begin{aligned}
\max_{B^{(1,3,4)}} r(B^*A^* - B^*A^*ABB^{(1,3,4)}A^{(1,3,4)}) &= \min\left\{r\begin{pmatrix} B^*B & B^*A^{(1,3,4)} \\ B^*A^*AB & B^*A^* \end{pmatrix} - r(B), \ r\begin{pmatrix} BB^* & A^{(1,3,4)} \\ B^*A^*ABB^* & B^*A^* \end{pmatrix} - r(B)\right\} \\
&= \min\left\{r\begin{pmatrix} B^*B & B^*A^{(1,3,4)} \\ B^*A^*AB & B^*A^* \end{pmatrix} - r(B), \ r\begin{pmatrix} B & A^{(1,3,4)} \\ B^*A^*AB & B^*A^* \end{pmatrix} - r(B)\right\} \\
&= r\begin{pmatrix} B^*B & B^*A^{(1,3,4)} \\ B^*A^*AB & B^*A^* \end{pmatrix} - r(B) \\
&= r(B^*A^* - B^*A^*ABB^\dagger A^{(1,3,4)}). \tag{4.46}
\end{aligned}
$$

Again using the formula (1.11) in Lemma 1.1, we have

$$
\max_{A^{(1,3,4)}} r(B^*A^*A - B^*A^*ABB^\dagger A^{(1,3,4)})
$$

$$
= \min\left\{ r\begin{pmatrix} A^*A & A^* \\ B^*A^*ABB^\dagger & B^*A^* \end{pmatrix} - r(A), \ r\begin{pmatrix} AA^* & I_m \\ B^*A^*ABB^\dagger A^* & B^*A^* \end{pmatrix} - r(A) \right\}
$$

$$
= \min\left\{ r(B^*A^*ABB^\dagger - B^*A^*A), \ r(B^*A^*ABB^\dagger A^* - B^*A^*AA^*) + m - r(A) \right\}. \tag{4.47}
$$

By the formula (1.12) in Lemma 1.1, we have

$$
r(B^*A^*ABB^\dagger - B^*A^*A) = r\begin{pmatrix} B^*BB^* & B^* \\ B^*A^*ABB^* & B^*A^*A \end{pmatrix} - r(B) = r\begin{pmatrix} B^* \\ B^*A^*A \end{pmatrix} - r(B) \tag{4.48}
$$

and

$$
r(B^*A^*ABB^\dagger A^* - B^*A^*AA^*) = r\begin{pmatrix} B^*BB^* & B^*A^* \\ B^*A^*ABB^* & B^*A^*AA^* \end{pmatrix} - r(B) = r\begin{pmatrix} B^*B & B^*A^* \\ B^*A^*AB & B^*A^*AA^* \end{pmatrix} - r(B). \tag{4.49}
$$

Combining (4.46), (4.47), (4.48) with (4.49), we have

$$
\max_{A^{(1,3,4)}, \ B^{(1,3,4)}} r(B^*A^* - B^*A^*ABB^{(1,3,4)}A^{(1,3,4)})
$$

$$
= \max_{A^{(1,3,4)}} r(B^*A^*A - B^*A^*ABB^\dagger A^{(1,3,4)})
$$

$$
= \min\left\{ r\begin{pmatrix} B^* \\ B^*A^*A \end{pmatrix} - r(B), \ r\begin{pmatrix} B^*B & B^*A^* \\ B^*A^*AB & B^*A^*AA^* \end{pmatrix} + m - r(A) - r(B) \right\}. \tag{4.50}
$$

It follows from (4.44), (4.45) and (4.50) that the mixed-type reverse order law $B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1,3\}$ holds if and only if

$$
\min\left\{ r\begin{pmatrix} B^* \\ B^*A^*A \end{pmatrix} - r(B), \ r\begin{pmatrix} B^*B & B^*A^* \\ B^*A^*AB & B^*A^*AA^* \end{pmatrix} + m - r(A) - r(B) \right\} = 0. \tag{4.51}
$$

By Lemma 1.2 (1.14) and (1.15), $G \in A\{1,4\}$ if and only if $G^* \in A^*\{1,3\}$. So from the results obtained in Lemma 4.1, we can get the necessary and sufficient conditions for the mixed-type reverse order law (1.7), which are stated below without proofs.

Lemma 4.2 *Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times m}$. Then $B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1,4\}$ if and only if the following condition holds:*

$$
\min\left\{ r\begin{pmatrix} A \\ ABB^* \end{pmatrix} - r(A), \ r\begin{pmatrix} AA^* & ABB^*A^* \\ B^*A^* & B^*BB^*A^* \end{pmatrix} + m - r(A) - r(B) \right\} = 0.
$$

It is obvious that the reverse order law $B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1,3,4\}$ holds if and only if the mixed-type reverse order laws (1.6) and (1.7) hold. Then from Lemma 4.1 and Lemma 4.2, we immediately obtain the following theorem.

Theorem 4.1 *Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times m}$. Then $B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1,3,4\}$ if and only if*

$$
\min\left\{ r\begin{pmatrix} B^* \\ B^*A^*A \end{pmatrix} - r(B), \ r\begin{pmatrix} B^*B & B^*A^* \\ B^*A^*AB & B^*A^*AA^* \end{pmatrix} + m - r(A) - r(B) \right\} = 0,
$$

and

$$
\min\left\{ r\begin{pmatrix} A \\ ABB^* \end{pmatrix} - r(A), \ r\begin{pmatrix} AA^* & ABB^*A^* \\ B^*A^* & B^*BB^*A^* \end{pmatrix} + m - r(A) - r(B) \right\} = 0.
$$

Example 3. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

It is easy to know that

$$\min \left\{ r \begin{pmatrix} B^* \\ B^*A^*A \end{pmatrix} - r(B), \ r \begin{pmatrix} B^*B & B^*A^* \\ B^*A^*AB & B^*A^*AA^* \end{pmatrix} + m - r(A) - r(B) \right\} = \min\{0, \ 1\} = 0,$$

and

$$\min \left\{ r \begin{pmatrix} A \\ ABB^* \end{pmatrix} - r(A), \ r \begin{pmatrix} AA^* & ABB^*A^* \\ B^*A^* & B^*BB^*A^* \end{pmatrix} + m - r(A) - r(B) \right\} = \min\{0, \ 1\} = 0.$$

By the definition of $\{1, 3, 4\}$-inverse in Lemma 1.3, we have

$$A\{1, 3, 4\} = \{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} - a & a \\ 0 & a & \frac{1}{2} - a \end{pmatrix} \mid a \in \mathbb{C} \}, \quad B\{1, 3, 4\} = \{ \begin{pmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & b & -b \end{pmatrix} \mid b \in \mathbb{C} \}$$

and

$$(AB)\{1, 3, 4\} = \{ \begin{pmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{4} & \frac{1}{4} \\ 0 & e & -e \end{pmatrix} \mid e \in \mathbb{C} \}.$$

we easily know

$$B\{1, 3, 4\}A\{1, 3, 4\} = \{ \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1/4 & 1/4 \\ 0 & \frac{b}{2} - 2ab & 2ab - \frac{b}{2} \end{pmatrix} \mid a, \ b \in \mathbb{C} \} \subseteq (AB)\{1, 3, 4\}.$$

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