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Absolutes of Hausdorff spaces and cardinal invariants F_θ and t_θ II

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Abstract. The research in this article continues the investigation of the cardinal functions F_θ and t_θ for absolutes of Hausdorff spaces. The relationship among the cardinal functions t, t_θ, F and F_θ are obtained for both of the Iliadis and Banaschewski absolutes for some well-known spaces.

1. Introduction

Absolutes play a major role in the study of Hausdorff spaces. The connection between an Hausdorff space and its absolute is enhanced when the relationship between the cardinal functions of the Hausdorff space and its absolute are known. The study of the connections for the two cardinal functions t_θ and F_θ (introduced and first studied in [8, 9]; see also [16]) and the established cardinal functions t and F was started in [6] and continued in [7]. In this paper, the relationships are further developed for the Iliadis and Banaschewski absolutes for some well-known subspaces of $\mathbb{R}^\kappa$ where κ is a cardinal.

2. Notations, terminologies and basic properties

Throughout this paper X will denote a Hausdorff space and $\tau(X)$ the topology on X. Our notation and terminology are mainly as in [12] (for general topological notions), [2, 14, 15] (for cardinal functions), [19, 21] (for H-closed spaces, H-closed extensions and absolutes of Hausdorff spaces) and in [6].

The Greek letters $\alpha, \beta, \gamma, ...$ are used to denote the infinite ordinal numbers and $\kappa, \lambda, \mu, ...$ are used to denote the infinite cardinal numbers. With $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{J}$, $\mathbb{R}$ we respectively denote the sets of positive integer, rational, irrational and real numbers with the usual topology. Also, by $\mathbb{I}^\kappa$ and $\mathbb{D}^\kappa$ we respectively denote the Tychonoff cube and the Cantor cube of weight κ.

For a space X, recall that $\tau(X)(s)$ is the topology generated by the base $RO(X) = \{U \in \tau(X) : U = int_X(cl_X(U))\}$ (semiregularization of X). A space X is *semiregular* if its topology $\tau(X)$ coincides with the topology $\tau(X)(s)$ and we denote it by $X(s)$ (or X_s). Clearly, *every T_3-space X is semiregular* (the converse is not true).

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A function $f : X \to Y$ is *θ-continuous* if for each $x \in X$ and open neighborhood V of $f(x)$, there is an open neighborhood U of x such that $f(cl_X(U)) \subseteq cl_Y(V)$. It easy to see that every continuous function is θ-continuous (the converse is not true). A surjection $f : X \to Y$ is *irreducible* if for each closed set $A \subseteq X$, if $A \neq X$, then $f(A) \neq Y$. Equivalently, f is irreducible if and only if for each nonempty open set $U \in \tau(X)$, there is $y \in Y$ such that $f^{\leftarrow}(y) \subseteq U$.

A space X is *H-closed* if X is closed in every Hausdorff space containing X as a subspace. Equivalently, X is H-closed if every open cover $\mathcal{U}$ of X has a finite subfamily $\mathcal{V}$ whose union is dense in X (i.e. $X \subseteq cl_X(\bigcup_{V\in\mathcal{V}} V)$).

We need this well-known result (see [19]):

X is H-closed Urysohn if and only if X_s is compact Hausdorff.

A space X is *extremally disconnected* (or *ED* for short) if the closure of every open set is open or, equivalently, if the closure of every open subset is clopen in X, i.e., in symbols $CLOP(X) = RO(X)$.

It is easy to verify that

X is ED if and only if X_s is ED and Tychonoff.

([19]) For a space X, let $X^* = X \cup \{\mathcal{U} : \mathcal{U}$ is a free open ultrafilter on $X\}$. Let κX be the set X^* with the topology generated by the base $\tau(X) \cup \{U \cup \{\mathcal{U}\} : U \in \mathcal{U} \in X^* \setminus X\}$, and σX be the set X^* with the topology generated by the base $\{o(U) : U \in \tau(X)\}$ where $o(U) = U \cup \{\mathcal{U} \in X^* \setminus X : U \in \mathcal{U}\}$. Both spaces κX and σX are H-closed extensions of X. κX is called the *Katětov H-closed extension of* X and σX is said the *Fomin H-closed extension of* X. The identity function $id : \kappa X \to \sigma X$ is continuous. The remainder of κX (= $\kappa X \setminus X$) is *discrete* and *closed* in κX, and the remainder of σX (= $\sigma X \setminus X$) is a *zero-dimensional subspace* of σX. If X is a Tychonoff space, then $\kappa X \geq_X \sigma X \geq_X \beta X$ where βX denote the Stone-Čech compactification of X. When X is Tychonoff, $\kappa X = \beta X$ if and only if X is compact and $\sigma X = \beta X$ if and only if every closed nowhere dense subset of X is compact. Also, we have that $(\kappa X)_s = (\sigma X)_s$.

([19]) Let X be a space and θX (called *the Stone space generated by* $RO(X)$ or *the Gleason cover of* X) denote the set of all open ultrafilters on X. For $U \in \tau(X)$ let $oU = \{\mathcal{U} \in \theta X : U \in \mathcal{U}\}$ and the topology on θX generated by $\{oU : U \in \tau(X)\}$ is ED and compact Hausdorff. The subspace $EX = \{\mathcal{U} \in \theta X : a(\mathcal{U}) \neq \emptyset\}$ (called the *the Iliadis absolute of* X) is dense, ED and T_3 (hence 0-dimensional). We define $k_X : EX \to X$ by $k_X(\mathcal{U}) = p$ where $a(\mathcal{U}) = \{p\}$. The function k_X is onto, perfect, irreducible and θ-continuous. Also, the function k_X is continuous if and only if X is T_3. Note that $EX = \bigcup_{p\in X} k_X^{\leftarrow}(p)$. In general, $\{oU \cap k_X^{\leftarrow}(V) : U, V \in \tau(X)\}$ is a base for a topology on EX (finer than $\tau(EX)$). The set EX with this finer topology is denoted by PX (called the *Banaschewski absolute of* X). The map $\Pi_X : PX \to X$ defined by $\Pi_X(\mathcal{U}) = k_X(\mathcal{U})$ is onto, perfect, irreducible and continuous. The space PX is ED but may not be T_3 (hence not 0-dimensional). Also, $\tau(PX)(s) = \tau(EX)$ and when X is T_3, $PX = EX$.

The following fact is well-known:

X is H-closed if and only if EX is compact if and only if PX is H-closed Urysohn.

For the Katětov H-closed extension $\kappa\omega$ of ω, note that $P(\kappa\omega) = \kappa\omega$ and $E(\kappa\omega) = (P(\kappa\omega))_s = (\kappa\omega)_s = \beta\omega$.

For the Fomin H-closed extension $\sigma\omega$ of ω, note that $P(\sigma\omega) = \sigma\omega = P(\beta\omega) = \beta\omega$ and $E(\sigma\omega) = (P(\sigma\omega))_s = (\beta\omega)_s = \beta\omega$.

For $x \in X$, $t(x, X) = \min\{\kappa : \forall\, A \subset X$ with $x \in \overline{A}\ \exists\, B \subset A$ s.t. $|B| \leq \kappa$ and $x \in \overline{B}\}$ is called the *tightness of X at x* and $t_\theta(x, X) = \min\{\kappa : \forall\, A \subset X$ with $x \in cl_\theta(A)\ \exists\, B \subset A$ s.t. $|B| \leq \kappa$ and $x \in cl_\theta(B)\}$ is called the *θ-tightness of X at x* ([8]). $t(X) = \sup_{x\in X}\{t(x, X)\} + \omega$ is called the *tightness of* X and $t_\theta(X) = \sup_{x\in X}\{t_\theta(x, X)\} + \omega$ is called the *θ-tightness of* X ([8]).

A sequence $(x_\alpha : \alpha \in \mu)$ in a space X is called a *free sequence of length* μ if for every $\alpha \in \mu$ we have $cl_X\{x_\beta : \beta < \alpha\} \cap cl_X\{x_\beta : \beta \geq \alpha\} = \emptyset$ and is called a *θ-free sequence of length* μ if for every $\alpha \in \mu$ we have $cl_\theta\{x_\beta : \beta < \alpha\} \cap cl_\theta\{x_\beta : \beta \geq \alpha\} = \emptyset$ ([8]). We define: $F(X) = \sup\{\mu :$ there is a free sequence of length μ in $X\} + \omega$ and $F_\theta(X) = \sup\{\mu :$ there is a θ-free sequence of length μ in $X\} + \omega$ ([8]).

We will need these properties.

Proposition 1. *Let X, Y be spaces.*

(a) ([7]) *Let $\{X_i\}_{i\in I}$ and $\{Y_i\}_{i\in I}$ be two collections of spaces, $X = \prod_{i\in I} X_i$, $Y = \prod_{i\in I} Y_i$. Suppose $f = \prod_{i\in I} f_i : X \to Y$ where $f_i : X_i \to Y_i$ (for each $i \in I$) are surjections. Then, f is irreducible if and only if f_i is irreducible for each $i \in I$.*

(b) ([7]) *Let $p \in X$ and $f : X \to Y$ be a perfect, irreducible, θ-continuous surjection. Then, p is isolated in X if and only if $f(p)$ is isolated in Y.*

(c) ([19]) *Let X and Y be spaces and $f : X \to Y$ be perfect, irreducible, θ-continuous surjection. Then, EX and EY are homeomorphic.*

Here are some preliminary results that will be needed in this paper. Statements (a)-(b) are straightforward to verify.

Proposition 2. *Let X be a space.*

(a) If σ is a topology in X such that $\sigma \supseteq \tau(X)$, then $F(X) \le F(X, \sigma)$ (in particular $F(X_s) \le F(X)$ and $F(EX) \le F(PX)$);

(b) If X is T_3, then $t_\theta(X) = t(X)$ and $F_\theta(X) = F(X)$;

(c) ([1]) *If X is compact Hausdorff, then $F(X) = t(X)$;*

(d) ([6]) *If X is H-closed Urysohn, then*

$$F_\theta(X) = F_\theta(X_s) = F(X_s) = t_\theta(X) = t_\theta(X_s) = t(X_s);$$

(e) ([14]) *If X is T_3, $t(X) \le w(X) \le 2^{d(X)}$.*

Note. In [6], we constructed an H-closed space H for which $F_\theta(H) < t_\theta(H)$.

Proposition 3. ([7]) *For a Hausdorff space X we have:*

(a) $F(PX) \ge F_\theta(PX) = F(EX) = F_\theta(EX) \ge F_\theta(X)$,

(b) $t(PX) = t_\theta(PX) = t(EX) = t_\theta(EX) \ge t_\theta(X)$,

(c) EX is ED Tychonoff (and therefore semiregular) and PX is ED,

(d) if $p \in X$, then $\tau(EX)|_{k_X^{\leftarrow}(p)} = \tau(PX)|_{\Pi_X^{\leftarrow}(p)}$.

Note 1.
(a) ([6, Example 12]) $\omega = t(\kappa\omega) < \mathfrak{c} = t_\theta(\kappa\omega) = F_\theta(\kappa\omega) < 2^{\mathfrak{c}} = F(\kappa\omega)$.
(b) ([14, Example 7.22]) As $\sigma\omega = \beta\omega$, $\mathfrak{c} = t(\sigma\omega) = F(\sigma\omega) = F_\theta(\sigma\omega) = t_\theta(\sigma\omega)$.

Proposition 4. ([7]) *Let X be a Hausdorff space and Y be ED.*

(a) EY is homeomorphic to Y_s and PY is homeomorphic to Y;

(b) If D is a discrete space such that $|D| = d(Y_s)$, then Y_s can be embedded in βD and $t(E) \le t(\beta E) \le t(\beta D) \le w(\beta D) \le 2^{|D|} = 2^{d(E)}$;

(c) A countable subset A of Y_s is C^*-embedded in Y_s. In particular, if B is an infinite compact subspace of Y_s, then B contains a copy of $\beta\omega$ (i.e. Y_s contains a subset $C \simeq \beta\omega$);

(d) if $\beta\omega \hookrightarrow Y$, then $|Y| \ge 2^{\mathfrak{c}}$, $t(Y) \ge \mathfrak{c}$ and $F(Y) \ge \mathfrak{c}$.

Proposition 5. ([7]) *Let X be a space.*

(a) If X is T_3, then $d(EX) = d(X)$,

(b) If X is separable, then EX is separable, $|EX| \le 2^{\mathfrak{c}}$ and $t(EX) \le \mathfrak{c}$,

(c) $w(EX) \le 2^{w(X)}$ and $w(PX) \le 2^{w(X)}$.

Here are additional preliminary results.

Proposition 6. *(a)* ([4]) *If X is compact ED, then $|X| = 2^{w(X)}$, $w(X) = t(X)$ and there is a continuous surjection $f : X \to \mathbb{D}^{w(X)}$;*

(b) ([6]) $t(\mathbb{D}^\kappa) = s(\mathbb{D}^\kappa) = F(\mathbb{D}^\kappa) = \kappa$ *and* $t(\mathbb{I}^\kappa) = s(\mathbb{I}^\kappa) = F(\mathbb{I}^\kappa) = \kappa$.

3. Some results and examples

We start with the following lemma:

Lemma 1. *Let X be a Hausdorff space and $(p_n)_{n\in\omega}$ a sequence converging to $p \in X$ where $p_n \neq p$ for $n \in \omega$. Then, $|k^{\leftarrow}_X(p)| \geq 2^{\mathfrak{c}}$.*

Proof. As X is Hausdorff, there is an open set $U_1 \in \tau(X)$ such that $p_{n_1} \in U_1$ and $p \notin cl_X U_1$. So, let $n_2 = \inf\{m : p_m \in X \setminus cl_X U_1\}$. Then, there is an open set $U_2 \in \tau(X)$ such that $p_{n_2} \in U_2$, $p \notin cl_X U_2$ and $U_2 \subseteq X \setminus cl_X U_1$. So, let $n_3 = \inf\{m : p_m \in X \setminus (cl_X U_1 \cup cl_X U_2)\}$. Then, there is an open set $U_3 \in \tau(X)$ such that $p_{n_3} \in U_3$, $p \notin cl_X U_3$ and $U_3 \subseteq X \setminus (cl_X U_1 \cup cl_X U_2)$. Continuing by induction we obtain a subsequence $(q_n)_{n\in\omega}$ of $(p_n)_{n\in\omega}$ and a family of pairwise disjoint sets $\{U_n : n \in \omega\}$ such that $q_n \in U_n$ and $p \notin cl_X U_n$ for $n \in \omega$. Now, let $\mathcal{U} \in \beta\omega \setminus \omega$ and for $A \in \mathcal{U}$, let $U_A = \bigcup_{n\in A} U_n$ and $\mathcal{F}_{\mathcal{U}} = \{U_A : A \in \mathcal{U}\}$. Note that

(i) $\mathcal{F}_{\mathcal{U}}$ is an open filterbase. If $A, B \in \mathcal{U}$, $U_A \cap U_B = U_{A\cap B}$ and $A \cap B \in \mathcal{U}$.

(ii) $a_X(\mathcal{F}_{\mathcal{U}}) = \{p\}$. Let $T \in \tau(X)$ and $p \in T$; there exists $m \in \omega$ such that $\{q_n : n \geq m\} \subseteq T$. In particular $T \cap U_n \neq \emptyset$ for all $n \geq m$. If $A \in \mathcal{U}$, A is infinite subset of ω and $\{q_n : n \in A\} \cap T \neq \emptyset$. Therefore $T \cap U_A \neq \emptyset$.

(iii) Let $\mathcal{G}_{\mathcal{U}}$ be an open ultrafilter on X such that $\mathcal{G}_{\mathcal{U}} \supseteq \mathcal{F}_{\mathcal{U}} \cup \mathcal{N}^o_p$. So, $a_X(\mathcal{G}_{\mathcal{U}}) = c_X(\mathcal{G}_{\mathcal{U}}) = \{p\}$ ($\mathcal{G}_{\mathcal{U}} \in EX$).

(iv) Let $\mathcal{V} \in \beta\omega \setminus \omega$ and $\mathcal{V} \neq \mathcal{U}$. Since $\mathcal{V} \neq \mathcal{U}$, there are $A \in \mathcal{U}$ and $B \in \mathcal{V}$ such that $A \cap B = \emptyset$. Now, $U_A \cap U_B = \emptyset$ and $U_A \in \mathcal{F}_{\mathcal{U}} \subseteq \mathcal{G}_{\mathcal{U}}$ and $U_B \in \mathcal{F}_{\mathcal{V}} \subseteq \mathcal{G}_{\mathcal{V}}$. Thus, $\mathcal{F}_{\mathcal{U}} \neq \mathcal{F}_{\mathcal{V}}$.

(v) Now, consider the following map $\beta\omega \setminus \omega \to k^{\leftarrow}_X(p)$ defined by $\mathcal{U} \mapsto \mathcal{F}_{\mathcal{U}}$. This function in 1-to-1 and thus $|k^{\leftarrow}_X(p)| \geq |\beta\omega \setminus \omega| = 2^{\mathfrak{c}}$. □

Proposition 7. *Let X be a Hausdorff space containing a convergent sequence $(p_n)_{n\in\omega} \to p$ where $p_n \neq p \in X$ for $n \in \omega$. Then:*

(a) *$\beta\omega \hookrightarrow k^{\leftarrow}_X(p) \subseteq EX$, $|EX| \geq 2^{\mathfrak{c}}$, $t(EX) \geq \mathfrak{c}$, and $F(EX) \geq \mathfrak{c}$;*

(b) *For $\kappa \geq \omega$, $\beta\omega \hookrightarrow EX^\kappa$;*

(c) *$|PX| \geq 2^{\mathfrak{c}}$, $t(PX) \geq \mathfrak{c}$, and $F(PX) \geq \mathfrak{c}$.*

Proof. (a) By previous lemma, $k^{\leftarrow}_X(p)$ is infinite. By Proposition 4(c,d) $\beta\omega \hookrightarrow k^{\leftarrow}_X(p)$, $|EX| \geq 2^{\mathfrak{c}}$, $t(EX) \geq \mathfrak{c}$, and $F(EX) \geq \mathfrak{c}$.

(b) Let $f \in X^\kappa$ be defined by $f(\alpha) = p$ for $\alpha < \kappa$. Also, for $n \in \omega$, let $f_n \in X^\kappa$ be defined by

$$f_n(\alpha) = \begin{cases} p & \text{if } \alpha \neq 0, \\ p_n & \text{if } \alpha = 0. \end{cases}$$

Then $f_n \to f$ in X^κ. By (a), $\beta\omega \hookrightarrow k^{\leftarrow}_X(f) \subseteq EX^\kappa$.

(c) By (a) and Proposition 3(d), $\beta\omega \hookrightarrow \Pi^{\leftarrow}_X(p)$. By Proposition 4(d), $|PX| \geq 2^{\mathfrak{c}}$, $t(PX) \geq \mathfrak{c}$, and $F(PX) \geq \mathfrak{c}$. □

Corollary 1. *For a second countable space X, we have two cases:*

(a) If X is discrete, $EX = PX = X$ and $F_\theta(EX) = F(EX) = t_\theta(EX) = t(EX) = F_\theta(PX) = F(PX) = t_\theta(PX) = t(PX) = \omega$;

(b) If X is not discrete, $F_\theta(EX) = F(EX) = t_\theta(EX) = t(EX) = F_\theta(PX) = F(PX) = t_\theta(PX) = t(PX) = \mathfrak{c}$.

Proof. (*a*) As X is second countable and discrete, $X = \mathbb{N}$ is both ED and semiregular. By Proposition 4(a), we have that $PX = EX = X$.

(*b*) Follows from Propositions 5(b) and 7(a,c) □

An immediate application of Corollary 1 is the computation of $F_\theta(EX)$, $F_\theta(PX)$, $t_\theta(EX)$ and $t_\theta(PX)$ when X is one of the well-known spaces of $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{J}$ and $\mathbb{R}$.

Corollary 2.
(a) $F_\theta(E\mathbb{N}) = F(E\mathbb{N}) = t_\theta(E\mathbb{N}) = t(E\mathbb{N}) = F_\theta(P\mathbb{N}) = F(P\mathbb{N}) = t_\theta(P\mathbb{N}) = t(P\mathbb{N}) = \omega$,

(b) $F_\theta(E\mathbb{Q}) = F(E\mathbb{Q}) = t_\theta(E\mathbb{Q}) = t(E\mathbb{Q}) = F_\theta(P\mathbb{Q}) = F(P\mathbb{Q}) = t_\theta(P\mathbb{Q}) = t(P\mathbb{Q}) = \mathfrak{c}$,

(c) $F_\theta(E\mathbb{J}) = F(E\mathbb{J}) = t_\theta(E\mathbb{J}) = t(E\mathbb{J}) = F_\theta(P\mathbb{J}) = F(P\mathbb{J}) = t_\theta(P\mathbb{J}) = t(P\mathbb{J}) = \mathfrak{c}$,

(d) $F_\theta(E\mathbb{R}) = F(E\mathbb{R}) = t_\theta(E\mathbb{R}) = t(E\mathbb{R}) = F_\theta(P\mathbb{R}) = F(P\mathbb{R}) = t_\theta(P\mathbb{R}) = t(P\mathbb{R}) = \mathfrak{c}$.

Now, the next goal is to compute the cardinal functions F_θ and t_θ for $E\mathbb{I}^\kappa$ ($\mathbb{I}^\kappa$ is the Tychonoff cube of weight κ) and $E\mathbb{D}^\kappa$ ($\mathbb{D}^\kappa$ is the Cantor cube of weight κ) where $\kappa \geq \omega$.

The function $f : \mathbb{D}^\omega \to \mathbb{I}$ defined by $x \mapsto \sum_{i\in\omega} \frac{x(i)}{2^{i+2}}$ is a continuous closed and open surjection (see 4.3 in [10]). By 6.5(c) in [19], there is a closed subset $A \subseteq \mathbb{D}^\omega$ such that $f|_A : A \to \mathbb{I}$ is an irreducible perfect surjection. Now, by Proposition 1(b) A has no isolated points. So, A is second countable and compact Hausdorff. By 3.3(e) in [19], A is homeomorphic to $\mathbb{D}^\omega$. Thus, we have a continuous, perfect irreducible surjection $g : \mathbb{D}^\omega \to \mathbb{I}$.

Theorem 1. *For $\kappa \geq \omega$, $E\mathbb{D}^\kappa$ and $E\mathbb{I}^\kappa$ are homeomorphic.*

Proof. By Proposition 1(a), there is a continuous, perfect irreducible surjection $\varphi : \mathbb{D}^\kappa \to \mathbb{I}^\kappa$. Then, by Proposition 1(c), $E\mathbb{D}^\kappa \simeq E\mathbb{I}^\kappa$. □

Proposition 8. *For $\kappa \geq \omega$, we have:*

(a) $\beta\omega \hookrightarrow E\mathbb{D}^\kappa$,

(b) $d(E\mathbb{D}^\kappa) = d(\mathbb{D}^\kappa) = \log\kappa = \inf\{\lambda : 2^\lambda \geq \kappa\}$,

(c) $\kappa = F(\mathbb{D}^\kappa) \leq F(E\mathbb{D}^\kappa) = t(E\mathbb{D}^\kappa) \leq 2^{d(E\mathbb{D}^\kappa)} = 2^{\log\kappa}$.

Proof. (a) By Proposition 7(b).

(b) As $\mathbb{D}^\kappa$ is T_3, by Proposition 5(b), we have that $d(E\mathbb{D}^\kappa) = d(\mathbb{D}^\kappa)$. By 11.8(d) in [14], $d(\mathbb{D}^\kappa) = \log\kappa = \inf\{\lambda : 2^\lambda \geq \kappa\}$.

(c) By Proposition 13(a) in [6], $t(\mathbb{D}^\kappa) = \kappa$. By Proposition 2(c), $t(\mathbb{D}^\kappa) = F(\mathbb{D}^\kappa)$. By Propositions 2(b) and 3(a), $F(\mathbb{D}^\kappa) \leq F(E\mathbb{D}^\kappa)$. By Proposition 2(c), $F(E\mathbb{D}^\kappa) = t(E\mathbb{D}^\kappa)$, and by Proposition 2(e) $t(E\mathbb{D}^\kappa) \leq 2^{d(E\mathbb{D}^\kappa)} = 2^{\log\kappa}$ (the last inequality by (b)). □

Corollary 3. **[GCH]** *If κ is a successor cardinal, then $t(E\mathbb{D}^\kappa) = F(E\mathbb{D}^\kappa) = \kappa$.*

Proof. If $\kappa = \mu^+$, $\log\kappa \leq \mu$ and $2^{\log\kappa} \leq 2^\mu$. By **[GCH]**, $2^\mu = \mu^+ = \kappa$. So, $\kappa \leq t(E\mathbb{D}^\kappa) \leq 2^{\log\kappa} \leq 2^\mu = \kappa$. □

Compare the next result with Proposition 6(b).

Proposition 9. *If $\omega \leq \kappa \leq \mathfrak{c}$, then $t(E\mathbb{D}^\kappa) = F(E\mathbb{D}^\kappa) = \mathfrak{c}$.*

Proof. Let $\omega \leq \kappa \leq \mathfrak{c}$. By the Hewitt-Marczewski-Pondiczery theorem, $\mathbb{D}^\kappa$ is separable as $\kappa \leq \mathfrak{c}$. By Proposition 5(b), $E\mathbb{D}^\kappa$ is separable too and $t(E\mathbb{D}^\kappa) \leq \mathfrak{c}$. Now, by Proposition 8(a), $\beta\omega \hookrightarrow E\mathbb{D}^\kappa$. So, $\mathfrak{c} = t(\beta\omega) \leq t(E\mathbb{D}^\kappa) =$ (by Theorem 1) $= t(E\mathbb{D}^\kappa)$. □

Corollary 4. $t(E\mathbb{D}^{\omega}) = t(E\mathbb{D}^{\omega_1}) = t(E\mathbb{D}^{\mathfrak{c}}) = F(E\mathbb{D}^{\omega}) = F(E\mathbb{D}^{\omega_1}) = F(E\mathbb{D}^{\mathfrak{c}}) = \mathfrak{c}$.

The following result indicates that $t(E\mathbb{D}^{\omega_2}) = F(E\mathbb{D}^{\omega_2})$ depends on the set theory model.

Note 2.
(*a*) **[¬CH]** $t(E\mathbb{D}^{\omega_2}) = F(E\mathbb{D}^{\omega_2}) = \mathfrak{c} \geq \omega_2$.
(*b*) **[GCH]** $t(E\mathbb{D}^{\omega_2}) = F(E\mathbb{D}^{\omega_2}) = \omega_2 > \mathfrak{c} = \omega_1$.

The next result is the ω-version of the combinatorial principle due to Tarski (for details see 2.1 in [14]).

Lemma 2. ([14]) *There is a family $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathcal{P}(\omega)$ such that $|\mathcal{A}| = \mathfrak{c}$, $|A_\alpha| = \omega$ for each $A_\alpha \in \mathcal{A}$ (with $\alpha \in \mathfrak{c}$), and the intersection of any two distinct elements of $\mathcal{A}$ is finite (i.e. $|A_\alpha \cap A_\beta| < \omega$ for $\alpha < \beta < \mathfrak{c}$).*

Is there a relationship between $F_\theta(EX)$ and $t_\theta(EX)$? By Proposition 2(b), as $F_\theta(EX) = F(EX)$ and $t_\theta(EX) = t(EX)$, this question is equivalent to asking if there is a relationship between $F(EX)$ and $t(EX)$? By Proposition 4(a), if X is ED and Tychonoff, EX is homeomorphic to X. Thus, the latter question is equivalent to asking if there is a relationship between $F(E)$ and $t(E)$ for an arbitrary ED, semiregular space E? In the first example, we construct an ED, semiregular space X such that $t(X) < F(X)$.

Example 1. The space X will be a dense subspace of $\beta\omega$. Apply Lemma 2 (with ω_1 instead of $\mathfrak{c}$) to obtain a family $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\} \subseteq [\omega]^\omega$ such that $A_\alpha \cap A_\beta$ is finite whenever $\alpha < \beta < \omega_1$. Now, we have that $cl_{\beta\omega}A_\alpha \cap cl_{\beta\omega}A_\beta \subseteq \omega$ is finite. So, for each $\alpha < \omega_1$ we select a point $p_\alpha \in cl_{\beta\omega}A_\alpha \setminus \omega$ and let $X = \omega \cup \{p_\alpha : \alpha < \omega_1\}$ (it is easy to see that $|X| = \omega_1$ and X is ED and semiregular). Also, $\omega \subseteq X \subseteq \beta\omega$ and $X \backslash \omega = \{p_\alpha : \alpha < \omega_1\}$ is closed. Moreover, as $cl_{\beta\omega}A_\alpha$ is clopen in $\beta\omega$, $cl_{\beta\omega}A_\alpha \cap X = \{p_\alpha\} \cup A_\alpha$. Then, $X\backslash\omega$ is closed and discrete and therefore $F(X) = |X\backslash\omega| = \omega_1$. To compute $t(X)$, let $B \subseteq X$ and $p \in cl_X B\backslash B$ and, as every point of ω is isolated, $p \notin \omega$ and therefore $p \in X\backslash\omega$. So, $p = p_\alpha$ for some $\alpha < \omega_1$ and we note $p_\alpha \in cl_{\beta\omega}A_\alpha$. So, $cl_{\beta\omega}A_\alpha \cap B \subseteq \omega$ and finally $t(X) = \omega$. That is, X is ED and semiregular and $t(X) < F(X)$.

We do not know whether there is an ED, semiregular space X such that $F(X) < t(X)$. We can find a ED space Y that is not semiregular such that $F(Y) < t(Y)$ and will present this result in the next example.

Example 2. The definition of Y starts with $\mathbb{I} = [0, 1]$ where $\tau(\mathbb{I})$ is the usual topology. Then $Y = E[0, 1]$ with this finer topology:

$$\tau(E[0, 1]) \text{ is generated by } \mathcal{B} = \{oU \setminus F : U \in \tau(\mathbb{I}), F \in [E\mathbb{I}]^{\leq\mathfrak{c}}\}.$$

First we need this fact:

<u>*Claim 1.*</u> $cl_Y(oU \setminus F) = oU$.

Proof. As oU (basic open set in $E\mathbb{I}$) is closed in $E\mathbb{I}$ (with $|oU| = 2^{\mathfrak{c}}$), oU is closed in Y. Let $\mathcal{U} \in oU$, then $U \in \mathcal{U}$. Now, let $\mathcal{U} \in oV \setminus G$ where $G \in [E\mathbb{I}]^{\leq\mathfrak{c}}$. Then, $V \in \mathcal{U}$, $U \cap V \in \mathcal{U}$ and $(oU \setminus F) \cap (oV \setminus G) = (oU \cap oV) \setminus (F \cup G) = o(U \cap V) \setminus (F \cup G)$. Now, $F \cup G \in [E\mathbb{I}]^{\leq\mathfrak{c}}$ and then $o(U \cap V) \setminus (F \cup G) \neq \emptyset$. Thus, $\mathcal{U} \in cl_Y(oU \setminus F)$ and $cl_Y(oU \setminus F) = oU$. △

By Claim 1, we have that $Y_s = E\mathbb{I}$ and note that Y is ED, H-closed and Urysohn (but not semiregular). Also, by By Proposition 2(d) and Corollary 1, $F_\theta(Y) = F_\theta(Y_s) = F(Y_s) = F(E\mathbb{I}) = \mathfrak{c} = t_\theta(Y) = t_\theta(Y_s) = t(Y_s) = t(E\mathbb{I}) = \mathfrak{c}$. So, it remains only to calculate $F(Y)$ and $t(Y)$. At first, we have that $\mathfrak{c} = F_\theta(Y) \leq F(Y)$ by Proposition 2(a). By Proposition 4(b) $E\mathbb{I} \subseteq \beta\omega$. As $hL(\beta\omega) \leq w(\beta\omega) = \mathfrak{c}$, it follows that $hL(E\mathbb{I}) = \mathfrak{c}$ and for each $B \subseteq E\mathbb{I}$, $L(B) \leq \mathfrak{c}$. Now let $\{x_\alpha\}_{\alpha\in\mathfrak{c}^+}$ be a free sequence in Y. Call $B = \{x_\alpha\}_{\alpha\in\mathfrak{c}^+}$, then $L(B) \leq \mathfrak{c}$. We are ready for the second step.

<u>*Claim 2.*</u> There is a point $b \in B$ such that if $b \in oU \in \tau(E\mathbb{I})$ then $|oU \cap B| = \mathfrak{c}^+$ (b is a complete accumulation point).

Proof. Assume the contrary: for each $p \in B$, there exists $oU_p \in \tau(E\mathbb{I})$ such that $p \in oU_p$ and $|oU_p \cap B| \leq \mathfrak{c}$. The open cover $\{oU_p\}_{p\in B}$ of B has a subcover of size $\mathfrak{c}$ and there is a subset $A \subset B$ with $|A| = \mathfrak{c}$ such that $B \subseteq \bigcup_{p\in A} oU_p$. Then $B = \bigcup_{p\in A}(oU_p \cap B)$ and, as A and $oU_p \cap B$ have size $\mathfrak{c}$, B have cardinality $\mathfrak{c}$. But this is not possible as $|B| = \mathfrak{c}^+$. $\triangle$

By Claim 2, there is a $\beta < \mathfrak{c}^+$ such that x_β is a complete accumulation of B in $E\mathbb{I}$. For $B' = \{x_\alpha : \alpha > \beta\}$, $x_\beta \notin cl_Y B'$. There exists $U \in \tau(\mathbb{I})$ and $F \in [E\mathbb{I}]^{\leq \mathfrak{c}}$ such that $(oU \setminus F) \cap B' = \emptyset$ and $x_\beta \in oU \setminus F$. Also, $(oU \setminus F) \cap B \subseteq B \setminus B' = \{x_\alpha : \alpha \leq \beta\}$. Now, $oU \cap B \subseteq ((oU \setminus F) \cap B) \cup F = \{x_\alpha : \alpha \leq \beta\} \cup F$ where $|\{x_\alpha : \alpha \leq \beta\}| = \mathfrak{c}$ and $|F| = \mathfrak{c}$. This contradicts that $|oU \cap B| = \mathfrak{c}^+$ and completes the proof that $F(Y) \leq \mathfrak{c}$. Now, let $B \subseteq Y$ where $B = oU \setminus \{p\}$ with $p \in oU$. It easy to see that $p \in cl_Y B$ and let $G \in [B]^{\leq \mathfrak{c}}$, $(oU \setminus G) \cap G \neq \emptyset$ and $p \notin cl_Y G$. Then, $t(Y) \geq \mathfrak{c}^+$. On the other hand, let $A \subseteq Y$ with $p \in cl_Y A \setminus A$. Suppose $|A| > \mathfrak{c}^+$ and, for each $U \in \tau(\mathbb{I})$ and $p \in oU$, $|oU \cap A| \geq \mathfrak{c}^+$. Now, let $B_U \subseteq oU \cap A$ such that $|B_U| = \mathfrak{c}^+$ and consider $B = \bigcup_{U\in\tau(\mathbb{I}), p\in oU} B_U$. We have that $B \subseteq A$, $|B| = \mathfrak{c}^+$ and $p \in cl_Y B$. Thus, $t(Y) = \mathfrak{c}^+$.

Is there a relationship between $F_\theta(PX)$ and $t_\theta(PX)$? Is there a relationship between $F(PX)$ and $t(PX)$? These two questions are related by Proposition 3(a,b) as $F(PX) \geq F_\theta(PX) = F(EX)$ and $t(PX) \geq t_\theta(PX) = t(EX)$. Again, using Proposition 4(a), the first question can be reformulated as whether there is a relationship between $F_\theta(E)$ and $t_\theta(E)$ for an arbitrary ED space E and the second question is whether there is a relationship between $F(E)$ and $t(E)$ for an arbitrary ED space E. The ED space X in Example 1 shows that $t(X) = t_\theta(X) < F(X) = F_\theta(X)$ and there is no relationship in one direction in response to both questions. The ED space Y in Example 2 shows that $F(Y) < t(Y)$ and completes the proof that there is no relationship between $F(PX)$ and $t(PX)$. The relationship question of whether $t_\theta(PX) \leq F_\theta(PX)$ is true for all spaces X is unanswered.

The relationship stated in Proposition 3(a,b) that $F(PX) \geq F_\theta(PX)$ and $t(PX) \geq t_\theta(PX)$ maybe strict. For $X = \kappa\omega$, X is ED, $PX = X$, $EX = X_s = \beta\omega$, and $\mathfrak{c} = F_\theta(X) < F(X) = 2^{\mathfrak{c}}$. For $Y = (E[0,1], \tau(\mathcal{B}))$ constructed in Example 2, Y is ED, $PY = Y$, $EY = Y_s = E\mathbb{I}$, and $\mathfrak{c} = t_\theta(Y) < t(Y) = \mathfrak{c}^+$.

By Corollary 1, we know that for second countable space X, $F_\theta(EX) = F(EX) = t_\theta(EX) = t(EX) = F_\theta(PX) = F(PX) = t_\theta(PX) = t(PX) = \kappa$ where κ is ω (if X is discrete) or $\mathfrak{c}$ (if X is not discrete). So, the natural question is whether in ZFC, there is a non second countable space X for which $F_\theta(EX) = F(EX) = t_\theta(EX) = t(EX) = F_\theta(PX) = F(PX) = t_\theta(PX) = t(PX) = \omega_1$?

Example 3. We start with the space $X = \omega \cup \{p_\alpha : \alpha < \omega_1\}$ from Example 1 and note that $\omega \subseteq E \subseteq \beta\omega$ and $t(X) = \omega < \omega_1 = F(X)$. The subset $B = X \backslash \omega$ has a complete accumulation point $q \in \beta\omega$. Also, if $q \in U \in CLOP(\beta\omega)$ then, $|U \cap B| = \omega_1$ and $U = cl_{\beta\omega}(U \cap \omega)$. In particular $U \cap X = cl_{\beta\omega}(U \cap \omega) \cap X$. Now, let $X' = X \cup \{q\} \subseteq \beta\omega$ (it is easy to see that $|X'| = \omega_1$ and X' is ED, semiregular). Also, $F(X') = t(X') = \omega_1$. That is, by adding only one point, namely q, to X, we increase the tightness from ω to ω_1.

We conclude this paper with this interesting H-closed result and a comment about the absolute of H-closed spaces in general.

Example 4. In this example our goal is to determine the behavior of the cardinal functions F, F_θ, t and t_θ for the well-known Urysohn's H-closed example (see [20] or 4.8(d) in [19] for details) and for its Iliadis and Banaschewski absolutes. Let $Y = \mathbb{N} \times \mathbb{Z} \subseteq \mathbb{R}^2$ with the subspace topology inherited from the usual topology on the plane $\mathbb{R}^2$. Now, let $X = Y \cup \{p, q\}$ where $p = (0, 1)$ and $q = (0, -1)$ are points in the plane.

A subset $U \subseteq X$ is defined to be open if:

(i) $p \in U$ implies there is $m \in \mathbb{N}$ such that $\{n\} \times \mathbb{N} \subseteq U$ for all $n \geq m$,

(ii) $q \in U$ implies there is $m \in \mathbb{N}$ such that $\{n\} \times \mathbb{N}^- \subseteq U$ for all $n \geq m$, and

(iii) $(n, 0) \in U$ implies there is $m \in \mathbb{N}$ such that $(n, k) \in U$ for all $|k| \geq m$.

Thus, all the points of $D = X \setminus ((\mathbb{N} \times \{0\}) \cup \{p, q\})$ are isolated. The space X is H-closed and semiregular (i.e., minimal Hausdorff) but neither compact nor Urysohn. Also, as all open sets in X are countable, we have that $F(X) = F_\theta(X) = t(X) = t_\theta(X) = \omega$. As the space X is second countable but not discrete, by Corollary 1(b), we have that $F(EX) = F_\theta(EX) = F(PX) = F_\theta(PX) = t(EX) = t_\theta(EX) = t(PX) = t_\theta(PX) = \mathfrak{c}$.

We are able to obtain some cardinality results about the Iliadis absolute of H-closed spaces in general. Let X be an H-closed space. Then EX is a compact ED space. By Proposition 6, $F_\theta(EX) = F(EX) = t_\theta(EX) = t(EX) = w(EX)$, $|EX| = |PX| = 2^{w(EX)}$ and there is a continuous surjection $f : EX \to \mathbb{D}^{w(EX)}$.

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