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New Extended Weyl Type Theorems and Polaroid Operators

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Abstract. In this paper we introduce the notion of s-polaroid and compare it with the related notions of right polaroid and a-polaroid. We establish for a bounded linear operator defined on a Banach space several sufficient and necessary conditions for which properties (b), (ab), (gb), and (gab) hold.

1. Introduction

Throughout this note we assume that $\mathcal{X}$ is an infinite dimensional complex Banach space. Let $B(\mathcal{X})$, $B_0(\mathcal{X})$ denote, respectively, the algebra of bounded linear operators, the ideal of compact operators acting on $\mathcal{X}$. If $T \in B(\mathcal{X})$ we shall write $N(T)$ and $R(T)$ for the null space and range of T. Also, let $\alpha(T) := \dim N(T)$, $\beta(T) := \dim \mathcal{X}/R(T)$, and let $\sigma(T)$, $\sigma_a(T)$, $\sigma_s(T)$, $\sigma_p(T)$, $p_0(T)$, $\pi_0(T)$ denote the spectrum, approximate point spectrum, surjective spectrum, point spectrum of T, the set of poles of the resolvent of T, the set of all eigenvalues of T which are isolated in $\sigma(T)$, respectively. For $T \in B(\mathcal{X})$, the smallest nonnegative integer p such that $N(T^p) = N(T^{p+1})$ is called the *ascent* of T and denoted by $p(T)$. If no such integer exists, we set $p(T) = \infty$. The smallest nonnegative integer q such that $R(T^q) = R(T^{q+1})$ is called the *descent* of T and denoted by $q(T)$. If no such integer exists, we set $q(T) = \infty$. An operator $T \in B(\mathcal{X})$ is called *upper semi-Fredholm* if it has closed range and finite dimensional null space and is called *lower semi-Fredholm* if it has closed range and its range has finite co-dimension. If $T \in B(\mathcal{X})$ is either upper or lower semi-Fredholm, then T is called *semi-Fredholm*, and *index of a semi-Fredholm operator* $T \in B(\mathcal{X})$ is defined by

$$i(T) := \alpha(T) - \beta(T).$$

If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called *Fredholm*. $T \in B(\mathcal{X})$ is called *Weyl* if it is Fredholm of index zero, and *Browder* if it is Fredholm of finite ascent and descent. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ of $T \in B(\mathcal{X})$ are defined by ([16])

$$\sigma_e(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\},$$

$$\sigma_w(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},$$

$$\sigma_b(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\},$$

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respectively. For $T \in B(\mathcal{X})$ and a nonnegative integer n define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_0 = T$). If for some integer n the range $R(T^n)$ is closed and T_n is upper (resp. lower) semi-Fredholm, then T is called *upper* (resp. *lower*) *semi-B-Fredholm*. Moreover, if T_n is Fredholm, then T is called *B-Fredholm*. T is called *semi-B-Fredholm* if it is upper or lower semi-B-Fredholm.

Definition 1.1. Let $T \in B(\mathcal{X})$ and let

$$\Delta(T) := \{n \in \mathbb{N} : m \in \mathbb{N} \text{ and } m \geq n \Rightarrow (R(T^n) \cap N(T)) \subseteq (R(T^m) \cap N(T))\}.$$

Then the *degree of stable iteration* $\operatorname{dis}(T)$ of T is defined as $\operatorname{dis}(T) := \inf \Delta(T)$.

Let T be semi-B-Fredholm and let d be the degree of stable iteration of T. It follows from [11, Proposition 2.1] that T_m is semi-Fredholm and $i(T_m) = i(T_d)$ for each $m \geq d$. This enables us to define the *index of semi-B-Fredholm* T as the index of semi-Fredholm T_d. In [7] he studied this class of operators and he proved [7, Theorem 2.7] that an operator $T \in B(\mathcal{X})$ is B-Fredholm if and only if $T = T_1 \oplus T_2$, where T_1 is Fredholm and T_2 is nilpotent. It appears that the concept of Drazin invertibility plays an important role for the class of B-Fredholm operators. It is well known that T is Drazin invertible if and only if it has finite ascent and descent, which is also equivalent to the fact that

$$T = T_1 \oplus T_2, \text{ where } T_1 \text{ is invertible and } T_2 \text{ is nilpotent.}$$

An operator $T \in B(\mathcal{X})$ is called *B-Weyl* if it is B-Fredholm of index 0. The *B-Fredholm spectrum* $\sigma_{BF}(T)$, the *B-Weyl spectrum* $\sigma_{BW}(T)$, and the *Drazin spectrum* of T are defined by

$$\sigma_{BF}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } B\text{-Fredholm}\},$$

$$\sigma_{BW}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } B\text{-Weyl}\},$$

$$\sigma_D(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Drazin invertible}\}.$$

Now we consider the following sets:

$$BF_+(\mathcal{X}) := \{T \in B(\mathcal{X}) : T \text{ is upper semi-}B\text{-Fredholm}\},$$
$$BF_+^-(\mathcal{X}) := \{T \in B(\mathcal{X}) : T \in BF_+(\mathcal{X}) \text{ and } i(T) \leq 0\},$$
$$LD(\mathcal{X}) := \{T \in B(\mathcal{X}) : p(T) < \infty \text{ and } R(T^{p(T)+1}) \text{ is closed}\},$$
$$RD(\mathcal{X}) := \{T \in B(\mathcal{X}) : q(T) < \infty \text{ and } R(T^{q(T)}) \text{ is closed}\}.$$

By definition,

$$\sigma_{Bea}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin BF_+^-(\mathcal{X})\}$$

is the upper semi-B-essential approximate point spectrum and

$$\sigma_{LD}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin LD(\mathcal{X})\}$$

is the left Drazin spectrum, and

$$\sigma_{RD}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin RD(\mathcal{X})\}$$

is the right Drazin spectrum. It is well known that

$$\sigma_{Bea}(T) \subseteq \sigma_{LD}(T) = [\sigma_{Bea}(T) \cup \operatorname{acc} \sigma_a(T)] \subseteq \sigma_D(T),$$

where we write acc K for the accumulation points of $K \subseteq \mathbb{C}$.

Definition 1.2. An operator $T \in B(\mathcal{X})$ has the *single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0) if for every open neighborhood U of λ_0 the only analytic function $f : U \longrightarrow \mathcal{X}$ which satisfies the equation

$$(T - \lambda)f(\lambda) = 0$$

is the constant function $f \equiv 0$ on U. The operator T is said to have SVEP if T has SVEP at every $\lambda \in \mathbb{C}$.

Evidently, every operator T, as well as its dual T^*, has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$, in particular, at every isolated point of $\sigma(T)$. We also have (see [1, Theorem 3.8])

$$p(T-\lambda) < \infty \implies T \text{ has SVEP at } \lambda, \tag{1.1}$$

and dually

$$q(T-\lambda) < \infty \implies T^* \text{ has SVEP at } \lambda. \tag{1.2}$$

Remark 1.3. If $T-\lambda$ is semi-Fredholm then the implications (1.1) and (1.2) are equivalences, see [1, Chapter 3].

By definition,

$$\sigma_{ea}(T) := \cap\{\sigma_a(T+K) : K \in B_0(\mathcal{X})\}$$

is the essential approximate point spectrum, and

$$\sigma_{ab}(T) := \cap\{\sigma_a(T+K) : TK = KT \text{ and } K \in B_0(\mathcal{X})\}$$

is the Browder essential approximate point spectrum. If we write iso $K := K \setminus$ acc K then we let

$$\pi_{00}(T) := \{\lambda \in \text{iso}\,\sigma(T) : 0 < \alpha(T-\lambda) < \infty\},$$

$$\pi_{00}^a(T) := \{\lambda \in \text{iso}\,\sigma_a(T) : 0 < \alpha(T-\lambda) < \infty\},$$

$$p_{00}(T) := \sigma(T) \setminus \sigma_b(T),$$

$$p_{00}^a(T) := \sigma_a(T) \setminus \sigma_{ab}(T),$$

$$p_0^a(T) := \{\lambda \in \sigma_a(T) : T-\lambda \in LD(\mathcal{X})\},$$

$$p_0^s(T) := \{\lambda \in \sigma_s(T) : T-\lambda \in RD(\mathcal{X})\},$$

$$\pi_0^a(T) := \{\lambda \in \text{iso}\,\sigma_a(T) : \lambda \in \sigma_p(T)\}.$$

We say that *Weyl's theorem holds for $T \in B(\mathcal{X})$*, in symbol (W), if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

Browder's theorem holds for $T \in B(\mathcal{X})$, in symbol (B), if

$$\sigma(T) \setminus \sigma_w(T) = p_{00}(T).$$

The following variants of Weyl's theorem has been introduced by Rakočević, Berkani, and Zariouh in ([10],[11],[12],[13],[17]).

Definition 1.4. Let $T \in B(\mathcal{X})$.

(1) *a-Weyl's theorem holds for T*, in symbol (aW), if $\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}^a(T)$ and *a-Browder's theorem holds for T*, in symbol (aB), if $\sigma_a(T) \setminus \sigma_{ea}(T) = p_{00}^a(T)$.
(2) *Generalized Weyl's theorem holds for T*, in symbol (gW), if $\sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T)$ and *generalized Browder's theorem holds for T*, in symbol (gB), if $\sigma(T) \setminus \sigma_{BW}(T) = p_0(T)$.
(3) *Generalized a-Weyl's theorem holds for T*, in symbol (gaW), if $\sigma_a(T) \setminus \sigma_{Bea}(T) = \pi_0^a(T)$ and *generalized a-Browder's theorem holds for T*, in symbol (gaB), if $\sigma_a(T) \setminus \sigma_{Bea}(T) = p_0^a(T)$.
(4) T satisfies *property* (w) if $\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}(T)$ and satisfies *property* (b) if $\sigma_a(T) \setminus \sigma_{ea}(T) = p_{00}(T)$.
(5) T satisfies *property* (gw) if $\sigma_a(T) \setminus \sigma_{Bea}(T) = \pi_0(T)$ and satisfies *property* (gb) if $\sigma_a(T) \setminus \sigma_{Bea}(T) = p_0(T)$.
(6) T satisfies *property* (aw) if $\sigma(T) \setminus \sigma_w(T) = \pi_{00}^a(T)$ and satisfies *property* (ab) if $\sigma(T) \setminus \sigma_w(T) = p_{00}^a(T)$.

(7) T satisfies *property* (gaw) if $\sigma(T) \setminus \sigma_{BW}(T) = \pi_0^a(T)$ and satisfies *property* (gab) if $\sigma(T) \setminus \sigma_{BW}(T) = p_0^a(T)$.

It is well known ([5],[6],[10],[11],[12],[13],[15],[17]) that if $T \in B(\mathcal{X})$ then we have:

$$\begin{array}{ccccccccccccc}
(gW) & & (gaB) & & (gaw) & \Rightarrow & (aw) & & & & & & (gB) \\
\Uparrow & & \Uparrow & & \Downarrow & & & & & & & & \Uparrow \\
(gw) & \Rightarrow & (gb) & \Rightarrow & (gab) & \Rightarrow & (gB) & \Leftarrow & (gW) & \Leftarrow & (gaW) & \Rightarrow & (gaB) \\
\Downarrow & & \Downarrow & & \Downarrow & & \Updownarrow & & \Downarrow & & \Downarrow & & \Updownarrow \\
(w) & \Rightarrow & (b) & \Rightarrow & (ab) & \Rightarrow & (B) & \Leftarrow & (W) & \Leftarrow & (aW) & \Rightarrow & (aB) \\
\Downarrow & & \Downarrow & & \Uparrow & & & & & & & & \Downarrow \\
(W) & & (aB) & & (aw) & & & & & & & & (B)
\end{array}$$

In the next section we give the structural properties for operators satisfying properties (b), (ab), (gb), and (gab), respectively. Also, we show that properties (b), (ab), (gb), and (gab) can be characterized by means of localized SVEP.

2. New extended Weyl type theorems

Theorem 2.1. Let $T \in B(\mathcal{X})$. Then the following statements are equivalent:

(1) T satisfies property (ab);

(2) $\sigma_{ab}(T) = \sigma_a(T) \cap \sigma_w(T)$.

Proof. (1)⇒(2): Suppose that property (ab) holds for T. Then Browder's theorem holds for T by [13, Theorem 2.4], and so $\sigma_w(T) = \sigma_b(T)$. Therefore $\sigma_{ab}(T) \subseteq \sigma_a(T) \cap \sigma_w(T)$. Conversely, let $\lambda \notin \sigma_{ab}(T)$. Then either $\lambda \in \sigma_a(T) \setminus \sigma_{ab}(T)$ or $\lambda \notin \sigma_a(T)$. Since T satisfies property (ab), we know that if $\lambda \in \sigma_a(T) \setminus \sigma_{ab}(T)$, then $\lambda \in \sigma(T) \setminus \sigma_w(T)$, which means that $\lambda \notin \sigma_a(T) \cap \sigma_w(T)$. Therefore $\sigma_a(T) \cap \sigma_w(T) \subseteq \sigma_{ab}(T)$, and hence $\sigma_{ab}(T) = \sigma_a(T) \cap \sigma_w(T)$.

(2)⇒(1): Suppose that $\sigma_{ab}(T) = \sigma_a(T) \cap \sigma_w(T)$. Let $\lambda \in \sigma(T) \setminus \sigma_w(T)$. Then $\lambda \in \sigma(T) \setminus \sigma_{ab}(T)$. Since $T - \lambda$ is Weyl but not invertible, it is not bounded below. Therefore $\lambda \in p_{00}^a(T)$. Conversely, let $\lambda \in p_{00}^a(T)$. Then $\lambda \in \sigma_a(T) \setminus \sigma_{ab}(T)$. Since $\sigma_{ab}(T) = \sigma_a(T) \cap \sigma_w(T)$, $\lambda \notin \sigma_w(T)$. Therefore $\lambda \in \sigma(T) \setminus \sigma_w(T)$, and hence T satisfies property (ab). □

We give necessary and sufficient conditions for a Banach space operator T to satisfy property (b).

Theorem 2.2. Let $T \in B(\mathcal{X})$. Then the following statements are equivalent:

(1) T satisfies property (b);

(2) $\sigma_{ea}(T) = \sigma_b(T) \cap \sigma_a(T)$;

(3) $\sigma_a(T) \setminus \sigma_{ea}(T) \subseteq p_{00}(T)$;

(4) $\sigma_a(T) = \sigma_{ea}(T) \cup \partial\sigma(T)$;

(5) $\sigma_a(T) \setminus \sigma_{ea}(T) \subseteq \text{iso}\,\sigma(T)$;

(6) $\sigma_a(T) \cap \text{acc } \sigma(T) \subseteq \sigma_{ea}(T)$.

Proof. The statements (1), (2), and (3) are equivalent from [18, Theorem 2.2]. Now we show that (1) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1) and (5) $\Leftrightarrow$ (6).

(1)$\Rightarrow$(4): Suppose that T satisfies property (b). Then $\sigma_a(T) \setminus \sigma_{ea}(T) = p_{00}(T)$. Let $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$. Then $\lambda \in p_{00}(T)$, and so $\lambda \in \text{iso } \sigma(T) \subseteq \partial\sigma(T)$. Therefore $\sigma_a(T) \subseteq \sigma_{ea}(T) \cup \partial\sigma(T)$. But $\sigma_{ea}(T) \cup \partial\sigma(T) \subseteq \sigma_a(T)$, hence $\sigma_a(T) = \sigma_{ea}(T) \cup \partial\sigma(T)$.

(4)$\Rightarrow$(5): Suppose that $\sigma_a(T) = \sigma_{ea}(T) \cup \partial\sigma(T)$. Let $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$. Then $\lambda \in \partial\sigma(T)$, and so T and T^* have SVEP at λ. Since $T - \lambda$ is upper semi-Fredholm, $T - \lambda$ is Browder by Remark 1.3. Therefore $\lambda \in \text{iso } \sigma(T)$, and hence $\sigma_a(T) \setminus \sigma_{ea}(T) \subseteq \text{iso } \sigma(T)$.

(5)$\Rightarrow$(1): Suppose that $\sigma_a(T) \setminus \sigma_{ea}(T) \subseteq \text{iso } \sigma(T)$. Let $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$. Then λ is an isolated point of $\sigma(T)$, and so T and T^* have SVEP at λ. Therefore $\lambda \in \sigma(T) \setminus \sigma_b(T) = p_{00}(T)$ by Remark 1.3. Conversely, let $\lambda \in p_{00}(T)$. Then λ is an isolated point of $\sigma(T)$ and $T - \lambda$ is Browder. So $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$, and hence $\sigma_a(T) \setminus \sigma_{ea}(T) = p_{00}(T)$. Therefore T satisfies property (b).

(5) $\Leftrightarrow$ (6): Suppose that $\sigma_a(T) \setminus \sigma_{ea}(T) \subseteq \text{iso } \sigma(T)$. Let $\lambda \notin \sigma_{ea}(T)$. If $\lambda \notin \sigma_a(T)$, then clearly, $\lambda \notin \sigma_a(T) \cap \text{acc } \sigma(T)$. If $\lambda \in \sigma_a(T)$, then $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$, which means that λ is an isolated point of $\sigma(T)$. Therefore $\lambda \notin \sigma_a(T) \cap \text{acc } \sigma(T)$.

Conversely, suppose that $\sigma_a(T) \cap \text{acc } \sigma(T) \subseteq \sigma_{ea}(T)$. Let $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$. Then $\lambda \notin \text{acc } \sigma(T)$, and hence $\lambda \in \text{iso } \sigma(T)$. Therefore $\sigma_a(T) \setminus \sigma_{ea}(T) \subseteq \text{iso } \sigma(T)$. □

Corollary 2.3. Let T be quasinilpotent. Then T satisfies property (b).

Proof. Straightforward from Theorem 2.2 and the fact that $\text{acc } \sigma(T) = \emptyset$ whenever T is quasinilpotent. □

Theorem 2.4. Let $T \in B(\mathcal{X})$. Then the following statements are equivalent:

(1) T satisfies property (gab) ;

(2) $\sigma_{LD}(T) = \sigma_a(T) \cap \sigma_{BW}(T)$.

Proof. (1)$\Rightarrow$(2): Suppose that property (gab) holds for T. Then T satisfies generalized Browder's theorem by [13, Corollary 2.6], and so $\sigma_{BW}(T) = \sigma_D(T)$. Therefore $\sigma_{LD}(T) \subseteq \sigma_a(T) \cap \sigma_{BW}(T)$. Conversely, let $\lambda \notin \sigma_{LD}(T)$. Then either $\lambda \in \sigma_a(T) \setminus \sigma_{LD}(T)$ or $\lambda \notin \sigma_a(T)$. If $\lambda \notin \sigma_a(T)$, then clearly $\lambda \notin \sigma_a(T) \cap \sigma_{BW}(T)$. Since T satisfies property (gab), we know that if $\lambda \in \sigma_a(T) \setminus \sigma_{LD}(T)$, then $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$, which means that $\lambda \notin \sigma_a(T) \cap \sigma_{BW}(T)$. Therefore $\sigma_a(T) \cap \sigma_{BW}(T) \subseteq \sigma_{LD}(T)$, and hence $\sigma_{LD}(T) = \sigma_a(T) \cap \sigma_{BW}(T)$.

(2)$\Rightarrow$(1): Suppose that $\sigma_{LD}(T) = \sigma_a(T) \cap \sigma_{BW}(T)$. Let $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Then $\lambda \in \sigma(T) \setminus \sigma_{LD}(T)$. Since $T - \lambda$ is B-Weyl but not invertible, it is not bounded below. Therefore $\lambda \in p_0^a(T)$. Conversely, let $\lambda \in p_0^a(T)$. Then $\lambda \in \sigma_a(T) \setminus \sigma_{LD}(T)$. Since $\sigma_{LD}(T) = \sigma_a(T) \cap \sigma_{BW}(T)$, $\lambda \notin \sigma_{BW}(T)$. Therefore $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$, and hence T satisfies property (gab). □

In analogy with Theorem 2.2, we obtain the following.

Theorem 2.5. Let $T \in B(\mathcal{X})$. Then the following statements are equivalent:

(1) T satisfies property (gb);

(2) $\sigma_{Bea}(T) = \sigma_D(T) \cap \sigma_a(T)$;

(3) $\sigma_a(T) = \sigma_{Bea}(T) \cup p_0(T)$;

(4) $\sigma_a(T) \setminus \sigma_{Bea}(T) \subseteq \pi_0(T)$;

(5) $\sigma_a(T) \setminus \sigma_{Bea}(T) \subseteq \text{iso}\,\sigma(T)$;

(6) $\sigma_a(T) = \sigma_{Bea}(T) \cup \partial\sigma(T)$.

Proof. (1)⇒(2) : Suppose that T satisfies property (gb). Since $\sigma_{Bea}(T) \subseteq \sigma_{LD}(T) \subseteq \sigma_D(T)$ and $\sigma_{Bea}(T) \subseteq \sigma_a(T)$, $\sigma_{Bea}(T) \subseteq \sigma_D(T) \cap \sigma_a(T)$. Conversely, let $\lambda \notin \sigma_{Bea}(T)$. If $\lambda \notin \sigma_a(T)$, then clearly, $\lambda \notin \sigma_D(T) \cap \sigma_a(T)$. If $\lambda \in \sigma_a(T)$, then $\lambda \in \sigma_a(T) \setminus \sigma_{Bea}(T)$. Since T satisfies property (gb), $\lambda \in \sigma(T) \setminus \sigma_D(T)$. Therefore $\lambda \notin \sigma_D(T) \cap \sigma_a(T)$, and hence $\sigma_D(T) \cap \sigma_a(T) \subseteq \sigma_{Bea}(T)$.

(2)⇒(3): Suppose that $\sigma_{Bea}(T) = \sigma_D(T) \cap \sigma_a(T)$. Let $\lambda \in \sigma_a(T) \setminus \sigma_{Bea}(T)$. Then $\lambda \notin \sigma_D(T)$, and so $\lambda \in p_0(T)$. But $\sigma_{Bea}(T) \cup p_0(T) \subseteq \sigma_a(T)$, hence $\sigma_a(T) = \sigma_{Bea}(T) \cup p_0(T)$.

(3)⇒(4): Since $p_0(T) \subseteq \pi_0(T)$, it is clear.

(4)⇒(5): Since $\pi_0(T) \subseteq \text{iso}\,\sigma(T)$, it is clear.

(5)⇒(6): Let $\lambda \in \sigma_a(T) \setminus \sigma_{Bea}(T)$. Then λ is an isolated point of $\sigma(T)$, and so $\lambda \in \partial\sigma(T)$. Conversely, since $\sigma_{Bea}(T) \subseteq \sigma_a(T)$ and $\partial\sigma(T) \subseteq \sigma_a(T)$, $\sigma_{Bea}(T) \cup \partial\sigma(T) \subseteq \sigma_a(T)$. Therefore $\sigma_a(T) = \sigma_{Bea}(T) \cup \partial\sigma(T)$.

(6)⇒(1): Suppose that $\sigma_a(T) = \sigma_{Bea}(T) \cup \partial\sigma(T)$. Let $\lambda \in \sigma_a(T) \setminus \sigma_{Bea}(T)$. Then λ is a boundary point of $\sigma(T)$, and so T and T^* have SVEP at λ. Therefore $T - \lambda$ is B-Weyl. But $\lambda \in \partial\sigma(T)$, hence $T - \lambda$ is Drazin invertible by [8, Theorem 2.3], which implies that $\lambda \in p_0(T)$. Conversely, let $\lambda \in p_0(T)$. Then $\lambda \in \text{iso}\,\sigma(T) \setminus \sigma_D(T)$, and hence $\lambda \in \sigma_a(T) \setminus \sigma_{Bea}(T)$. Therefore T satisfies property (gb). □

Let $H(\sigma(T))$ denote the set of all analytic functions defined on an open neighborhood of $\sigma(T)$. From Theorem 2.5, we obtain the following corollary.

Corollary 2.6. Suppose T^* has SVEP. Then $f(T)$ satisfies property (gb) for each $f \in H(\sigma(T))$.

Proof. Since T^* has SVEP, $\sigma_{Bea}(T) = \sigma_D(T)$ and $\sigma_a(T) = \sigma(T)$. So $\sigma_{Bea}(T) = \sigma_D(T) \cap \sigma_a(T)$, and hence T satisfies property (gb) by Theorem 2.5. Since $f(T^*) = f(T)^*$, $f(T)^*$ has SVEP for each $f \in H(\sigma(T))$. Therefore $f(T)$ satisfies property (gb) for each $f \in H(\sigma(T))$. □

The following example shows that the converse of Corollary 2.6 does not hold in general.

Example 2.7. Let $U \in B(\ell_2)$ be the unilateral shift. Then $\sigma_{Bea}(U) = \sigma_a(U) = \mathbf{\Gamma}$ and $\sigma_D(U) = \overline{\mathbb{D}}$, where $\mathbf{\Gamma}$ is the unit circle and $\mathbb{D}$ is the open unit disk. Therefore $\sigma_{Bea}(U) = \sigma_D(U) \cap \sigma_a(U)$, and hence U satisfies property (gb) by Theorem 2.5. However, U^* does not have SVEP.

Now we characterize the bounded linear operators T satisfying properties (b), (gb), (ab), and (gab) by means of localized SVEP.

Theorem 2.8. Let $T \in B(\mathcal{X})$. Then the following equivalences hold:

(1) T satisfies property $(b) \Leftrightarrow T^*$ has SVEP at every $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$.

(2) T satisfies property $(gb) \Leftrightarrow T^*$ has SVEP at every $\lambda \in \sigma_a(T) \setminus \sigma_{Bea}(T)$.

(3) T satisfies property $(ab) \Leftrightarrow T^*$ has SVEP at every $\lambda \in [\sigma(T) \setminus \sigma_w(T)] \cup p_{00}^a(T)$.

(4) T satisfies property $(gab) \Leftrightarrow T^*$ has SVEP at every $\lambda \in [\sigma(T) \setminus \sigma_{BW}(T)] \cup p_0^a(T)$.

Proof. (1): Suppose that T satisfies property (b). Then $\sigma_a(T) \setminus \sigma_{ea}(T) = p_{00}(T)$. Let $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$. Then $\lambda \in p_{00}(T)$, and so λ is an isolated point of $\sigma(T^*)$. Therefore T^* has SVEP at λ. Conversely, suppose that T^* has SVEP at every $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$. Let $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$. Since $\lambda \notin \sigma_{ea}(T)$, $T - \lambda$ is upper semi-Fredholm and $i(T - \lambda) \leq 0$. But T^* has SVEP at λ, hence $i(T - \lambda) \geq 0$. So $T - \lambda$ is Weyl. Since $q(T - \lambda) < \infty$ by Remark 1.3, $T - \lambda$ is Browder. Therefore $\lambda \in p_{00}(T)$, and hence $\sigma_a(T) \setminus \sigma_{ea}(T) \subseteq p_{00}(T)$. But $p_{00}(T) \subseteq \sigma_a(T) \setminus \sigma_{ea}(T)$ for any T, hence $\sigma_a(T) \setminus \sigma_{ea}(T) = p_{00}(T)$. Therefore T satisfies property (b).

(2): Suppose that T satisfies property (gb). Then $\sigma_a(T) \setminus \sigma_{Bea}(T) = p_0(T)$. Let $\lambda \in \sigma_a(T) \setminus \sigma_{Bea}(T)$. Then $\lambda \in p_0(T)$, and so λ is an isolated point of $\sigma(T^*)$. Therefore T^* has SVEP at λ. Conversely, suppose that T^* has SVEP at every $\lambda \in \sigma_a(T) \setminus \sigma_{Bea}(T)$. Let $\lambda \in \sigma_a(T) \setminus \sigma_{Bea}(T)$. Since $\lambda \notin \sigma_{Bea}(T)$, $T - \lambda$ is upper semi-B-Fredholm and $i(T - \lambda) \leq 0$. But T^* has SVEP at λ, hence $i(T - \lambda) \geq 0$. So $T - \lambda$ is B-Weyl, and hence $T - \lambda$ is quasi-Fredholm. It follows form [2, Theorem 2.11] that $q(T - \lambda) < \infty$. Since $T - \lambda = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$, where T_1 is Weyl and T_2 is nilpotent by [9, Lemma 4.1], T_1 is Browder and T_2 is nilpotent. Therefore $T - \lambda$ is Drazin invertible, and hence $\lambda \in p_0(T)$. So we have $\sigma_a(T) \setminus \sigma_{Bea}(T) \subseteq p_0(T)$. Conversely, let $\lambda \in p_0(T)$. Then $\lambda \in \sigma(T) \setminus \sigma_D(T)$, and so $\lambda \in \text{iso}\,\sigma(T)$ and $T - \lambda$ is Drazin invertible. Therefore $\lambda \in \sigma_a(T) \setminus \sigma_{Bea}(T)$, and hence $p_0(T) \subseteq \sigma_a(T) \setminus \sigma_{Bea}(T)$. So T satisfies property (gb).

(3): Suppose that T satisfies property (ab). Then $\sigma(T) \setminus \sigma_w(T) = p_{00}^a(T)$. Let $\lambda \in \sigma(T) \setminus \sigma_w(T)$. Then $\lambda \in p_{00}^a(T)$, and so λ is an isolated point of $\sigma_a(T)$, which implies that T has SVEP at λ. Since $T - \lambda$ is Weyl, it follows from Remark 1.3 that $p(T - \lambda) < \infty$. Therefore $T - \lambda$ is Browder, and so λ is an isolated point of $\sigma(T^*)$. Hence T^* has SVEP at λ. Conversely, suppose that T^* has SVEP at every $\lambda \in [\sigma(T) \setminus \sigma_w(T)] \cup p_{00}^a(T)$. Let $\lambda \in \sigma(T) \setminus \sigma_w(T)$. Since $\lambda \notin \sigma_w(T)$, $T - \lambda$ is Weyl. Since T^* has SVEP at λ, it follows from Remark 1.3 that $q(T - \lambda) < \infty$. Therefore $T - \lambda$ is Browder, and hence $\lambda \in p_{00}^a(T)$. So $\sigma(T) \setminus \sigma_w(T) \subseteq p_{00}^a(T)$. Conversely, let $\lambda \in p_{00}^a(T)$. Then T and T^* have SVEP at λ. Since $T - \lambda$ is upper semi-Fredholm, it follows from Remark 1.3 that $T - \lambda$ is Weyl. So $\lambda \in \sigma(T) \setminus \sigma_w(T)$, and hence $p_{00}^a(T) \subseteq \sigma(T) \setminus \sigma_w(T)$. Therefore T satisfies property (ab).

(4): Suppose that T satisfies property (gab). Then $\sigma(T) \setminus \sigma_{BW}(T) = p_0^a(T)$. Let $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Then $\lambda \in p_0^a(T)$, and so λ is an isolated point of $\sigma_a(T)$. Therefore T has SVEP at λ. Since $T - \lambda$ is B-Weyl, it follows from [9, Lemma 4.1] that $T - \lambda = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$, where T_1 is Weyl and T_2 is nilpotent. Since T_1 has SVEP at 0, T_1 is Browder by Remark 1.3. Hence $T - \lambda$ is Drazin invertible, and so T^* has SVEP at λ. Conversely, suppose that T^* has SVEP at every $\lambda \in [\sigma(T) \setminus \sigma_{BW}(T)] \cup p_0^a(T)$. Let $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Since $\lambda \notin \sigma_{BW}(T)$, $T - \lambda$ is B-Weyl. But T^* has SVEP at λ, hence $T - \lambda$ is Drazin invertible. Therefore $\lambda \in p_0^a(T)$, and hence $\sigma(T) \setminus \sigma_{BW}(T) \subseteq p_0^a(T)$. Conversely, let $\lambda \in p_0^a(T)$. Then T and T^* have SVEP at λ. Since $T - \lambda$ is upper semi-B-Fredholm, it follows from [2, Theorems 2.7 and 2.11] that $T - \lambda$ is B-Weyl. So $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$, and hence $p_0^a(T) \subseteq \sigma(T) \setminus \sigma_{BW}(T)$. Therefore T satisfies property (gab). □

Now we introduce the concept of s-polaroid and compare it with the related notions of right polaroid and a-polaroid.

Definition 2.9. Let $T \in B(\mathcal{X})$. An operator T is called *a-polaroid* if iso $\sigma_a(T) \subseteq p_0(T)$. T is *s-polaroid* if iso $\sigma_s(T) \subseteq p_0(T)$. T is called *polaroid* if iso $\sigma(T) \subseteq p_0(T)$. T is called *isoloid* if iso $\sigma(T) \subseteq \sigma_p(T)$. T is said to be *left polaroid* if iso $\sigma_a(T) \subseteq p_0^a(T)$ and T is said to be *right polaroid* if iso $\sigma_s(T) \subseteq p_0^s(T)$.

From these definitions, if $T \in B(\mathcal{X})$ then we have:

$$T\ a\text{-polaroid} \implies T \text{ polaroid} \implies T \text{ isoloid}$$

$$T\ a\text{-polaroid} \implies T \text{ left polaroid}$$

$$T \text{ left polaroid or right polaroid} \implies T \text{ polaroid}$$

The concept of s-polaroid and a-polaroid are dual each other:

Theorem 2.10. Let $T \in B(\mathcal{X})$.

(1) Suppose T is s-polaroid. Then it is right polaroid.

(2) T is s-polaroid if and only if T^* is a-polaroid.

Proof. (1) Suppose T is s-polaroid. Let λ is an isolated point of $\sigma_s(T)$. Since T is s-polaroid, $0 < p := p(T-\lambda) = q(T-\lambda) < \infty$. So $p = q(T-\lambda) \in \mathbb{N}$ and $(T-\lambda)^p(\mathcal{X})$ is closed. Therefore $\lambda \in p_0^s(T)$, and hence T is right polaroid.

(2) Recall that

$$\sigma_s(T) = \sigma_a(T^*) \text{ and } \sigma_D(T) = \sigma_D(T^*).$$

Therefore $p_0(T) = p_0(T^*)$, and hence

$$\text{iso } \sigma_s(T) \subseteq p_0(T) \iff \text{iso } \sigma_a(T^*) \subseteq p_0(T^*).$$

So T is s-polaroid if and only if T^* is a-polaroid. $\square$

The following example shows that the converse of the statement (1) of Theorem 2.10, in general, does not hold.

Example 2.11. Let U be the unilateral shift on ℓ_2 and let $A \in B(\ell_2)$ be given by

$$A(x_1, x_2, x_3, \dots) := (0, x_2, x_3, x_4, \dots).$$

Define $T := U^* \oplus A$. Then $\sigma_s(T) = \mathbf{\Gamma} \cup \{0\}$, and so iso $\sigma_s(T) = \{0\}$. Since $q(T) = 1$ and $R(T)$ is closed, $0 \in p_0^s(T)$. Therefore T is right polaroid. However, since $p(T) = \infty$, T is not s-polaroid.

The following result gives a very simple framework for establishing property (gaw) if T is a-polaroid.

Theorem 2.12. Let $T \in B(\mathcal{X})$. Suppose T is a-polaroid. Then the following statements are equivalent:

(1) T has SVEP at every $\lambda \notin \sigma_{BW}(T)$;

(2) T satisfies property (gaw);

(3) T satisfies property (gab);

(4) T satisfies property (ab);

(5) T satisfies property (aw);

(6) Weyl's theorem holds for T;

(7) Generalized Browder's theorem holds for T.

Proof. (1) ⇔ (2): Suppose T has SVEP at every $\lambda \notin \sigma_{BW}(T)$. Let $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Then $T - \lambda$ is B-Weyl. Since T has SVEP at λ, $T - \lambda$ is Drazin invertible. So $\lambda \in \text{iso}\, \sigma(T) \setminus \sigma_D(T)$, and hence $\lambda \in \pi_0^a(T)$. Conversely, let $\lambda \in \pi_0^a(T)$. Then λ is an isolated point of $\sigma_a(T)$ and $\alpha(T - \lambda) > 0$. Since T is a-polaroid, $\lambda \in p_0(T)$. Therefore $T - \lambda$ is Drazin invertible, and hence $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Hence $\sigma(T) \setminus \sigma_{BW}(T) = \pi_0^a(T)$, and so T satisfies property (gaw).
Conversely, suppose T satisfies property (gaw). Then $\sigma(T) \setminus \sigma_{BW}(T) = \pi_0^a(T)$. If $\lambda \notin \sigma(T)$, then clearly, T has SVEP at λ. If $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$, then $\lambda \in \pi_0^a(T)$. Therefore λ is an isolated point of $\sigma_a(T)$, and hence T has SVEP at λ.

(2)⇒(3) and (3)⇒(4): These statements hold by [13, Theorems 3.5 and 2.2].

(4)⇒(5): Suppose T satisfies property (ab). Then $\sigma(T) \setminus \sigma_w(T) = p_{00}^a(T)$. To show that T satisfies property (aw) it is sufficient to show that $p_{00}^a(T) = \pi_{00}^a(T)$. Let $\lambda \in \pi_{00}^a(T)$. Then λ is an isolated point of $\sigma_a(T)$ and $0 < \alpha(T - \lambda) < \infty$. Since T is a-polaroid, $\lambda \in p_0(T)$. Therefore $T - \lambda$ has finite ascent and descent, and hence $T - \lambda$ is Browder, which implies that $\lambda \in p_{00}^a(T)$. But $p_{00}^a(T) \subseteq \pi_{00}^a(T)$, hence $p_{00}^a(T) = \pi_{00}^a(T)$. So T satisfies property (aw).

(5)⇒(6): Suppose T satisfies property (aw). Then $\sigma(T) \setminus \sigma_w(T) = \pi_{00}^a(T)$. We first show that Browder's theorem holds for T. Let $\lambda \in \sigma(T) \setminus \sigma_w(T)$. Then $T - \lambda$ is Weyl and $\lambda \in \pi_{00}^a(T)$. So T has SVEP at λ, and so $p(T - \lambda) < \infty$. Since $0 < \alpha(T - \lambda) = \beta(T - \lambda) < \infty$, $0 < p(T - \lambda) = q(T - \lambda) < \infty$. Therefore $T - \lambda$ is Browder, and hence $\lambda \in \sigma(T) \setminus \sigma_b(T)$. So $\sigma_w(T) = \sigma_b(T)$, and hence Browder's theorem holds for T. To show that Weyl's theorem holds for T it suffices to prove that the equality $\pi_{00}(T) = p_{00}(T)$ holds. Let $\lambda \in \pi_{00}(T)$. Since the inclusion $\pi_{00}(T) \subseteq \pi_{00}^a(T)$ is clear, $\lambda \in \pi_{00}^a(T)$. Since T satisfies property (aw), $\lambda \in \sigma(T) \setminus \sigma_w(T)$. But T has SVEP at λ, hence $0 < p(T - \lambda) = q(T - \lambda) < \infty$. Therefore $\lambda \in p_{00}(T)$, and so $\pi_{00}(T) \subseteq p_{00}(T)$. Since the opposite inclusion holds for every $T \in B(\mathcal{X})$, $\pi_{00}(T) = p_{00}(T)$. Therefore Weyl's theorem holds for T.

(6)⇒(7): Suppose Weyl's theorem holds for T. Then Browder's theorem holds for T. Since T satisfies Browder's theorem if and only if T satisfies generalized Browder's theorem, hence generalized Browder's theorem holds for T.

(7)⇒(1): Suppose generalized Browder's theorem holds for T. Then $\sigma(T) \setminus \sigma_{BW}(T) = p_0(T)$. Since $p_0(T) = \sigma(T) \setminus \sigma_D(T)$, $\sigma_{BW}(T) = \sigma_D(T)$. Therefore T has SVEP at every $\lambda \notin \sigma_{BW}(T)$. □

In Theorem 2.12, the condition "a-polaroid" cannot be replaced by the weaker condition "polaroid".

Example 2.13. Let U be the unilateral shift on ℓ_2 and let $A \in B(\ell_2)$ be given by

$$A(x_1, x_2, x_3, \dots) := (0, x_2, x_3, x_4, \dots).$$

Define $T := U \oplus A$. Then $\sigma(T) = \sigma_w(T) = \overline{\mathbb{D}}$, and hence $\pi_{00}(T) = \emptyset$. Moreover, since $\sigma_a(T) = \mathbf{\Gamma} \cup \{0\}$, $\text{iso}\, \sigma_a(T) = \{0\}$. Since $N(T) = N(U) \oplus N(A)$ and $\alpha(A) = 1$, $\alpha(T) = 1$. Hence $\pi_{00}^a(T) = \{0\}$. Therefore Weyl's theorem holds for T, while T does not satisfy property (aw). Since $\sigma(T) = \overline{\mathbb{D}}$, T is polaroid. However, since $p(T) = \infty$, T is not a-polaroid.

In analogy with Theorem 2.12, we obtain the following.

Theorem 2.14. Let $T \in B(\mathcal{X})$. Suppose T is a-polaroid. Then the following statements are equivalent:

(1) T^* has SVEP at every $\lambda \in \sigma_a(T) \setminus \sigma_{Bea}(T)$;

(2) T satisfies property (gw);

(3) T satisfies property (gb);

(4) Generalized a-Weyl's theorem holds for T;

(5) Generalized a-Browder's theorem holds for T;

(6) T satisfies property (b);

(7) T satisfies property (w).

Proof. (1) $\Leftrightarrow$ (2): Suppose T^* has SVEP at every $\lambda \in \sigma_a(T) \setminus \sigma_{Bea}(T)$. Let $\lambda \in \sigma_a(T) \setminus \sigma_{Bea}(T)$. Then $T - \lambda$ is upper semi-B-Fredholm and $i(T - \lambda) \leq 0$. Since T^* has SVEP at λ, $i(T - \lambda) \geq 0$. Therefore $T - \lambda$ is B-Weyl, and hence $T - \lambda$ is Drazin invertible. So $\lambda \in \pi_0(T)$, and hence $\lambda \in \sigma_a(T) \setminus \sigma_{Bea}(T) \subseteq \pi_0(T)$. Conversely, let $\lambda \in \pi_0(T)$. Then λ is an isolated point of $\sigma(T)$ and $\alpha(T - \lambda) > 0$. Since T is a-polaroid, $\lambda \in p_0(T)$, which implies that $T - \lambda$ is Drazin invertible. Therefore $\lambda \in \sigma_a(T) \setminus \sigma_{Bea}(T)$, and hence $\pi_0(T) \subseteq \sigma_a(T) \setminus \sigma_{Bea}(T)$. So T satisfies property (gw). Conversely, suppose T satisfies property (gw). Let $\lambda \in \sigma_a(T) \setminus \sigma_{Bea}(T)$. Then $\lambda \in \pi_0(T)$, and so λ is an isolated point of $\sigma(T^*)$. Therefore T^* has SVEP at λ.

(2)$\Rightarrow$(3): It follows form [12, Theorem 2.15].

(3)$\Rightarrow$(4): Suppose T satisfies property (gb). Then $\sigma_a(T) \setminus \sigma_{Bea}(T) = p_0(T)$. Let $\lambda \in \sigma_a(T) \setminus \sigma_{Bea}(T)$. Then $\lambda \in p_0(T)$, and so λ is an isolated point of $\sigma(T)$ and $T - \lambda$ is Drazin invertible. Therefore λ is an isolated point of $\sigma_a(T)$ and $\alpha(T - \lambda) > 0$. Hence $\lambda \in \pi_0^a(T)$. Conversely, let $\lambda \in \pi_0^a(T)$. Then λ is an isolated point of $\sigma_a(T)$ and $\alpha(T - \lambda) > 0$. Since T is a-polaroid, $\lambda \in p_0(T)$. Therefore $\lambda \in \sigma_a(T) \setminus \sigma_{Bea}(T)$, and hence generalized a-Weyl's theorem holds for T.

(4)$\Rightarrow$(5): It follows form [10, Corollary 3.3].

(5)$\Rightarrow$(1): Suppose generalized a-Browder's theorem holds for T. Then $\sigma_a(T) \setminus \sigma_{Bea}(T) = p_0^a(T)$. Let $\lambda \in \sigma_a(T) \setminus \sigma_{Bea}(T)$. Then $\lambda \in p_0^a(T)$, and so $\lambda \in \sigma_a(T) \setminus \sigma_{LD}(T)$. Since $\sigma_{LD}(T) = \sigma_{Bea}(T) \cup \operatorname{acc} \sigma_a(T)$, λ is an isolated point of $\sigma_a(T)$. Since T is a-polaroid, $\lambda \in p_0(T)$. Therefore λ is an isolated point of $\sigma(T^*)$, and hence T^* has SVEP at λ.

(5) $\Leftrightarrow$ (6): Suppose generalized a-Browder's theorem holds for T. Then $\sigma_a(T) \setminus \sigma_{Bea}(T) = p_0^a(T)$. Let $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$. Then $\lambda \in \sigma_a(T) \setminus \sigma_{LD}(T)$, and so λ is an isolated point of $\sigma_a(T)$. Since T is a-polaroid, $\lambda \in p_0(T)$. Therefore T is an isolated point of $\sigma(T)$. But $T - \lambda$ is upper semi-Fredholm, hence it is Browder. So $\lambda \in p_{00}(T)$, and hence $\sigma_a(T) \setminus \sigma_{ea}(T) \subseteq p_{00}(T)$. But $p_{00}(T) \subseteq \sigma_a(T) \setminus \sigma_{ea}(T)$ holds for any operator T, hence T satisfies property (b). Conversely, suppose T satisfies property (b). Then $\sigma_a(T) \setminus \sigma_{ea}(T) = p_{00}(T)$. Since $p_{00}(T) \subseteq p_{00}^a(T)$, $\sigma_a(T) \setminus \sigma_{ea}(T) \subseteq p_{00}^a(T)$, which means that a-Browder's theorem holds for T. It follows form [5, Theorem 3.2] that generalized a-Browder's theorem holds for T.

(6) $\Leftrightarrow$ (7): Suppose T satisfies property (b). Then $\sigma_a(T) \setminus \sigma_{ea}(T) = p_{00}(T)$. To show that T satisfies property (w) it suffices to prove that $\pi_{00}(T) \subseteq p_{00}(T)$. Let $\lambda \in \pi_{00}(T)$. Then λ is an isolated point of $\sigma(T)$ and $0 < \alpha(T-\lambda) < \infty$. Since T is a-polaroid, $\lambda \in p_0(T)$. Therefore there exists a positive integer $p := p(T-\lambda) = q(T-\lambda)$, and hence $R(T-\lambda)^p$ is closed. But $\alpha(T-\lambda)^p < \infty$, hence $(T-\lambda)^p$ is upper semi-Fredholm, which implies that $T-\lambda$ is upper semi-Fredholm. Since λ is an isolated point of $\sigma(T)$, $T-\lambda$ is Browder. Therefore $\lambda \in p_{00}(T)$, and hence $\pi_{00}(T) \subseteq p_{00}(T)$. So T satisfies property (w). $\square$

Let $H_1(\sigma(T))$ denote the set of all analytic functions on an open neighborhood of $\sigma(T)$ such that f is nonconstant on each of the components of its domain. Let $T \in B(\mathcal{X})$. Then it is well known that the inclusion $\sigma_w(f(T)) \subseteq f(\sigma_w(T))$ holds for every $f \in H(\sigma(T))$ with no other restriction on T.

Theorem 2.15. Let $T \in B(\mathcal{X})$ and $f \in H_1(\sigma(T))$.

(1) Suppose T satisfies property (gb). Then $f(T)$ satisfies property (gb) $\Leftrightarrow f(\sigma_{Bea}(T)) = \sigma_{Bea}(f(T))$.

(2) Suppose T satisfies property (ab). Then $f(T)$ satisfies property (ab) $\Leftrightarrow f(\sigma_w(T)) = \sigma_w(f(T))$.

(3) Suppose T satisfies property (gab). Then $f(T)$ satisfies property (gab) $\Rightarrow f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$.

Proof. (1):($\Rightarrow$) Suppose $f(T)$ satisfies property (gb). Then $\sigma_a(f(T)) \setminus \sigma_{Bea}(f(T)) = p_0(f(T))$. To show that $\sigma_{Bea}(f(T)) = f(\sigma_{Bea}(T))$ it suffices to show that $f(\sigma_{Bea}(T)) \subseteq \sigma_{Bea}(f(T))$. Suppose $\lambda \notin \sigma_{Bea}(f(T))$. Then $f(T)-\lambda$ is upper semi-B-Fredholm and $i(f(T)-\lambda) \leq 0$. We consider two cases.

Case I. Suppose $f(T)-\lambda$ is bounded below. Then $\lambda \notin \sigma_a(f(T)) = f(\sigma_a(T))$, and hence $\lambda \notin f(\sigma_{Bea}(T))$.

Case II. Suppose $\lambda \in \sigma_a(f(T)) \setminus \sigma_{Bea}(f(T))$. Since $f(T)$ satisfies property (gb), $\lambda \in p_0(f(T))$, which implies that $\lambda \in \sigma(f(T)) \setminus \sigma_D(f(T))$. Since $\sigma_D(f(T)) = f(\sigma_D(T))$ by [14, Theorem 2.7], $\lambda \notin f(\sigma_D(T))$. Therefore $\lambda \notin f(\sigma_{Bea}(T))$, and hence $f(\sigma_{Bea}(T)) \subseteq \sigma_{Bea}(f(T))$. It follows from Cases I and II that $f(\sigma_{Bea}(T)) = \sigma_{Bea}(f(T))$.

($\Leftarrow$) Suppose $f(\sigma_{Bea}(T)) = \sigma_{Bea}(f(T))$. Since T has property (gb),

$$\begin{aligned}\sigma_a(f(T)) \setminus \sigma_{Bea}(f(T)) &= f(\sigma_a(T)) \setminus f(\sigma_{Bea}(T)) \\ &\subseteq f(\sigma_a(T) \setminus \sigma_{Bea}(T)) \\ &= f(p_0(T)) \\ &\subseteq p_0(f(T)).\end{aligned}$$

Therefore $\sigma_a(f(T)) \setminus \sigma_{Bea}(f(T)) \subseteq p_0(f(T))$, and hence $f(T)$ satisfies property (gb).

(2):($\Rightarrow$) Suppose $f(T)$ satisfies property (ab). Then $\sigma(f(T)) \setminus \sigma_w(f(T)) = p_{00}^a(f(T))$. To show that $\sigma_w(f(T)) = f(\sigma_w(T))$ it suffices to show that $f(\sigma_w(T)) \subseteq \sigma_w(f(T))$. Suppose $\lambda \notin \sigma_w(f(T))$. Then $f(T)-\lambda$ is Weyl. We consider two cases.

Case I. Suppose $f(T)-\lambda$ is invertible. Then $\lambda \notin \sigma(f(T)) = f(\sigma(T))$, and hence $\lambda \notin f(\sigma_w(T))$.

Case II. Suppose $\lambda \in \sigma(f(T)) \setminus \sigma_w(f(T))$. Since $f(T)$ satisfies property (ab), $\lambda \in p_{00}^a(f(T))$, which means that $\lambda \in \sigma_a(f(T)) \setminus \sigma_{ab}(f(T))$. Therefore $f(T)-\lambda$ is Weyl and $p(f(T)-\lambda) < \infty$. Therefore $f(T)-\lambda$ is Browder, and hence $\lambda \notin \sigma_b(f(T)) = f(\sigma_b(T))$. So $\lambda \notin f(\sigma_w(T))$, and hence $f(\sigma_w(T)) \subseteq \sigma_w(f(T))$. It follows from Cases I and II that $f(\sigma_w(T)) = \sigma_w(f(T))$.

($\Leftarrow$) Suppose $f(\sigma_w(T)) = \sigma_w(f(T))$. Let $\lambda \in \sigma(f(T)) \setminus \sigma_w(f(T))$. Write

$$f(T)-\lambda = c_0(T-\lambda_1)(T-\lambda_2)\cdots(T-\lambda_n)g(T),$$

where $c_0, \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ and $g(T)$ is invertible. Since $\lambda \notin \sigma_w(f(T)) = f(\sigma_w(T))$, $T - \lambda_i$ is Weyl for each $i = 1, 2, \ldots, n$. Since T satisfies property (ab), Browder's theorem holds for T. Therefore $\sigma_w(T) = \sigma_b(T)$, and hence $T - \lambda_i$ is Browder for each $i = 1, 2, \ldots, n$. So $f(T) - \lambda$ is Browder, and hence $\lambda \in p_{00}^a(f(T))$.

Conversely, suppose $\lambda \in p_{00}^a(f(T))$. Then $\lambda \in \sigma_a(f(T)) \setminus \sigma_{ab}(f(T))$. Write

$$f(T) - \lambda = c_0(T - \lambda_1)(T - \lambda_2)\cdots(T - \lambda_n)g(T),$$

where $c_0, \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ and $g(T)$ is invertible. Since $\lambda \in f(\sigma_a(T)) \setminus f(\sigma_{ab}(T))$, $\lambda_i \in \sigma_a(T) \setminus \sigma_{ab}(T)$ for each $i = 1, 2, \ldots, n$. But T satisfies property (ab), hence every $T - \lambda_i$ is Weyl. So $\lambda \in \sigma(f(T)) \setminus \sigma_w(f(T))$. Therefore $f(T)$ satisfies property (ab).

(3): Suppose $f(T)$ satisfies property (gab). Then $\sigma(f(T)) \setminus \sigma_{BW}(f(T)) = p_0^a(f(T))$. Since T satisfies property (gab), generalized Browder's theorem holds for T. So $\sigma_{BW}(f(T)) \subseteq f(\sigma_{BW}(T))$ by [14, Corollary 2.8]. Now we show that $f(\sigma_{BW}(T)) \subseteq \sigma_{BW}(f(T))$. Suppose that $\lambda \notin \sigma_{BW}(f(T))$. Then $f(T) - \lambda$ is B-Weyl. We consider two cases.

Case I. Suppose $f(T) - \lambda$ is invertible. Then $\lambda \notin \sigma(f(T)) = f(\sigma(T))$, and hence $\lambda \notin f(\sigma_{BW}(T))$.

Case II. Suppose $\lambda \in \sigma(f(T)) \setminus \sigma_{BW}(f(T))$. Since $f(T)$ satisfies property (gab), $\lambda \in p_0^a(f(T))$, which means that $\lambda \in \sigma_a(f(T)) \setminus \sigma_{LD}(f(T))$. Therefore $f(T) - \lambda$ is B-Weyl and $p(f(T) - \lambda) < \infty$. Therefore $f(T) - \lambda$ is Drazin invertible, and hence $\lambda \notin \sigma_D(f(T)) = f(\sigma_D(T))$. So $\lambda \notin f(\sigma_{BW}(T))$, and hence $f(\sigma_{BW}(T)) \subseteq \sigma_{BW}(f(T))$. It follows from Cases I and II that $f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$. $\square$

In [3], it was shown that if T is left polaroid then $f(T)$ is also left polaroid for each $f \in H_1(\sigma(T))$. We obtain similar results for a-polaroid and s-polaroid, respectively.

Theorem 2.16. Let $T \in B(\mathcal{X})$ and $f \in H_1(\sigma(T))$.

(1) Suppose T is a-polaroid. Then $f(T)$ is a-polaroid.

(2) Suppose T is s-polaroid. Then $f(T)$ is s-polaroid.

Proof. (1) Suppose T is a-polaroid. We shall show that iso $\sigma_a(f(T)) \subseteq p_0(f(T))$. Let $\lambda_0 \in$ iso $\sigma_a(f(T))$. Since the spectral mapping theorem holds for the approximate point spectrum, $\lambda_0 \in$ iso $f(\sigma_a(T))$. We let $\mu_0 \in \sigma_a(T)$ such that $f(\mu_0) = \lambda_0$. Denote by Ω the connected component of the domain of f which contains μ_0. Now we show that $\mu_0 \in$ iso $\sigma_a(T)$. Assume to the contrary that $\mu_0 \in$ acc $\sigma_a(T)$. Then there exists a sequence (μ_n) in $\Omega \cap \sigma_a(T)$ of distinct scalars such that $\mu_n \longrightarrow \mu_0$ as $n \to \infty$. Since $C := \{\mu_0, \mu_1, \mu_2, \mu_3, \ldots\}$ is a compact subset of Ω, f may assume the value $\lambda_0 = f(\mu_0)$ only a finite number of points of C, so for n sufficiently large $f(\mu_n) \neq f(\mu_0) = \lambda_0$, and since $f(\mu_n) \longrightarrow f(\mu_0) = \lambda_0$ as $n \to \infty$, $\lambda_0 \in$ acc $f(\sigma_a(T))$. This is a contradiction. Therefore $\mu_0 \in$ iso $\sigma_a(T)$. Since T is a-polaroid, $\mu_0 \in p_0(T)$. It follows from [4, Theorem 2.9] that $\lambda_0 \in p_0(f(T))$. Hence $f(T)$ is a-polaroid.

(2) Suppose T is s-polaroid. Then T^* is a-polaroid by Theorem 2.10. So $f(T)^* = f(T^*)$ is a-polaroid by (1), and hence $f(T)$ is s-polaroid. $\square$

From Theorem 2.12, if T is a-polaroid and T has SVEP, then T satisfies property (gaw). We can prove more:

Theorem 2.17. Let $T \in B(\mathcal{X})$ and $f \in H_1(\sigma(T))$.

(1) Suppose T is a-polaroid and has SVEP. Then $f(T)$ satisfies property (gaw).

(2) Suppose T is s-polaroid and T^* has SVEP. Then $f(T^*)$ satisfies property (gaw).

Proof. (1) Since T is a-polaroid, $f(T)$ is a-polaroid by Theorem 2.16. Also, since T has SVEP, $f(T)$ has SVEP. It follows from Theorem 2.12 that $f(T)$ satisfies property (gaw).

(2) Since T is s-polaroid, $f(T)$ is s-polaroid by theorem 2.16, which means that $f(T^*) = f(T)^*$ is a-polaroid. Since T^* has SVEP, $f(T^*) = f(T)^*$ has SVEP. Therefore $f(T^*)$ satisfies property (gaw) by Theorem 2.12. $\square$

References

[1] P. Aiena, Fredholm and Local Spectral Theory, with Applications to Multipliers, Kluwer, 2004.
[2] P. Aiena, *Quasi Fredholm operators and localized SVEP*, Acta Sci. Math. (Szeged), **73** (2007), 251–263.
[3] P. Aiena, E. Aponte and E. Balzan, *Weyl type theorems for left and right polaroid operators*, Integral Equations Operator Theory **66** (2010), 1–20.
[4] P. Aiena and J.E. Sanabria, *On left and right poles of the resolvent*, Acta Sci. Math. (Szeged), **74** (2008), 669–687.
[5] P. Aiena, M.T. Biondi. and C. Carpintero, *On Drazin invertibility*, Proc. Amer. Math. Soc. **136** (2008), no. 8, 2839–2848.
[6] M. Amouch and M. Berkani, *On the property (gw)*, Mediterr. J. Math. **5** (2008), 371–378.
[7] M. Berkani, *On a class of quasi-Fredholm operators*, Integral Equations Operator Theory **34** (1999), no. 2, 244–249.
[8] M. Berkani, *B-Weyl spectrum and poles of the resolvent*, J. Math. Anal. Appl. **272** (2002), 596–603.
[9] M. Berkani, *Index of B-Fredholm operators and generalization of a Weyl theorem*, Proc. Amer. Math. Soc. **130** (2002), 1717–1723.
[10] M. Berkani and J.J. Koliha, *Weyl type theorems for bounded linear operators*, Acta Sci. Math. (Szeged) **69** (2003), 359–376.
[11] M. Berkani and M. Sarih, *On semi B-Fredholm operators*, Glasg. Math. J. **43** (2001), no. 3, 457–465.
[12] M. Berkani, Oujda, H. Zariouh and Meknes, *Extended Weyl type theorems*, Math. Bohemica. **134** (2009), no.4, 369–378.
[13] M. Berkani and H. Zariouh, *New extended Weyl type theorems*, Mat. Vesnik **62** (2010) 145–154.
[14] R.E. Curto and Y.M. Han, *Generalized Browder's and Weyl's theorems for Banach spaces operators*, J. Math. Anal. Appl. **336** (2007), 1424–1442.
[15] S.V. Djordjević and Y.M. Han, *Browder's theorems and spectral continuity*, Glasgow Math. J. **42** (2000), 479–486.
[16] R. Harte, Invertibility and Singularity for Bounded Linear Operators, Dekker, New York, 1988.
[17] V. Rakočević, *On a class of operators*, Mat. Vesnik **37** (1985) 423–426.
[18] C. Sun, X. Cao and L. Dai, *Property* (ω_1) *and Weyl type theorem*, J. Math. Anal. Appl. **363** (2010), 1–6.