Filomat 27:6 (2013), 995–1009
DOI 10.2298/FIL1306995L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

p-variation of an integral functional associated with bi-fractional Brownian motion*

Junfeng Liu[a], Litan Yan[b], Donglei Tang[a]

[a] *Department of Mathematics, Nanjing Audit University*
86 West Yushan Rd., Pukou, Nanjing 211815, P.R. China
[b] *Department of Mathematics, Donghua University*
2999 North Renmin Rd., Songjiang, Shanghai, 201620, P.R. China

Abstract. In this paper we consider the functionals

$$A_1(t,x) = \int_0^t 1_{[0,\infty)}(x - B_s^{H,K})ds$$
$$A_2(t,x) = \int_0^t 1_{[0,\infty)}(x - B_s^{H,K})s^{2HK-1}ds,$$

where $B^{H,K}$ is a bifractional Brownian motion with indices $H \in (0,1), K \in (0,1]$. We find a constant $p_{H,K} \in (1,2)$ such that p-variation of the process $A_j(t, B_s^{H,K}) - \int_0^t \mathscr{L}_j(s, B_s^{H,K})dB_s^{H,K}$ $(j = 1,2)$ equals to 0 if $p > p_{H,K}$, where $\mathscr{L}_j$, $j = 1,2$, are the local times of $B_t^{H,K}$. This extends the classical results for Brownian motion (Rogers-Walsh [17]).

1. Introduction

Given $H \in (0,1), K \in (0,1]$. The bifractional Brownian motion on $\mathbb{R}$ with indices H and K is a mean zero Gaussian process $B^{H,K} = \left\{B_t^{H,K}, t \geq 0\right\}$ such that

$$E\left[B_t^{H,K}B_s^{H,K}\right] = \frac{1}{2^K}\left[\left(t^{2H} + s^{2H}\right)^K - |t-s|^{2HK}\right] \tag{1}$$

for all $s,t \geq 0$. Clearly, if $K = 1$, the process is a fractional Brownian motion with Hurst parameter H. This process was first introduced by Houdré and Villa [8]. Russo and Tudor [20] have established some properties on the strong variations, local times and stochastic calculus of real-valued bifractional Brownian motion.

2010 *Mathematics Subject Classification*. Primary 60G15; 60J55; Secondary 60H05
Keywords. Bifractional Brownian motion, local time, self-intersection local time, p-variation, stochastic area integrals.
Received: 28 May 2012; Accepted: 23 February 2013
Communicated by Miljana Jovanović
* Research supported by Mathematical Tianyuan Foundation of China (Grant No.11226198), NSFC (11171062), NSFC (11201232), NSFC (11271020), NSFC (81001288), NSRC (10023), Innovation Program of Shanghai Municipal Education Commission (12ZZ063), NSF of Jiangsu Educational Committee (12KJB110008), Humanity and Social Science Youth foundation of Ministry of Education (12YJCZH128), Priority Academic Program Development of Jiangsu Higher Education Institutions and and Major Program of Key Research Center in Financial Risk Management of Jiangsu Universities Philosophy Social Sciences (No:2012JDXM009).
Email addresses: jordanjunfeng@163.com (Junfeng Liu), litanyan@dhu.edu.cn (Litan Yan), tdonglei@nau.edu.cn (Donglei Tang)

An interesting property is that the quadratic variation of this process on $[0,t]$ equals to $2^{1-K}t$ provided $2HK = 1$. Tudor-Xiao [23] studied some sample path properties of bifractional Brownian motion. A Tanaka formula for multidimensional bifractional Brownian motion was given by Es-sebaiy and Tudor [6].

The self-intersection local time of $B^{H,K}$ is formally defined, for any $t > 0$ and $x \in \mathbb{R}$, by

$$\alpha(t,x) = \int_0^t \int_0^s \delta(B_s^{H,K} - B_r^{H,K} - x)drds,$$

which exists as a limit in $L^2(\Omega)$ if we approximate the "δ" by f_ε. Intuitively, for $x = 0$, the random variable $\alpha(t,0)$ measures the amount of time the bifractional Brownian motion intersects with itself in the real time interval $[0,t]$ (see Jiang and Wang [13]).

In this paper we consider the integral functionals of the forms

$$A_1(t,x) = \int_0^t 1_{[0,\infty)}(x - B_s^{H,K})ds,$$
$$A_2(t,x) = \int_0^t 1_{[0,\infty)}(x - B_s^{H,K})s^{2HK-1}ds.$$

Our main aim is to study the p-variation of the following processes:

$$X_t^{(1)} := A_1(t, B_t^{H,K}) - \int_0^t \mathscr{L}_1(s, B_s^{H,K})dB_s^{H,K},$$

$$X_t^{(2)} := A_2(t, B_t^{H,K}) - \int_0^t \mathscr{L}_2(s, B_s^{H,K})dB_s^{H,K},$$

with $HK \in (0, \frac{2}{3})$, where the stochastic integral is of the Skorohod type and

$$\mathscr{L}_1(t,x) = \int_0^t \delta(B_s^{H,K} - x)ds, \qquad \mathscr{L}_2(t,x) = \int_0^t \delta(B_s^{H,K} - x)s^{2H-1}ds,$$

are the local time and weighted local time of $B^{H,K}$, respectively. In particular, by using Itô's formula for bifractional Brownian motion $B^{H,K}$, one can connect the processes $X_t^{(i)}$, $i = 1,2$ with the derivative of self-intersection local time (DSLT) of $B^{H,K}$. See Section 3 for more details.

For $K = 1$ and $H = \frac{1}{2}$, the process $B^{H,K}$ is classical Brownian motion B. In the study of stochastic area integral for standard Brownian motion B, Rogers and Walsh [17], [18], [19] were led to analyze the functional of the form

$$A(t, B_t) = \int_0^t 1_{[0,\infty)}(B_t - B_s)ds$$

In particular, by using the classical Itô formula, essentially the Burkholder-Davis-Gundy inequalities of martingales and the decomposition of the expression

$$\sum_{j=1}^{2^n} |X_{j/2^n} - X_{(j-1)/2^n}|^p,$$

with $X_t = A(t, B_t)$, they showed that the process $A(t, B_t)$ is not a semimartingale, and in fact showed that the process

$$A(t, B_t) - \int_0^t \mathscr{L}(s, B_s)dB_s \tag{2}$$

has finite non-zero 4/3-variation. Here $\mathscr{L}(t, B_s)$ is the local time of B at x, which is formally defined by $\mathscr{L}(t,x) = \int_0^t \delta(B_s - x)ds$. Hu *et al* [12] also compute the $\frac{4}{3}$-variation of the derivative of self-intersection Brownian local time by using techniques from the theory of fractional martingales (Hu *et al* [11]).

Recently, Yan *et al.* [24] considered the similar integral functional driven by fractional Brownian motion B^H with Hurst index $H \in (0,1)$ which arises in the study of integration with respect to fractional local times of fractional Brownian motion (see Yan *et al.* [28]). In the general case, that is $K \neq 1$, this question has not been studied. On the other hand, in recent years the fBm has become an object of intense study, due to its interesting properties and its applications in various scientific areas including telecommunications, turbulence, image processing and finance. However, contrast to the extensive studies on fractional Brownian motion, there has been little systematic investigation on other self-similar Gaussian processes. The main reasons for this are the complexity of dependence structures and the non-availability of convenient stochastic integral representations for self-similar Gaussian processes which do not have stationary increments. Therefore, it seems interesting to study the problem.

This paper is organized as follows. In Section 2 we presents some preliminaries for bifractional Brownian motion. In order to study the functionals $A_1(t, B_t^{H,K})$ and $A_2(t, B_t^{H,K})$ as above, in Section 3 we define the so-call weighted self-intersection local times and consider their derivatives (see Rosen [22]). In Section 4 we will use the results established in Section 3 to give our main theorem. Some technical estimates are included in the appendix.

2. Preliminaries for bifractional Brownian motion

In this section, we briefly recall some basic definitions and results of bifractional Brownian motion. For simplicity we let $C_{H,K} > 0$ stand for a positive constant depending only on H, K and its value may be different in different appearance. As we pointed out before, bifractional Brownian motion (bi-fBm) $B^{H,K} = \left\{B_t^{H,K}, 0 \le t \le T\right\}$, on the probability space $(\Omega, \mathscr{F}, P)$ with indices $H \in (0,1)$ and $K \in (0,1]$ is a rather special class of self-similar Gaussian processes such that $B_0^{H,K} = 0$ and

$$R(t,s) := E\left[B_t^{H,K} B_s^{H,K}\right] = \frac{1}{2^K}\left[\left(t^{2H} + s^{2H}\right)^K - |t-s|^{2HK}\right], \quad \forall s,t \ge 0. \tag{3}$$

The process is HK-self similar and satisfies the following estimates (the quasi-helix property)

$$2^{-K}|t-s|^{2HK} \le E\left[\left(B_t^{H,K} - B_s^{H,K}\right)^2\right] \le 2^{1-K}|t-s|^{2HK}. \tag{4}$$

Thus, Kolmogorov's continuity criterion implies that bifractional Brownian motion is Hôlder continuous of order δ for any $\delta < HK$. More works for bi-fBm can be found in Houdré and Villa [8], Tudor-Xiao [23], Russo-Tudor [20], Es-sebaiy and Tudor [6], Kruk *et al* [15], Yan *et al* [25–27], and the references therein.

As a Gaussian process, it is possible to construct a stochastic calculus of variations with respect to $B^{H,K}$. We refer to Nualart [16] and Alós *et al* [4] for a complete description of stochastic calculus with respect to Gaussian processes, in particular, we refer to Es-sebaiy and Tudor [6] for stochastic calculus with respect to $B^{H,K}$ with $2HK \ge 1$ and Yan and Xiang [25] with $0 < 2HK < 1$. We will use the notation

$$\int_0^T u_s dB_s^{H,K},$$

to express the Skorohod integral of an adapted process u.

Theorem 2.1 (Itô's formula [6] and [25]). *Let $f \in C^{2,1}(\mathbb{R}\times\mathbb{R}_+)$. Suppose that $HK \in (0,1)$, then we have*

$$f\left(B_t^{H,K}, t\right) = f(0,0) + \int_0^t \frac{\partial f}{\partial x}(B_s^{H,K}, s)dB_s^{H,K} + \int_0^t \frac{\partial f}{\partial t}(B_s^{H,K}, s)ds + HK\int_0^t \frac{\partial^2 f}{\partial x^2}(B_s^{H,K}, s)s^{2HK-1}ds.$$

Recall that bi-fBm $B^{H,K}$ has a local time $\mathscr{L}^{H,K}(x,t)$ continuous in $(x,t) \in \mathbb{R} \times [0,\infty)$ which satisfies the occupation formula (see Geman-Horowitz [7])

$$\int_0^t \phi(B_s^{H,K})ds = \int_{\mathbb{R}} \phi(x)\mathscr{L}_1^{H,K}(x,t)dx, \tag{5}$$

for every continuous and bounded function $\phi(x) : \mathbb{R} \to \mathbb{R}$ and any $t \geqslant 0$, and such that

$$\mathscr{L}_1^{H,K}(x,t) = \int_0^t \delta(B_s^{H,K} - x)ds = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon}\lambda\Big(s \in [0,t], |B_s^{H,K} - x| < \epsilon\Big),$$

where λ denotes Lebesgue measure and $\delta(\cdot)$ is the Dirac delta function. Set

$$\begin{aligned}\mathscr{L}_2^{H,K}(x,t) &= 2HK \int_0^t s^{2HK-1}\mathscr{L}_1^{H,K}(x,ds),\\ &\equiv 2HK \int_0^t \delta(B_s^{H,K} - x)s^{2HK-1}ds.\end{aligned}$$

It follows from (5) that

$$2HK \int_0^t \phi(B_s^{H,K})s^{2HK-1}ds = \int_{\mathbb{R}} \phi(x)\mathscr{L}_2^{H,K}(x,t)dx.$$

At the end of the section, we give some estimates for the following expresses:

$$\lambda = E\left[(B_s^{H,K} - B_r^{H,K})^2\right], \qquad \rho = E\left[(B_{s'}^{H,K} - B_{r'}^{H,K})^2\right],$$

and

$$\mu = E\left[(B_s^{H,K} - B_r^{H,K})(B_{s'}^{H,K} - B_{r'}^{H,K})\right],$$

for $0 < r < s < T, 0 < r^{'} < s^{'} < T$.

Lemma 2.2. *Let* $2HK \neq 1$. *For* $0 < r < s < T$, $0 < r' < s' < T$ *we we have*

$$0 \leq \mu \leq 2^{-K}\mu^*, \tag{6}$$

where

$$\mu^* = |s' - r|^{2HK} + |s - r'|^{2HK} - |s' - s|^{2HK} - |r - r'|^{2HK}.$$

Proof. The inequalities (6) are two calculus exercises. By symmetry one may assume that $s \leq s'$, and we have

$$\begin{aligned}\mu &= E\left[(B_s^{H,K} - B_r^{H,K})(B_{s'}^{H,K} - B_{r'}^{H,K})\right]\\ &= 2^{-K}\left(\nu + |s' - r|^{2HK} + |s - r'|^{2HK} - |s' - s|^{2HK} - |r - r'|^{2HK}\right),\end{aligned}$$

where

$$\nu = (s'^{2H} + s^{2H})^K - (s'^{2H} + r^{2H})^K - (s^{2H} + r'^{2H})^K + (r^{2H} + r'^{2H})^K.$$

Let now $0 < r' < r < s < s' < T$. Set

$$G_{s',r'}(x) = (s'^{2H} + x^{2H})^K - (r'^{2H} + x^{2H})^K$$

and

$$F_{s',r'}(x) = (x - r')^{2HK} - (s' - x)^{2HK}.$$

Mean value theorem implies that there are $\xi, \eta \in (r, s)$ such that

$$\begin{aligned}
\nu &= (s'^{2H} + s^{2H})^K - (s'^{2H} + r^{2H})^K - (s^{2H} + r'^{2H})^K + (r^{2H} + r'^{2H})^K \\
&= G_{s',r'}(s) - G_{s',r'}(r) = (s-r)\frac{d}{dx}G_{s',r'}(\xi) \\
&= 2HK(s-r)\xi^{2H-1}\frac{(r'^{2H} + \xi^{2H})^{1-K} - (s'^{2H} + \xi^{2H})^{1-K}}{(s'^{2H} + \xi^{2H})^{1-K}(r'^{2H} + \xi^{2H})^{1-K}} \le 0,
\end{aligned}$$

and

$$\begin{aligned}
&\nu + (s'-r)^{2HK} + (s-r')^{2HK} - (s'-s)^{2HK} - (r-r')^{2HK} \\
&= G_{s',r'}(s) - G_{s',r'}(r) + F_{s',r'}(s) - F_{s',r'}(r) \\
&= 2HK(s-r)\left(\frac{\eta^{2H-1}}{(s'^{2H} + \eta^{2H})^{1-K}} - \frac{\eta^{2H-1}}{(r'^{2H} + \eta^{2H})^{1-K}} + (\eta - r')^{2HK-1} + (s'-\eta)^{2HK-1}\right) \\
&\ge 0.
\end{aligned}$$

Similarly, one can show that (6) hold for $0 < r < r' < s < s' < T$ and $0 < r < s < r' < s' < T$, respectively. This gives the estimates (6). □

Lemma 2.3. *Let* $2HK = 1$.

(1) *For all* $0 < r < s < r' < s' \le T$, *we have*

$$0 \le -\mu \le (2H-1)2^{-K}\frac{(s-r)(s'-r')}{r'}; \tag{7}$$

(2) *For all* $0 < r' < r < s < s' \le T$, *we have*

$$0 \le \mu \le 2^{1-K}(s-r); \tag{8}$$

(3) *For all* $0 < r < r' < s < s' \le T$, *we have*

$$0 \le \mu \le 2^{1-K}(s-r'). \tag{9}$$

Proof. (1) For $m > 0$ we define the function $x \mapsto G_m(x)$ on $[0, T]$ by

$$G_m(x) = \left(m^{2H} + x^{2H}\right)^{K-1}.$$

Thanks to mean value theorem, we see that there are $\xi \in (r, s)$ and $\eta \in (r', s')$ such that

$$G_m(s) - G_m(r) = 2H(K-1)\xi^{2H-1}(s-r)\left(m^{2H} + \xi^{2H}\right)^{K-2}$$

and

$$G_m(s') - G_m(r') = 2H(K-1)\eta^{2H-1}(s'-r')\left(m^{2H} + \eta^{2H}\right)^{K-2}.$$

It follows from the duality relationship that

$$\begin{aligned}
-E(B_s^{H,K} - B_r^{H,K})(B_{s'}^{H,K} - B_{r'}^{H,K}) &= -\int_{r'}^{s'}\int_r^s \frac{\partial^2}{\partial u \partial v} R(u,v)dudv \\
&= 2H(1-K)2^{-K}\int_{r'}^{s'}\int_r^s \left(u^{2H}+v^{2H}\right)^{K-2} u^{2H-1}v^{2H-1}dudv \\
&= 2H(1-K)2^{-K}\int_{r'}^{s'} u^{2H-1}du\int_r^s \left(u^{2H}+v^{2H}\right)^{K-2} v^{2H-1}dv \\
&= -2^{-K}\int_{r'}^{s'} u^{2H-1}\left\{G_u(s)-G_u(r)\right\}du \\
&= -2H(K-1)2^{-K}\xi^{2H-1}(s-r)\int_{r'}^{s'} u^{2H-1}\left(u^{2H}+\xi^{2H}\right)^{K-2}du \\
&= -2^{-K}\xi^{2H-1}(s-r)\left\{G_\xi(s')-G_\xi(r')\right\} \\
&= 2H(1-K)2^{-K}(s-r)(s'-r')(\xi\eta)^{2H-1}\left(\xi^{2H}+\eta^{2H}\right)^{K-2} \geq 0.
\end{aligned}$$

Combining this with

$$(\xi^{2H}+\eta^{2H})^{K-2} \leq \eta^{1-4H}$$

and $\xi\eta \leq \eta^2$, we obtain (7). Moreover, (8) and (9) follow from the proof of Lemma 2.2. Thus, we completes the proof. □

According to the property of strong local nondeterminism of bi-fBm and (4), we can obtain the next lemma.

Lemma 2.4. *There is a constant $\kappa > 0$ such that the following statements hold:*
(1) for any $0 \leq r < r' < s < s' \leq T$,

$$\lambda\rho - \mu^2 \geq \kappa[(s-r)^{2HK}(s'-s)^{2HK} + (s'-r')^{2HK}(r'-r)^{2HK}]; \tag{10}$$

(2) for any $0 \leq r' < r < s < s' \leq T$,

$$\lambda\rho - \mu^2 \geq \kappa(s-r)^{2HK}(s'-r')^{2HK}; \tag{11}$$

(3) for any $0 \leq r < s < r' < s' \leq T$,

$$\lambda\rho - \mu^2 \geq \kappa(s-r)^{2HK}(s'-r')^{2HK}. \tag{12}$$

3. Derivative of weighted self-intersection Local Times

To study our main aim, in this section we consider the weighted self-intersection local times of bi-fBm, defined as

$$\alpha_1(t) = \int_0^t\int_0^s \delta(B_s^{H,K} - B_r^{H,K})s^{2HK-1}drds$$
$$\alpha_2(t) = \int_0^t\int_0^s \delta(B_s^{H,K} - B_r^{H,K})(sr)^{2HK-1}drds$$

Suppose that $\mathscr{L}_1(t,x) = \int_0^t \delta(B_s^{H,K} - x)ds$ and $\mathscr{L}_2(t,x) = \int_0^t \delta(B_s^{H,K} - x)s^{2HK-1}ds$ are the local time and weighted local time of $B^{H,K}$, respectively, then we have

$$\alpha_j(t) = \int_0^t \mathscr{L}_j(s, B_s^{H,K})s^{2HK-1}ds, \, j = 1, 2.$$

which leads to the existence of α_j, $j = 1, 2$ in L^2. Define the functionals A_1 and A_2 as follow

$$A_1(t,x) = \int_0^t 1_{[0,\infty)}(x - B_s^{H,K})ds, \tag{13}$$

$$A_2(t,x) = \int_0^t 1_{[0,\infty)}(x - B_s^{H,K})s^{2HK-1}ds. \tag{14}$$

Then we have

$$A_1(t,x) = \int_{-\infty}^x \mathscr{L}_1(t,y)dy, \qquad A_2(t,x) = \int_{-\infty}^x \mathscr{L}_2(t,y)dy,$$

which imply that $A_j \in C^{1,1}(\mathbb{R}_+ \times \mathbb{R})$ for $j = 1, 2$. A formal application of Itô's formula for bi-fBm, using $\frac{d}{dx}1_{[0,\infty)}(x) = \delta(x)$ and $\frac{d^2}{dx^2}1_{[0,\infty)}(x) = \delta'(x)$, yields

$$A_1(t, B_t^{H,K}) = t + \int_0^t \mathscr{L}_1(s, B_s^{H,K})dB_s^{H,K} + HK\int_0^t\int_0^s \delta'(B_s^{H,K} - B_r^{H,K})s^{2HK-1}drds,$$

and

$$A_2(t, B_t^{H,K}) = \frac{t^{2HK}}{2HK} + \int_0^t \mathscr{L}_2(s, B_s^{H,K})dB_s^{H,K} + HK\int_0^t\int_0^s \delta'(B_s^{H,K} - B_r^{H,K})(sr)^{2HK-1}drds$$

in the setting of distributional sense, where δ' is the distributional derivative of the Dirac-delta function, and the stochastic integral is of the Skorohod type. For $0 \le t \le T$, define two processes as follow

$$\alpha_1'(t) = \int_0^t\int_0^s \delta'(B_s^{H,K} - B_r^{H,K})s^{2HK-1}drds,$$

$$\alpha_2'(t) = \int_0^t\int_0^s \delta'(B_s^{H,K} - B_r^{H,K})(sr)^{2HK-1}drds.$$

Thus, the expressions A_1 and A_2 can be rewritten as

$$A_1(t, B_t^{H,K}) = t + \int_0^t \mathscr{L}_1(s, B_s^{H,K})dB_s^{H,K} + HK\alpha_1'(t) \tag{15}$$

and

$$A_2(t, B_t^{H,K}) = \frac{t^{2HK}}{2HK} + \int_0^t \mathscr{L}_2(s, B_s^{H,K})dB_s^{H,K} + HK\alpha_2'(t). \tag{16}$$

Now, let us prove the existence of the processes $\alpha_j'(t)$, $j = 1, 2$. Consider the heat kernel

$$f_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}}e^{-\frac{x^2}{2\varepsilon}} = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}e^{-\varepsilon\frac{\xi^2}{2}}d\xi, \qquad x \in \mathbb{R},$$

and define

$$\begin{aligned}\alpha_1'(t,\varepsilon) :&= \int_0^t\int_0^s f_\varepsilon'(B_s^{H,K} - B_r^{H,K})s^{2HK-1}drds \\ &\equiv \frac{i}{2\pi}\int_0^t\int_0^s s^{2HK-1}drds\int_{\mathbb{R}} \xi e^{i(B_s^{H,K} - B_r^{H,K})\xi}e^{-\varepsilon\frac{\xi^2}{2}}d\xi\end{aligned}$$

$$\alpha_2'(t,\varepsilon) := \int_0^t \int_0^s f_\varepsilon'(B_s^{H,K} - B_r^{H,K})(sr)^{2HK-1} drds$$
$$\equiv \frac{i}{2\pi} \int_0^t \int_0^s (sr)^{2HK-1} drds \int_{\mathbb{R}} \xi e^{i(B_s^{H,K} - B_r^{H,K})\xi} e^{-\varepsilon \frac{\xi^2}{2}} d\xi.$$

Proposition 3.1. *The processes* $\alpha_j'(t)$, $j = 1,2$ *exist in* L^2 *if* $0 < HK < \frac{2}{3}$.

Proof. Denote

$$\mathbb{T} := \{(r,s,r',s') : 0 < r < s < t, 0 < r' < s' < t\}.$$

Suppose that $\zeta_1 = (rsr's')^{2HK-1}$, $\zeta_2 = (ss')^{2HK-1}$ and

$$\lambda = \mathrm{Var}(B_s^{H,K} - B_r^{H,K}), \quad \rho = \mathrm{Var}(B_{s'}^{H,K} - B_{r'}^{H,K}),$$

and

$$\mu = \mathrm{Cov}(B_s^{H,K} - B_r^{H,K}, B_{s'}^{H,K} - B_{r'}^{H,K})$$

for all $(s,r,s',r') \in \mathbb{T}$. Then we have for $j = 1,2$,

$$E\alpha_j'(t,\varepsilon) = \frac{i}{2\pi} \int_0^t \int_0^s \zeta_j drds \int_{\mathbb{R}} \xi E\left(e^{i(B_s^{H,K} - B_r^{H,K})\xi}\right) e^{-\varepsilon \frac{\xi^2}{2}} d\xi = 0$$

and

$$\begin{aligned} E(\alpha_j'(t,\varepsilon)^2) &= \frac{-1}{(2\pi)^2} \int_{\mathbb{T}} \zeta_j drdsdr'ds' \cdot \\ &\quad \int_{\mathbb{R}^2} \xi\eta E[\exp(i\xi(B_s^{H,K} - B_r^{H,K}) + i\eta(B_{s'}^{H,K} - B_{r'}^{H,K}))] e^{-\varepsilon \frac{\xi^2+\eta^2}{2}} d\xi d\eta \\ &= \frac{-1}{(2\pi)^2} \int_{\mathbb{T}} \zeta_j drdsdr'ds' \cdot \\ &\quad \int_{\mathbb{R}^2} \xi\eta \exp(-\frac{1}{2}(\lambda+\varepsilon)\xi^2 - \mu\xi\eta - \frac{1}{2}(\rho+\varepsilon)\eta^2) d\xi d\eta \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\mu\zeta_j}{[(\lambda+\varepsilon)(\rho+\varepsilon) - \mu^2]^{\frac{3}{2}}} drdsdr'ds'. \end{aligned}$$

Thus, by Lemma 3.2, for $0 < HK < \frac{2}{3}$, we can define, in L^2-space,

$$\alpha_j'(t) = \lim_{\varepsilon \to 0} \alpha_j'(t,\varepsilon), \quad j = 1,2. \tag{17}$$

□

Lemma 3.2. *For* $0 < HK < \frac{2}{3}$ *and* $j = 1,2$, *we have*

$$\int_{\mathbb{T}} \frac{\mu\zeta_j}{(\lambda\rho - \mu^2)^{\frac{3}{2}}} dsdrds'dr' < \infty. \tag{18}$$

The proof of the lemma will be given at Appendix.

4. *p*-variations

In this section, we give the main result of this note. The idea used here is essentially due to Rosen [22]. Fix $t > 0$ and let $\{0 = t_0 < t_1 < t_2 < \cdots t_n = t\}$ be a partition of $[0, t]$ such that $|\triangle_n| = \max_j |t_j - t_{j-1}| \to 0$ as n tends to infinity. For a stochastic process $X = \{X_t; t \geq 0\}$, we denote

$$V_p^n(X, t) = \sum_{j=0}^{n-1} |X_{t_{j+1}} - X_{t_j}|^p,$$

where $p > 0$. Recall that the process X is of bounded p-variation if the limit of $V_p^n(X, t)$ exists in L^1 as n tends to infinity. We denote this limit by $V_p(X, t)$ and call it p-variation of X on $[0, t]$. For any $t, t' \in [0, T], t < t'$, we denote

$$\mathscr{D}_1 = \{\, 0 < r < r' < s < s' < t',\ t < s, s' < t' \,\},$$
$$\mathscr{D}_2 = \{\, 0 < r' < r < s < s' < t',\ t < s, s' < t' \,\}.$$

Lemma 4.1. *For $\frac{1}{2} \leq HK < \frac{2}{3}$ and $j = 1, 2$, we have*

$$\int_{\mathscr{D}_1} \zeta_j drdsdr'ds' \int_{\mathbb{R}^2} |\xi\eta| e^{-\frac{1}{2}\mathrm{Var}\{\xi(B_s^{H,K} - B_r^{H,K}) + \eta(B_{s'}^{H,K} - B_{r'}^{H,K})\}} d\xi d\eta \leq C(t' - t)^{3-3HK} \tag{19}$$

and

$$\int_{\mathscr{D}_2} \frac{\mu\zeta_j}{(\lambda\rho - \mu^2)^{\frac{3}{2}}} drdsdr'ds' \leq C(t' - t)^{2-HK}, \tag{20}$$

where $\zeta_1 = (ss')^{2HK-1}$ and $\zeta_2 = (rsr's')^{2HK-1}$.

This lemma will be proved in Appendix.

Theorem 4.2. *For $\frac{1}{2} \leq HK < \frac{2}{3}$, we have*

$$V_p(\alpha'_j, t) = 0, \qquad j = 1, 2$$

if $p > \frac{2}{3-3HK}$

Proof. For $\frac{1}{2} \leq HK < \frac{2}{3}$ and $j = 1, 2$, we have that

$$\begin{aligned}
E[(\alpha'_j(t', \varepsilon) - (\alpha'_j(t,\varepsilon))^2] &= -\frac{1}{(2\pi)^2} \int_t^{t'} \int_0^s \int_t^{t'} \int_0^{s'} \zeta_j drdsdr'ds' \\
&\quad \times \int_{\mathbb{R}^2} \xi\eta E e^{i\xi(B_s^{H,K} - B_r^{H,K}) + i\eta(B_{s'}^{H,K} - B_{r'}^{H,K})} \cdot e^{-\frac{\varepsilon}{2}(\xi^2+\eta^2)} d\xi d\eta \\
= -\frac{1}{(2\pi)^2} \int_t^{t'} \int_0^s \int_t^{t'} \int_0^{s'} \zeta_j drdsdr'ds' & \\
&\quad \times \int_{\mathbb{R}^2} \xi\eta e^{-\frac{1}{2}\mathrm{Var}[\xi(B_s^{H,K} - B_r^{H,K}) + \eta(B_{s'}^{H,K} - B_{r'}^{H,K})]} \cdot e^{-\frac{\varepsilon}{2}(\xi^2+\eta^2)} d\xi d\eta \\
= -\frac{1}{(2\pi)^2} \int_{\mathscr{D}_1} \zeta_j drdsdr'ds' & \\
&\quad \times \int_{\mathbb{R}^2} \xi\eta e^{-\frac{1}{2}\mathrm{Var}[\xi(B_s^{H,K} - B_r^{H,K}) + \eta(B_{s'}^{H,K} - B_{r'}^{H,K})]} \cdot e^{-\frac{\varepsilon}{2}(\xi^2+\eta^2)} d\xi d\eta \\
\quad + \frac{1}{2\pi} \int_{\mathscr{D}_2} \frac{\mu\zeta_j}{[(\lambda + \varepsilon)(\rho + \varepsilon) - \mu^2]^{\frac{3}{2}}} drdsdr'ds'. &
\end{aligned}$$

Combining this with $\alpha'_j(t,\varepsilon) \to \alpha'_j(t)$ $(j = 1,2)$ in L^2 as ε tends to zero, we get

$$\begin{aligned} E[(\alpha'_j(t') - \alpha'_j(t))^2] &\le \frac{1}{2\pi}\int_{\mathscr{D}_2} \frac{\mu\zeta_j drdsdr'ds'}{(\lambda\rho-\mu^2)^{\frac{3}{2}}} \\ &+ \frac{1}{(2\pi)^2}\int_{\mathscr{D}_1} \zeta_j drdsdr'ds' \int_{\mathbb{R}^2} |\xi\eta| e^{-\frac{1}{2}\mathrm{Var}[\xi(B_s^{H,K}-B_r^{H,K})+\eta(B_{s'}^{H,K}-B_{r'}^{H,K})]} d\xi d\eta \\ &\le C(t'-t)^{2-HK} + C(t'-t)^{3-3HK} \le C(t'-t)^{3-3HK} \end{aligned}$$

by Lemma 4.1 because of $2 - HK \ge 3 - 3HK$. It follows that for $0 < p \le 2$ and $j = 1,2$,

$$\begin{aligned} E[V_p^n(\alpha'_j,t)] &= \sum_{k=0}^{n-1} E|\alpha'_{j,t_{k+1}} - \alpha'_{j,t_k}|^p \le \sum_{k=0}^{n-1} (E|\alpha'_{j,t_{k+1}} - \alpha'_{j,t_k}|^2)^{\frac{p}{2}} \\ &\le c\sum_{k=0}^{n-1} |t_{k+1}-t_k|^{\frac{p(3-3HK)}{2}}, \end{aligned}$$

which shows that the p-variation of the process α'_j $(j = 1,2)$ is zero provided $p > \frac{2}{3-3HK}$. This completes the proof. □

For the case $0 < HK < \frac{1}{2}$. We first give a lemma which is similar to Lemma 4.1.

Lemma 4.3. *For $0 < HK < \frac{1}{2}$ and $i,j = 1,2$, we have*

$$\int_{\mathscr{D}_j} \frac{\mu\zeta_i drdsdr'ds'}{(\lambda\rho-\mu^2)^{\frac{3}{2}}} \le C(t'-t)^{2-HK}. \tag{21}$$

Thanks to the above lemma, one can easily obtain the following theorem by similar proof of Theorem 4.2.

Theorem 4.4. *For $0 < HK < \frac{1}{2}$ and $j = 1,2$, we have*

$$V_p(\alpha'_j,t) = 0.$$

if $p > \frac{2}{2-HK}$.

Remark 4.5. *From the above proof of Theorem 4.2 and Theorem 4.4, we obtain that,*

1. *for* $\frac{1}{2} \le HK < \frac{2}{3}$,
$$E\left[|\alpha'_j(t') - \alpha'_j(t)|^2\right] \le C|t'-t|^{3-3HK}, \quad j = 1,2.$$
That means the DSLT $\alpha'_j(t)$, $j = 1,2$, has a modification which is a.s. Hölder continuous in t of any order less than $\frac{3-3HK}{2}$.
2. *for* $0 < HK < \frac{1}{2}$,
$$E\left[|\alpha'_j(t') - \alpha'_j(t)|^2\right] \le C|t'-t|^{2-HK}, \quad j = 1,2.$$
That means the DSLT $\alpha'_j(t)$, $j = 1,2$, has a modification which is a.s. Hölder continuous in t of any order less than $\frac{2-HK}{2}$.
Hence (15) *and* (16) *are identities as functions of t when* $0 < HK < \frac{2}{3}$.

Remark 4.6. *On the other hand, the Hölder continuity of the DSLT $\alpha'_j(t)$, $j = 1,2$ with $0 < HK < \frac{1}{2}$ can also be proved by the methods developed in Jung and Markowsky [14] with the covariance structure and strong local nondeterminism of bifractional Brownian motion. It is therefore omitted.*

We now can consider the p-variations of the processes $X^{(j)}$, $j = 1, 2$ given by

$$X_t^{(j)} := A_j(t, B_t^{H,K}) - \int_0^t \mathscr{L}_j(s, B_s^{H,K}) dB_s^{H,K},$$

for $0 < HK < \frac{2}{3}$.

Corollary 4.7. *For $\frac{1}{2} \le HK < \frac{2}{3}$, we have*

$$V_p(X^{(j)}, t) = 0, \qquad j = 1, 2,$$

if $p > \frac{2}{3-3HK}$.

Corollary 4.8. *For $0 < HK < \frac{1}{2}$, we have*

$$V_p(X^{(j)}, t) = 0, \qquad j = 1, 2,$$

if $p > \frac{2}{2-HK}$.

Appendix A. Some technical estimates

In the appendix, we will give the proofs of the estimates (18), (19), (20) and (21). Denote

$$\mathbb{T} := \{(r, s, r', s') : 0 < r < s < t, 0 < r' < s' < t\}.$$

Suppose that $\zeta_1 = (ss')^{2HK-1}$, $\zeta_2 = (rsr's')^{2HK-1}$. For any $(r, s, r', s') \in \mathbb{T}$, we set $s < s'$ and denote

$$\mathbb{T}_1 = \{0 \le r < r' < s < s' \le t\},$$

$$\mathbb{T}_2 = \{0 \le r' < r < s < s' \le t\},$$

$$\mathbb{T}_3 = \{0 \le r < s < r' < s' \le t\}.$$

Let us first obtain the inequalities (20) and (21). Set

$$\Xi_{j,i} := \int_{\mathscr{D}_j} \frac{\mu \zeta_i dr ds dr' ds'}{(\lambda\rho - \mu^2)^{\frac{3}{2}}}$$

for $i, j = 1, 2$. We claim that

$$\Xi_{2,i} \le C(t' - t)^{2-HK}, \frac{1}{2} \le HK < \frac{2}{3}. \tag{A.1}$$

and

$$\Xi_{j,i} \le C(t' - t)^{2-HK}, 0 < HK < \frac{1}{2}. \tag{A.2}$$

It follows from (6), (11) and the inequality

$$s' - r' \ge (s' - s) + (r - r') \ge (s' - s)^{\frac{2}{5}} (r - r')^{\frac{3}{5}}$$

that for all $(r,s,r',s') \in \mathbb{T}_2$ and $\frac{1}{2} \le HK < \frac{2}{3}$.

$$\begin{aligned}\frac{\mu}{(\lambda\rho-\mu^2)^{\frac{3}{2}}} &\le \kappa\frac{(s-r)^{\frac{3}{2}HK}(s-r)^{1-\frac{3}{2}HK}(s'-r')^{2HK-1}}{(s-r)^{3HK}(s'-r')^{3HK}}\\ &\le \kappa(s-r)^{-\frac{3}{2}HK}(s'-r')^{-\frac{5}{2}HK}\\ &\le \kappa(s-r)^{-\frac{3}{2}HK}(s'-s)^{-HK}(r-r')^{-\frac{3}{2}HK}\end{aligned}$$

which yields

$$\begin{aligned}\Xi_{2,i} := &\int_{\mathscr{D}_2}\frac{\mu\zeta_i drdsdr'ds'}{(\lambda\rho-\mu^2)^{\frac{3}{2}}}\\ \le &\, C\int_{\mathscr{D}_2}\frac{\zeta_i drdsdr'ds'}{(s-r)^{\frac{3}{2}HK}(s'-s)^{HK}(r-r')^{\frac{3}{2}HK}}\\ \le &\, C\int_{\mathscr{D}_2}\frac{drdsdr'ds'}{(s-r)^{\frac{3}{2}HK}(s'-s)^{HK}(r-r')^{\frac{3}{2}HK}}\\ \le &\, C(t'-t)^{2-HK}\end{aligned}$$

for $\frac{1}{2} \le HK < \frac{2}{3}$. This gives the inequality (A.1) for $i = 1,2$.

Now suppose $0 < HK < \frac{1}{2}$, let us consider the inequalities (A.2). For $(r,s,r',s') \in \mathscr{D}_1$ by applying Young inequality(see Beckenbach *et al.* [5]) and (10), we have

$$\lambda\rho-\mu^2 \ge \kappa(s-r)^{\frac{2}{3}HK}(s'-s)^{\frac{2}{3}HK}(s'-r')^{\frac{4}{3}HK}(r'-r)^{\frac{4}{3}HK}$$

Combining this with (11) and the Schwartz inequality $\mu^2 \le \lambda\rho$ we obtain

$$\begin{aligned}\Xi_{j,1} := &\int_{\mathscr{D}_j}\frac{\mu(ss')^{2HK-1}drdsdr'ds'}{(\lambda\rho-\mu^2)^{\frac{3}{2}}} \le C\int_{\mathscr{D}_j}\frac{\mu(rr')^{2HK-1}drdsdr'ds'}{(\lambda\rho-\mu^2)^{\frac{3}{2}}}\\ \le &\, C\int_t^{t'}ds'\int_t^{s'}(s'-s)^{-HK}ds \le C(t'-t)^{2-HK}\end{aligned}$$

and

$$\begin{aligned}\Xi_{1,2} := &\int_{\mathscr{D}_1}\frac{\mu(rsr's')^{2HK-1}drdsdr'ds'}{(\lambda\rho-\mu^2)^{\frac{3}{2}}} \le C\int_{\mathscr{D}_1}\frac{(rsr's')^{2HK-1}drdsdr'ds'}{(s'-s)^{HK}(s'-r')^{HK}(r'-r)^{2HK}}\\ \le &\, C\int_{\mathscr{D}_1}\frac{(rsr's')^{2HK-1}drdsdr'ds'}{(s'-s)^{HK}(s-r')^{HK}(r'-r)^{2HK}}\\ \le &\, C\int_t^{t'}ds'\int_t^{s'}(ss')^{2HK-1}(s'-s)^{-HK}ds\int_0^{s}\int_0^{r'}\frac{(rr')^{2HK-1}}{(s-r')^{HK}(r'-r)^{2HK}}drdr'\\ \le &\, C\int_t^{t'}(s')^{2HK-1}ds'\int_t^{s'}s^{3HK-1}(s'-s)^{-HK}ds \le C(t'-t)^{2-HK}\end{aligned}$$

Similar, one can obtain the estimate

$$\Xi_{2,2} \le C(t'-t)^{2-HK}, 0 < HK < \frac{1}{2}.$$

These show the inequalities (A.2) hold.

Next, we obtain the inequality (19). Suppose $\frac{1}{2} \le HK < \frac{2}{3}$. By the strong local nondeterminism property of the bi-fractional Brownian motion proved by Tudor and Xiao [23], we have

$$\begin{aligned}
&\mathrm{Var}[\xi(B_s^{H,K} - B_r^{H,K}) + \eta(B_{s'}^{H,K} - B_{r'}^{H,K})] \\
&= \mathrm{Var}[\xi(B_{r'}^{H,K} - B_r^{H,K}) + (\xi+\eta)(B_s^{H,K} - B_{r'}^{H,K}) + \eta(B_{s'}^{H,K} - B_s^{H,K})] \\
&\ge \kappa[\xi^2(r'-r)^{2HK} + (\xi+\eta)^2(s-r')^{2HK} + \eta^2(s'-s)^{2HK}]
\end{aligned}$$

for some constant $\kappa > 0$ and $(r,s,r',s') \in \mathscr{D}_1$. It follows that

$$\begin{aligned}
\Upsilon_i :&= \int_{\mathscr{D}_1} \zeta_i drdsdr'ds' \\
&\quad \times \int_{\mathbb{R}^2} |\xi\eta| \exp\{-\frac{\kappa}{2}[\xi^2(r'-r)^{2HK} + (\xi+\eta)^2(s-r')^{2HK} + \eta^2(s'-s)^{2HK}]\} d\xi d\eta \\
&\le C \int_{\mathbb{R}^2} |\xi\eta| d\xi d\eta \int_{\mathscr{D}_3} \exp\{-\frac{\kappa}{2}[\xi^2 r_2^{2HK} + (\xi+\eta)^2 r_3^{2HK} + \eta^2 r_4^{2HK}]\} dr_1dr_2dr_3dr_4
\end{aligned}$$

where $\mathscr{D}_3 = \{t \le \sum_{j=1}^3 r_j \le t', t \le \sum_{j=1}^4 r_j \le t', 0 < r_1, r_2, r_3, r_4 < T\}$. Noting that

$$\begin{aligned}
\int_{t-r_1-r_2-r_3}^{t'-r_1-r_2-r_3} e^{-\frac{\kappa}{2}\eta^2 r_4^{2HK}} dr_4 &\le (t'-t)^a \Big(\int_0^T e^{-\frac{\kappa}{2(1-a)}\eta^2 r_4^{2HK}} dr_4\Big)^{1-a} \\
&\le C(t'-t)^a \frac{1}{1+|\eta|^{\frac{1-a}{HK}}}
\end{aligned}$$

by Hölder inequality with parameters $\frac{1}{a}$ and $\frac{1}{1-a}$ and $0 < a < 1$, then we get

$$\begin{aligned}
&\int_{\mathscr{D}_3} \exp\{-\frac{\kappa}{2}[\xi^2 r_2^{2HK} + (\xi+\eta)^2 r_3^{2HK} + \eta^2 r_4^{2HK}]\} dr_1dr_2dr_3dr_4 \\
&\quad \le C(t'-t)(t'-t)^a \frac{1}{1+|\eta|^{\frac{1-a}{HK}}} \int_0^T \int_0^T \exp\{-\frac{\kappa}{2}(\xi^2 r_2^{2HK} + (\xi+\eta)^2 r_3^{2HK})\} dr_2dr_3 \\
&\quad \le C(t'-t)^{1+a} \frac{1}{1+|\eta|^{\frac{1-a}{HK}}} \frac{1}{1+|\xi|^{\frac{1}{HK}}} \frac{1}{1+|\xi+\eta|^{\frac{1}{HK}}}
\end{aligned}$$

where we have used a basic inequality

$$\int_0^1 e^{-t^{2HK}x^2} dt \le \frac{c}{1+|x|^{\frac{1}{HK}}}$$

for all $x \in \mathbb{R}$. Thus, as $a < 2 - 3HK$, i.e., when $\frac{1-a}{HK} - 1 + \frac{1}{HK} - 1 > 1$, we get

$$\begin{aligned}
\Upsilon_i &\le C(t'-t)^{1+a} \int_{\mathbb{R}^2} |\xi\eta| \frac{1}{1+|\eta|^{\frac{1-a}{HK}}} \frac{1}{1+|\xi|^{\frac{1}{HK}}} \frac{1}{1+|\xi+\eta|^{\frac{1}{HK}}} d\xi d\eta \\
&\le C(t'-t)^{1+a} \le C(t'-t)^{3-3HK}.
\end{aligned}$$

which shows the inequality (19) holds for all $\frac{1}{2} \le HK < \frac{2}{3}$.

Finally, let us prove the estimates (18). Because of the above proof, we just need to prove the estimate

$$\int_{\mathbb{T}_3} \frac{\mu\zeta_j drdsdr'ds'}{(\lambda\rho-\mu^2)^{\frac{3}{2}}} < \infty$$

for $j = 1, 2$ and $0 < HK < \frac{2}{3}$. This will be done in two cases.

Case 1. Let $HK \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{2}{3})$. The estimates (6) and the estimate for μ^* in Hu [9, 248] imply that

$$\mu \leq C(r'-s)^{2\alpha(HK-1)}[(s-r)(s'-r')]^{2\beta(HK-1)+1}$$

for $\alpha > 0, \beta > 0$ and $\alpha + 2\beta = 1$. Combining this with (12), we get

$$\frac{\mu}{(\lambda\rho-\mu^2)^{\frac{3}{2}}} \leq C(s-r)^{2\beta(HK-1)+1-3HK}(r'-s)^{2\alpha(HK-1)}(s'-r')^{2\beta(HK-1)+1-3HK}.$$

Taking $0 < \beta < \frac{2-3HK}{2-2HK}$ leads to $2\beta(HK-1)+1-3HK > -1$ and $2\alpha(HK-1) > -1$ and so that when $HK \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{2}{3})$, we have

$$\int_{\mathbb{T}_3} \frac{\mu\zeta_j}{(\lambda\rho-\mu^2)^{\frac{3}{2}}} drdsdr'ds' < \infty.$$

Case 2. Let $HK = \frac{1}{2}$. It follows from (7) and (12) that

$$\frac{|\mu|}{(\lambda\rho-\mu^2)^{\frac{3}{2}}} \leq \kappa \frac{1}{r'(s-r)^{\frac{1}{2}}(s'-r')^{\frac{1}{2}}},$$

which deduces

$$\int_{\mathbb{T}_3} \frac{|\mu|\zeta_j}{(\lambda\rho-\mu^2)^{\frac{3}{2}}} drdsdr'ds' \leq \kappa \int_{\mathbb{T}_3} \frac{drdsdr'ds'}{r'(s-r)^{\frac{1}{2}}(s'-r')^{\frac{1}{2}}} < \infty.$$

Thus we have proved (18).

Acknowledgments 1. *We want to thank the Editor and anonymous referees whose remarks and suggestions greatly improved the presentation of this paper.*

References

[1] W. Rudin, Real and Complex Analysis, (3rd edition), McGraw-Hill, New York, 1986.

[2] J. A. Goguen, L-fuzzy sets, Journal of Mathematical Analysis and Applications 18 (1967) 145–174.

[3] P. Erdös, S. Shelah, Separability properties of almost-disjoint families of sets, Israel Journal of Mathematics 12 (1972) 207–214.

[4] E. Alós, O. Mazet and D. Nualart, Stochastic calculus with respect to Gaussian processes, *Ann. Prob.* **29** (2001), 766-801.

[5] Beckenbach, E.F., Bellman, R., *Inequality*, (1983). Springer-Verlag

[6] K. Es-sebaiy and Ciprian A. Tudor, Multidimensional bifractional Brownian motion: Itô and Tanaka formulas, *Stochastics and Dynamics*, **7** (2007), 366-388.

[7] D. Geman, J. Horowitz, Occupation densities, *Ann. Probab.* **8** (1980), 1-67.

[8] C. Houdré and J. Villa, An example of infinite dimensional quasi-helix. *Stochastic models (Mexico City, 2002)*, pp.195-201, Contemp. Math., **336** (2003), Amer. Math. Soc., Providence, RI.

[9] Y. Hu, Self-intersection local time of fractional brownian motions-via chaos expansion. *J. Math. Kyoto Univ.* **41** (2001), 233-250.

[10] Y. Hu and D. Nulart, Renormalized Self-intersection local time for fractional brownian motions. *Ann. Prob.* **33** (2005), 948-983.

[11] Y. Hu, D. Nualart and J. Song, Fractional martingales and characterization of the fractional brownian motions, *Ann. Probab* **37** (2009), 2404-2430.

[12] Y. Hu, D. Nualart and J. Song, The $\frac{4}{3}$-variation of the derivative of the self-intersection Brownian local time and related processes, *J. Theor. Probab.* to appear in *J. Theor. Probab.* (2012) Doi 10.1007/s10959-012-0469-5.

[13] Y. Jiang and Y. Wang, Self-intersection local times and collision local times of bifractional Brownian motions, *Science in China Series A: Mathematics* **52(9)** (2009), 1905-1919.

[14] P. Jung and G. Markowsky, Hölder continuity and occupation-time formulas for fBm self-intersection local time and its derivative, to appear in *J. Theor. Probab.* (2013) Doi 10.1007/s10959-017-0474-8.

[15] I. Kruk, F. Russo and C. A. Tudor, Wiener integrals, Malliavin calculus and covariance measure structure, *J. Funct. Anal.* **249** (2007), 92-142.

[16] D. Nualart, *Malliavin Calculus and Related Topics.* 2nd edition Springer, New York (2006).

[17] C.G., Rogers and J.B. Walsh, $A(t, B_t)$ is not a semimartingale, , *Seminar on stochastic processes. Progress in Probability* **24** (1990), 457–482.

[18] C.G., Rogers and J.B. Walsh, Local time and stochastic area integral. *Ann. Probab.* **19** (1991), 457–482.

[19] C. G., Rogers and J. B. Walsh, The exact $\frac{4}{3}$-variation of a process arising from Brownian motion, *Stochastics and Stochastics Reports*, **51** (1994), 267-291.

[20] F. Russo and Ciprian A. Tudor, On the bifractional brownian motion, *Stoc. Proc. Appl*, **5** (2006), 1830-856.

[21] J. Rosen, The intersection local time of fractional Brownian motion in the plane, *J. Multivar. Anal.* **23** (1987), 37-46.

[22] J. Rosen, Derivative of self-intersection local time. *Lecture Notes in Mathematics* **1857** (2005), 263-281.

[23] C. A. Tudor and Y. Xiao, Some path properties of bifractional brownian motion, *Bernoulli*, **13** (2007), 1023-1052.

[24] L. Yan, X. Yang and Y. Lu, p-variation of an integral functional driven by fractional Brownian motion, *Statistics and Probability Letters* **78** (2008), 1148-1157.

[25] L. Yan and J. Xiang, The generalized quadratic covariation for a bi-fractional Brownian motion, *J. of Natural Science of Heilongjiang University* **28(5)** (2011), 587-603.

[26] L. Yan, B. Gao and J. Liu, The Bouleau-Yor identity for a bi-fractional Brownian motion, to appear in *Stochastics* (2012).

[27] L. Yan, J. Liu and C. Chen, On the collision local time of bifractional Brownian motion, *Stochastics and Dynamics*, **9(3)** (2009), 479-491.

[28] L.Yan, J. Liu and X. Yang, Integration with respect to fractional local time with Hurst index $1/2 < H < 1$. *Potential Anal.* **30** (2009), 115-138.