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On the regularity of topologies in the family of sets having the Baire property

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Abstract. The paper concerns the topologies introduced in the family of sets having the Baire property in a topological space (X, τ) and in the family generated by the sets having the Baire property and given a proper σ-ideal containing τ-meager sets. The regularity property of such topologies is investigated.

1. Introduction

Let $(X, \mathcal{S}, \mathcal{J})$ be a measurable space, where $\mathcal{S}$ is a σ-algebra in 2^X and $\mathcal{J} \subset \mathcal{S}$ is a σ-ideal. We shall say that $\mathcal{J}$ is a proper σ-ideal if $X \notin \mathcal{J}$. If τ is a topology on X then $Ba(\tau)$ and $\mathbb{K}(\tau)$ denote the family of sets having the Baire property and the family of meager sets, respectively, in a topological space (X, τ). If $\mathcal{A}$ is a family of subsets of X then the family $\{C \subset X : C = A \triangle B,\ A \in \mathcal{A},\ B \in \mathcal{J}\}$ will be denoted by $\mathcal{A} \triangle \mathcal{J}$. If a family $\mathcal{A}$ is a σ-algebra of subsets of 2^X then the family $\mathcal{A} \triangle \mathcal{J}$ is the smallest σ-algebra containing both families $\mathcal{A}$ and $\mathcal{J}$. We always assume that X is nonempty set. By $\mathbb{R}$ we shall denote the set of reals and by $\mathcal{L}$ the family of all Lebesgue measurable sets on $\mathbb{R}$. Let $\mathbb{L}$ be the family of all Lebesgue null sets on $\mathbb{R}$.

We introduce the following notion.

Definition 1. (cf. [8]) We shall say that an operator $\Phi : \mathcal{S} \to 2^X$ is a semi-lower density operator on $(X, \mathcal{S}, \mathcal{J})$ if the following conditions are satisfied:

1^0 $\Phi(\emptyset) = \emptyset, \qquad \Phi(X) = X,$

2^0 $\underset{A,B \in \mathcal{S}}{\forall} \Phi(A \cap B) = \Phi(A) \cap \Phi(B),$

3^0 $\underset{A,B \in \mathcal{S}}{\forall} A \triangle B \in \mathcal{J} \implies \Phi(A) = \Phi(B).$

Moreover, we set

$$\mathcal{T}_\Phi = \{A \in \mathcal{S} : A \subset \Phi(A)\}.$$

If $\mathcal{T}_\Phi$ is a topology then we say that $\mathcal{T}_\Phi$ is generated by Φ. Some ideas of this approach one can find in [4].

It is well observing the following fact.

Observation 1. *Let $(X, \mathcal{S}, \mathcal{J})$ be an arbitrary measurable space. Then there exists a semi-lower density operator $\Phi : \mathcal{S} \to 2^X$ on $(X, \mathcal{S}, \mathcal{J})$ generating topology $\mathcal{T}_\Phi$ if and only if $X \notin \mathcal{J}$.*

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Proof. Necessity. Suppose that $X \in \mathcal{J}$. Then, for every operator $\Phi : \mathcal{S} \to 2^X$ satisfying conditions 1^0-3^0 on $(X, \mathcal{S}, \mathcal{J})$ we have that $\Phi(A) = \emptyset$ for every $A \subset X$. It implies that family $\mathcal{T}_\Phi$ is not a topology on X.

Sufficiency. Let $X \notin \mathcal{J}$. Putting

$$\forall_{A \in \mathcal{S}} \ \Phi(A) = \begin{cases} \emptyset, & X - A \notin \mathcal{J}, \\ X, & X - A \in \mathcal{J}; \end{cases}$$

we see that Φ is a semi-lower density operator on $(X, \mathcal{S}, \mathcal{J})$ and $\mathcal{T}_\Phi = \{A \subset X : A = \emptyset \vee X - A \in \mathcal{J}\}$ is a topology on X.

It is also worth mentioning the following theorem contained in [8]. $\square$

Theorem 1. *Let $(X, \mathcal{S}, \mathcal{J})$ be a measurable space such that $\{x\} \in \mathcal{J}$ for every $x \in X$. Then there exists a semi-lower density operator $\Phi : \mathcal{S} \to 2^X$ on $(X, \mathcal{S}, \mathcal{J})$ not generating topology $\mathcal{T}_\Phi$ if and only if*

$$\mathcal{S}_0 \subsetneq \mathcal{S} \subsetneq 2^X,$$

where

$$\mathcal{S}_0 = \{A \subset X : A \in \mathcal{J} \vee X - A \in \mathcal{J}\}.$$

Definition 2. (cf. [10]) We shall say that a semi-lower density operator $\Phi : \mathcal{S} \to 2^X$ on $(X, \mathcal{S}, \mathcal{J})$ is a lower density operator on $(X, \mathcal{S}, \mathcal{J})$, if moreover

4^0 $\forall_{A \in \mathcal{S}} A \triangle \Phi(A) \in \mathcal{J}$.

Definition 3. (cf. [10]) If $(X, \mathcal{S}, \mathcal{J})$ is a measurable space then we shall say that a pair $(\mathcal{S}, \mathcal{J})$ possesses the hull property if for every set $A \subset X$ there exists a set $B \in \mathcal{S}$ such that $A \subset B$ and every $\mathcal{S}$-measurable set $C \subset B - A$ is a member of $\mathcal{J}$.

Theorem 2. (cf. [10]) *If $\Phi : \mathcal{S} \to \mathcal{S}$ is a semi-lower density operator on $(X, \mathcal{S}, \mathcal{J})$ and the pair $(\mathcal{S}, \mathcal{J})$ has the hull property then Φ generates topology $\mathcal{T}_\Phi$.*

The proof of this theorem is presented in [10, p. 208] for measurable space in the aspect of measure. The comments in [10, p. 213] underline that Theorem 1 is true in the case of an arbitrary measurable space possessing the hull property.

The classical application of this theorem is the density topology in the family of Lebesgue measurable sets on $\mathbb{R}$, which it turns out to be completely regular (cf. [15]).

It is interesting that replacing in Definition 1 condition 4^0 by its weaker form that $\Phi(A) - A \in \mathcal{J}$ for every $\mathcal{S}$-measurable set A, we get that Theorem 1 is still true for such operators on $(X, \mathcal{S}, \mathcal{J})$ (see [5], [6]). By this method, one can get several topologies on the real line: the ψ-density topology, which is not regular (see [2], [11]), f-density topology whose regularity depends on the properties of function f (see [2]), and the f-symmetrical density topology (see [3]), whose regularity has not been investigated yet.

2. The main results

Let (X, τ) be a topological space. The motivation of this paper is to investigate the regularity property of the topological spaces $(X, \mathcal{T}_\Phi)$, where $\mathcal{T}_\Phi$ is a topology generated by a semi-lower density operator Φ on $(X, \mathcal{S}, \mathcal{J})$, where $\mathcal{S} = Ba(\tau)$ and $\mathcal{J} \subset \mathbb{K}(\tau)$ or $\mathcal{S} = Ba(\tau) \triangle \mathcal{J}$ and $\mathcal{J} \supset \mathbb{K}(\tau)$.

First, it is easy to observe the following property.

Property 1. *Let (X, τ) be a topological space and $\mathcal{J} \supset \mathbb{K}(\tau)$ be a σ-ideal in 2^X. Then $Ba(\tau) \triangle \mathcal{J} = \tau \triangle \mathcal{J}$.*

Lemma 1. *Let (X, τ) be a topological space, $\mathcal{J} \supset \mathbb{K}(\tau)$ be a σ-ideal in 2^X and $\Phi : Ba(\tau) \triangle \mathcal{J} \to 2^X$ be a semi-lower density operator on $(X, Ba(\tau) \triangle \mathcal{J}, \mathcal{J})$ generating topology $\mathcal{T}_\Phi$ such that $\tau \subset \mathcal{T}_\Phi$. If a set $W \in \mathcal{T}_\Phi$ is τ-dense then $X - W \in \mathcal{J}$.*

Proof. Let $W \in \mathcal{T}_\Phi$ be τ-dense. Then, by Property 1, $W = V \triangle A$, where $V \in \tau$ and $A \in \mathcal{J}$. We show that the set V is τ-dense. Let us assume contrary, that there exists a nonempty set $C \in \tau$ such that $C \cap V = \emptyset$. Then, by property 1^0, 2^0 and 3^0 we get that $\emptyset = \Phi(C \cap V) = \Phi(C) \cap \Phi(V) = \Phi(C) \cap \Phi(W) = \Phi(C \cap W)$. By the assumption that $\tau \subset \mathcal{T}_\Phi$, we have that $C \cap W \in \mathcal{T}_\Phi$. Thus $C \cap W \subset \Phi(C \cap W)$. Hence $C \cap W = \emptyset$. This contradiction proves that the set V is τ-dense. It implies that V as τ-dense and τ-open is τ-residual. Then $X - V \in \mathbb{K}(\tau) \subset \mathcal{J}$ and finally $X - W \in \mathcal{J}$. □

Corollary 1. *Let (X, τ) be a topological space. If a semi-lower density operator $\Phi : Ba(\tau) \to 2^X$ on $(X, Ba(\tau), \mathbb{K}(\tau))$ generates topology $\mathcal{T}_\Phi$ such that $\tau \subset \mathcal{T}_\Phi$ then every $\mathcal{T}_\Phi$-open set and τ-dense is τ-residual.*

If the operators $\Phi_1, \Phi_2 : Ba(\tau) \to 2^X$ fulfils that $\Phi_1(A) \subset \Phi_2(A)$ for every set $A \in Ba(\tau)$ then we use notation $\Phi_1 \leq \Phi_2$.

Property 2. *Let (X, τ) be a topological space, $\mathcal{J} \subset \mathbb{K}(\tau)$ be a σ-ideal and $\Phi_1, \Phi_2 : Ba(\tau) \to 2^X$ be the semi-lower density operators on $(X, Ba(\tau), \mathcal{J})$ and $(X, Ba(\tau), \mathbb{K}(\tau))$, respectively. If Φ_2 generates topology $\mathcal{T}_{\Phi_2}$ and $\Phi_1 \leq \Phi_2$ then the operator Φ_1 generates topology $\mathcal{T}_{\Phi_1}$ such that $\mathcal{T}_{\Phi_1} \subset \mathcal{T}_{\Phi_2}$.*

Proof. By 1^0 we get that $\emptyset, X \in \mathcal{T}_{\Phi_1}$. The family $\mathcal{T}_{\Phi_1}$ is closed under finite intersections by 2^0. Let $\{A_t\}_{t \in T} \subset \mathcal{T}_{\Phi_1}$. By assumption $\Phi_1 \leq \Phi_2$ we get $\mathcal{T}_{\Phi_1} \subset \mathcal{T}_{\Phi_2}$. Hence $\{A_t\}_{t \in T} \subset \mathcal{T}_{\Phi_2}$ and $\bigcup_{t \in T} A_t \in \mathcal{T}_{\Phi_2}$, because $\mathcal{T}_{\Phi_2}$ is a topology. This implies that $\bigcup_{t \in T} A_t \in Ba(\tau)$ and $\bigcup_{t \in T} A_t \subset \Phi_1(\bigcup_{t \in T} A_t)$ by the monotonicity property of Φ_1. This means that $\mathcal{T}_{\Phi_1}$ is closed under arbitrary unions and the proof is completed. □

Theorem 3. *Let (X, τ) be a topological space, $\mathcal{J} \subset \mathbb{K}(\tau)$ be a σ-ideal and $\Phi_1, \Phi_2 : Ba(\tau) \to 2^X$ be the semi-lower density operators on $(X, Ba(\tau), \mathcal{J})$ and $(X, Ba(\tau), \mathbb{K}(\tau))$, respectively. If Φ_2 generates topology $\mathcal{T}_{\Phi_2}$ such that $\tau \subset \mathcal{T}_{\Phi_2}$, $\Phi_1 \leq \Phi_2$ and there exists a τ-dense set $A \in \mathcal{J}$ then the topological space $(X, \mathcal{T}_{\Phi_1})$ is not regular.*

Proof. By Property 2 operator Φ_1 generates topology $\mathcal{T}_{\Phi_1}$ such that $\mathcal{T}_{\Phi_1} \subset \mathcal{T}_{\Phi_2}$. The set A is $\mathcal{T}_{\Phi_1}$-closed and $\mathcal{T}_{\Phi_2}$-closed, as well. Let $x \notin A$. We will show that the sets $\{x\}$ and A cannot separated by $\mathcal{T}_{\Phi_2}$-open sets. Indeed, let $W, V \in \mathcal{T}_{\Phi_2}$ be such that $A \subset W$, $x \in V$. By Corollary 1, W is τ-residual and $V \notin \mathbb{K}(\tau)$. This implies that $V \cap W \neq \emptyset$ and finally $(X, \mathcal{T}_{\Phi_1})$ is not regular. □

Corollary 2. *Let (X, τ) be a topological space containing a τ-dense set $A \in \mathbb{K}(\tau)$. If a semi-lower density operator $\Phi : Ba(\tau) \to 2^X$ on $(X, Ba(\tau), \mathbb{K}(\tau))$ generates topology $\mathcal{T}_\Phi$ such that $\tau \subset \mathcal{T}_\Phi$ then the topological space $(X, \mathcal{T}_\Phi)$ is not regular.*

Example 1. If (X, τ) is a discrete topological space and $\Phi(A) = A$ for every set $A \subset X$ then Φ is a semi-lower density operator on $(X, Ba(\tau), \mathbb{K}(\tau))$ and $\mathcal{T}_\Phi = \tau$. Obviously $(X, \mathcal{T}_\Phi)$ is the regular topological space, but there is no τ-dense set of the first category in the space (X, τ).

Example 2. Let $(\mathbb{R}, \mathcal{T}_d)$ be the topological, where $\mathcal{T}_d$ is the density topology on $\mathbb{R}$. Then $Ba(\mathcal{T}_d) = \mathcal{L}$ and $\mathbb{K}(\mathcal{T}_d) = \mathbb{L}$ (see [15]). For every $A \in \mathcal{L}$, $\Phi_d(A)$ is the set of density points of A. Putting $\Phi_2(A) = \Phi_d(A)$ for every $A \in \mathcal{L}$ we get that Φ_2 is the lower density operator on $(\mathbb{R}, Ba(\mathcal{T}_d), \mathbb{K}(\mathcal{T}_d))$ generating topology $\mathcal{T}_{\Phi_2} = \mathcal{T}_d$. Putting $\Phi_1(A) = \Phi_d(A)$ for every $A \in \mathcal{L}$ and $\mathcal{J} = \{\emptyset\}$ we have that Φ_1 is the semi-lower density operator on $(\mathbb{R}, Ba(\mathcal{T}_d), \mathcal{J})$ such that $\Phi_1 \leq \Phi_2$ and $\mathcal{T}_{\Phi_1} = \mathcal{T}_d$. It is well known that $(R, \mathcal{T}_d)$ is regular space. Theorem 3 does not work in this case because $\mathcal{J}$ does not contain $\mathcal{T}_d$-dense set.

Property 3. (cf. [6]) *Let (X, τ) be a topological space and suppose that the pair $(Ba(\tau), \mathbb{K}(\tau))$ satisfies ccc. Then for every σ-ideal $\mathcal{J} \supset \mathbb{K}(\tau)$ the pair $(Ba(\tau) \triangle \mathcal{J}, \mathcal{J})$ also satisfies ccc.*

Theorem 4. *Let (X, τ) be a topological space. Let $\mathcal{J} \supset \mathbb{K}(\tau)$ be a proper σ-ideal in 2^X and $\Phi : Ba(\tau) \triangle \mathcal{J} \to 2^X$ be a semi-lower density operator on $(X, Ba(\tau) \triangle \mathcal{J}, \mathcal{J})$. If the pair $(Ba(\tau), \mathbb{K}(\tau))$ fulfills ccc and $V - \Phi(V) \in \mathcal{J}$ for every set $V \in \tau$ then the operator Φ generates topology $\mathcal{T}_\Phi$.*

Proof. If $A \in Ba(\tau)\triangle\mathcal{J}$ then, by Property 1, we get that $A = V\triangle B$, where $V \in \tau$ and $B \in \mathcal{J}$. Then $A - \Phi(A) = V\triangle B - \Phi(V) \subset (V - \Phi(V)) \cup B \in \mathcal{J}$. It follows that also $\Phi(A) - A \in \mathcal{J}$ for every set $A \in Ba(\tau)\triangle\mathcal{J}$ (see [1]). It means that Φ is the lower density operator on $(X, Ba(\tau)\triangle\mathcal{J}, \mathcal{J})$. By assumption and Property 3 the pair $(Ba(\tau)\triangle\mathcal{J}, \mathcal{J})$ fulfills ccc. Hence it has the hull property. Therefore by Theorem 1 the family $\mathcal{T}_\Phi$ is a topology on X. □

If (X, τ) is a topological space then it is well known that the pair $(Ba(\tau), \mathbb{K}(\tau))$ has the hull property (see [9, Corollary 1 p. 90)]. Thus following the proof of Theorem 4 we have the following theorem.

Theorem 5. *Let (X, τ) be a topological space and $\Phi : Ba(\tau) \to 2^X$ be a semi-lower density operator on $(X, Ba(\tau), \mathbb{K}(\tau))$. If $V - \Phi(V) \in \mathbb{K}(\tau)$ for every $V \in \tau$ then Φ is a lower density operator on the space $(X, Ba(\tau), \mathbb{K}(\tau))$ and Φ generates topology $\mathcal{T}_\Phi$.*

By applying this theorem, one can get the $\mathcal{I}$-density topology (see [13]), the $\langle s\rangle$-density topology with respect to category (see [7]), the density topology with respect to a sequence tending to zero related to category (see ([12]), and the simple density with respect to category (see [14]).

The above topologies on the real line, contained in the family of sets having the Baire property, are not regular.

Theorem 6. *Let (X, τ) be a topological space containing a τ-dense subset $A \in \mathbb{K}(\tau)$. If $\mathcal{J} \supset \mathbb{K}(\tau)$ is a σ-ideal in 2^X and $\Phi : Ba(\tau)\triangle\mathcal{J} \to 2^X$ is a semi-lower density operator on $(X, Ba(\tau)\triangle\mathcal{J}, \mathcal{J})$ generating topology $\mathcal{T}_\Phi$ such that $\tau \subset \mathcal{T}_\Phi$ then the topological space $(X, \mathcal{T}_\Phi)$ is not regular.*

Proof. Let $A \in \mathbb{K}(\tau)$ be a τ-dense set. Then $A \in \mathcal{J}$ and thus is $\mathcal{T}_\Phi$-closed. Let $x \notin A$. The sets A and $\{x\}$ can not be separated by $\mathcal{T}_\Phi$-open sets. Let $W, V \in \mathcal{T}_\Phi$ be such that $A \subset W$, $x \in V$. Then W is τ-dense and by Lemma 1 we have that $X - W \in \mathcal{J}$. The set $V \notin \mathcal{J}$, because otherwise $\Phi(V) = \emptyset$ and by the condition $V \subset \Phi(V)$ this would imply that $V = \emptyset$. Hence $W \cap V \neq \emptyset$. □

Example 3. Let $(\mathbb{R}, \mathcal{T}_d)$ be the topological space, where $\mathcal{T}_d$ is the density topology. Let $\mathcal{J} = \mathbb{L} = \mathbb{K}(\mathcal{T}_d)$. Then $Ba(\mathcal{T}_d)\triangle\mathbb{L} = \mathcal{L}\triangle\mathbb{L} = \mathcal{L} = Ba(\mathcal{T}_d)$. Putting $\Phi(A) = \Phi_d(A)$ for every $A \in \mathcal{L}$, where $\Phi_d(A)$ is the set of density points of A we get that Φ is the lower density operator on $(\mathbb{R}, Ba(\mathcal{T}_d)\triangle\mathcal{J}, \mathcal{J})$ generating topology $\mathcal{T}_d$. Since the topological space $(\mathbb{R}, \mathcal{T}_d)$ is regular, we have that Theorem 6 does not hold because $\mathbb{K}(\mathcal{T}_d)$ does not contain $\mathcal{T}_d$-dense set.

By Theorem 4 and Theorem 6 we get the following corollary.

Corollary 3. *Let (X, τ) be a topological Baire space containing τ-dense subset $A \in \mathbb{K}(\tau)$ and the pair $(Ba(\tau), \mathbb{K}(\tau))$ fulfills ccc . If $\mathcal{J} \supset \mathbb{K}(\tau)$ is a proper σ-ideal in 2^X and $\Phi : Ba(\tau)\triangle\mathcal{J} \to 2^X$ is a semi-lower density operator on $(X, Ba(\tau)\triangle\mathcal{J}, \mathcal{J})$ such that $\tau \subset \mathcal{T}_\Phi$ then the family $\mathcal{T}_\Phi$ is a topology on X and the topological space $(X, \mathcal{T}_\Phi)$ is not regular.*

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