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On hereditarily normal rectifiable spaces

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Abstract. We show, modifying very slightly the proofs of the recent results of R. Buzyakova [5] about topological groups, that all assertions in [5] are valid in the more general class of rectifiable spaces.

Recall that a topological group G is a group G with a (Hausdorff) topology such that the product maps of $G \times G$ into G is jointly continuous and the inverse map of G onto itself associating x^{-1} with arbitrary $x \in G$ is continuous. A paratopological group G is a group G with a topology such that the product maps of $G \times G$ into G is jointly continuous. A topological space G is said to be a rectifiable space provided that there exists a homeomorphism $\varphi : G \times G \to G \times G$ and an element $e \in G$ such that $\pi_1 \circ \varphi = \pi_1$ and for every $x \in G$ we have $\varphi(x, x) = (x, e)$, where $\pi_1 : G \times G \to G$ is the projection to the first coordinate. If G is a rectifiable space, then φ is called a rectification on G. It is well known that rectifiable spaces are good generalization of topological groups. In fact, for a topological group with the neutral element e, then it is easy to see that the map $\varphi(x, y) = (x, x^{-1}y)$ is a rectification on G. However, there exists a paratopological group which is not a rectifiable space. The Sorgenfrey line ([6, Example 1.2.2]) is such an example. Also, the 7-dimensional sphere S_7 is a rectifiable space but not a topological group [12]. Further, it is easy to see that paratopological groups and rectifiable spaces are all homogeneous. A series of results on rectifiable spaces have recently been obtained in [1, 2, 8–11].

Theorem 1. ([9]) *A topological space G is a rectifiable space if and only if there exist $e \in G$ and two continuous maps $p : G \times G \to G$, $q : G \times G \to G$ such that for any $x \in G$, $y \in G$ the next identities hold:*

$$p(x, q(x, y)) = q(x, p(x, y)) = y \ \textit{and} \ q(x, x) = e.$$

In fact, we can assume that $p = \pi_2 \circ \varphi^{-1}$ and $q = \pi_2 \circ \varphi$ in Theorem 1. Fixing a point $x \in G$, we get that the maps $f_x, g_x : G \to G$ defined by $f_x(y) = p(x, y)$ and $g_x(y) = q(x, y)$ for each $y \in G$, are homeomorphisms.

The above map $p : G \times G \to G$ will be called multiplication on G. Let G be a rectifiable space, and let p be the multiplication on G. We will write $x \cdot y$ instead of $p(x, y)$ and $A \cdot B$ instead of $p(A, B)$ for any $A, B \subset G$. Therefore, $q(x, y)$ is an element such that $x \cdot q(x, y) = y$. Since $x \cdot e = x \cdot q(x, x) = x$ and $x \cdot q(x, e) = e$, it follows that e is a right neutral element for G and $q(x, e)$ is a right inverse for x.

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A space has a G_δ-diagonal if the diagonal $\triangle = \{(g, g) : g \in G\}$ is the intersection of a countable family of its open neighborhoods in $G \times G$. The diagonal number $\triangle(G)$ is the minimal cardinality of a family of open sets of $G \times G$ such that the intersection of its elements is the diagonal $\triangle = \{(g, g) \in G \times G : g \in G\}$. A non-trivial convergent sequence means a subspace which is homeomorphic to the reals $\{0\} \cup \{1/n : n = 1, 2, 3, \ldots\}$. All spaces are assumed to be T_1. The letter e will always denote the right neutral element of a rectifiable space, $\psi(G)$ will denote the pseudocharacter of a space G.

Now, we recall an important result which is due to A.S. Gul'ko [7].

Theorem 2. ([7]) *Let G be a rectifiable space. Then $\psi(G) = \Delta(G)$.*

Using Theorems 1 and 2, one can easily modify the proofs of all assertions from [6] for extending them to the class of rectifiable spaces.

Theorem 3. *A hereditarily normal rectifiable space with a non-trivial convergent sequence has a G_δ-diagonal.*

Corollary 4. *Every countably compact subspace of a hereditarily normal rectifiable space G that contains a non-trivial convergent sequence is metrizable.*

Corollary 5. *Every countably compact hereditarily normal rectifiable space that contains a non-trivial convergent sequence is metrizable.*

Corollary 6. *Assume the Proper Forcing Axiom. Then every countably compact hereditarily normal rectifiable space G is metrizable.*

Lemma 7. *Let G be a hereditarily normal rectifiable space and let S and T be compact subspaces of G. Suppose that S is separable, s is a limit point of S, $e \in T$ and e has uncountable character in T. Then there exists a compactum $C \subset T$ such that $e \in G$, e has uncountable character in C and $s \cdot C \subset S$.*

Theorem 8. *Every compact subset of a hereditarily normal rectifiable space is metrizable.*

Theorem 9. *Let G be a hereditarily normal rectifiable space. Then either G has a non-trivial convergent sequence and a G_δ-diagonal, or G has no non-trivial convergent sequences and every compact subset of G is finite. In either case, every compact subset of G is metrizable.*

Since a monotonically normal space is hereditarily normal and a generalized order space (GO-space) is monotonically normal, we have the following two corollaries.

Corollary 10. *Let G be a monotonically normal rectifiable space. Then every compact subset of G is metrizable.*

Corollary 11. *Let G be a rectifiable space which is a GO-space. Then every compact subset of G is metrizable.*

Since the Sorgenfrey line is a hereditarily normal paratopological group and every compact subset is metrizable, we have the following open question.

Question 12. *Let G be a hereditarily normal(or even monotonically normal) paratopological group. Is every compact subset of G metrizable?*

A space X is called to be a *rotoid* space if there is a special point $e \in X$ and a homeomorphism H from X^2 onto itself with the following properties:

(1) for each $x \in X$, $H(x, x) = (x, e)$, and;

(2) for each $x \in X$, $H(e, x) = (e, x)$.

Obviously, each rectifiable space is rotoid. Moreover, the Sorgenfrey line is a rotoid [5]. Therefore, we have the following question.

Question 13. *Let G be a hereditarily normal (or even monotonically normal) rotoid space. Is every compact subset of G metrizable?*

A space X is called to be a *Choban* space if there is a special point $e \in X$ and a continuous function H from X^2 to X with the following properties:

(1) for each $x \in X$, $H(x, x) = e$, and;

(2) for each $x \in X$, the function H maps the subspace Vert(x)=$\{x\} \times X$ of X^2 onto X in a one-to-one way.

Obviously, each rectifiable space is also a *Choban* space. Moreover, the Sorgenfrey line is a *Choban* space [4]. Therefore, we have the following question.

Question 14. *Let G be a hereditarily normal (or even monotonically normal) Choban space. Is every compact subset of G metrizable?*

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