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Bornologies and bitopological function spaces

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Dedicated to Prof. Lj. D.R. Kočinac on the occasion of his 65th birthday

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Abstract. The aim of this paper is to study certain closure-type properties of function spaces over metric spaces endowed with two topologies: the topology of uniform convergence on a bornology and the topology of strong uniform convergence on a bornology. The study of function spaces with the strong uniform topology on a bornology was initiated by G. Beer and S. Levi in 2009, and then continued by several authors: A. Caserta, G. Di Maio and L'. Holá in 2010, A. Caserta, G. Di Maio, Lj.D.R. Kočinac in 2012. Properties that we consider in this paper are defined in terms of selection principles.

1. Introduction

Let (X,d) and (Y,ρ) be (infinite) metric spaces. We study some closure-type properties of the function spaces Y^X and $C(X,Y)$ endowed with two topologies: the topology $\tau_{\mathfrak{B}}$ of uniform convergence on a bornology $\mathfrak{B}$ on X and the topology $\tau^s_{\mathfrak{B}}$ of strong uniform convergence on $\mathfrak{B}$.

Our terminology and notation are standard as in [1, 5, 11].

The study of function spaces with the strong uniform topology on a bornology was initiated in [2], and then continued in [3, 4].

The properties we consider are defined in terms of the following two classical selection principles (see [8, 14] where the reader can find undefined notions concerning selection principles):

Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set X. Then:

$\mathsf{S}_1(\mathcal{A},\mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n, $B_n \subset A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

If (X,d) is a metric space, $x \in X$, $A \subset X$ and $\varepsilon > 0$ a real number, we denote by

$$S(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\},$$
$$A^\varepsilon := \textstyle\bigcup_{a\in A} S(a,\varepsilon),$$

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the open ε-ball with center x and the ε-enlargement of A, respectively.

If X and Y are given spaces, then we denote by Y^X (resp. $\mathsf{C}(X,Y)$) the set of all functions (resp. all continuous functions) from X into Y. By $\mathsf{C}(X)$ we denote the set $\mathsf{C}(X,\mathbb{R})$, and by $\mathsf{C}_p(X)$ and $\mathsf{C}_k(X)$ the set $\mathsf{C}(X)$ equipped with the pointwise topology τ_p and the compact-open topology τ_k, respectively.

A *bornology* on a metric space (X,d) is a family $\mathfrak{B}$ of nonempty subsets of X which is closed under finite unions, hereditary (i.e. closed under taking nonempty subsets) and forms a cover of X [6, 7]. To avoid some trivial situations we suppose that X does not belong to a bornology $\mathfrak{B}$ on X. A *base* for a bornology $\mathfrak{B}$ on (X,d) is a subfamily $\mathfrak{B}_0$ of $\mathfrak{B}$ such that for each $B\in\mathfrak{B}$ there is $B_0\in\mathfrak{B}_0$ with $B\subset B_0$. A base is called *closed* (*compact*) if all its members are closed (compact) subsets of X.

Examples of important bornologies on a metric space (X,d) are:

1. The family $\mathfrak{F}$ of all nonempty finite subsets of X (the smallest bornology on X and has a closed, in fact a compact, base);
2. The family of all nonempty subsets of X (the largest bornology on X);
3. The collection $\mathcal{K}_r$ of all nonempty relatively compact subsets (i.e. subsets with compact closure).

Let (X,d) and (Y,ρ) be metric spaces and $\mathfrak{B}$ a bornology on X. By $\tau_{\mathfrak{B}}$ we denote the *topology of uniform convergence on* $\mathfrak{B}$ generated by a uniformity on Y^X having as a base the sets of the form

$$[B,\varepsilon] := \{(f,g) : \forall x\in B, \rho(f(x),g(x)) < \varepsilon\} \ \ (B\in\mathfrak{B}, \varepsilon>0).$$

The following topology was introduced in [2].

Let (X,d) and (Y,ρ) be metric spaces and $\mathfrak{B}$ a bornology **with closed base** on X. The *topology of strong uniform convergence on* $\mathfrak{B}$, denoted by $\tau^s_{\mathfrak{B}}$, is generated by a uniformity on Y^X having as a base the sets of the form

$$[B,\varepsilon]^s := \{(f,g) : \exists\delta>0\,\forall x\in B^{\delta}, \rho(f(x),g(x)) < \varepsilon\} \ \ (B\in\mathfrak{B}, \varepsilon>0).$$

Therefore, if $f\in C(X,Y)$, then the standard local base of f in $(\mathsf{C}(X,Y),\tau_{\mathfrak{B}})$ is the family of sets

$$[B,\varepsilon](f) = \{g\in\mathsf{C}(X,Y) : \rho(g(x),f(x)) < \varepsilon,\ \forall x\in B\} \ \ (B\in\mathfrak{B}, \varepsilon>0),$$

and in $(\mathsf{C}(X,Y),\tau^s_{\mathfrak{B}})$ the family of sets

$$[B,\varepsilon]^s(f) = \{g\in(\mathsf{C}(X,Y),\tau^s_{\mathfrak{B}}) : \exists\delta>0,\ \rho(g(x),f(x)) < \varepsilon,\ \forall x\in B^{\delta}\} \ \ (B\in\mathfrak{B}, \varepsilon>0).$$

Observe that the spaces $(\mathsf{C}(X),\tau_{\mathfrak{B}})$ and $(\mathsf{C}(X),\tau^s_{\mathfrak{B}})$ are homogeneous so that it suffices to look at the constantly zero function $\underline{0}$ considering local properties of these spaces.

Throughout the paper we assume that all bornologies are with closed base. For each bornology $\mathfrak{B}$ with closed base on X the topology $\tau^s_{\mathfrak{B}}$ on Y^X is stronger than the topology $\tau_{\mathfrak{B}}$, and if $\mathfrak{B}$ has a compact base, then $\tau^s_{\mathfrak{B}} = \tau_{\mathfrak{B}} \le \tau_k$ on $\mathsf{C}(X,Y)$. In particular,

$$\tau_p \le \tau^s_{\mathfrak{F}} \le \tau^s_{\mathfrak{B}} \le \tau^s_{\mathfrak{K}} = \tau_k$$

on $\mathsf{C}(X)$.

2. Results

We begin with investigation of duality between covering properties of a space X and closure-type properties of bitopological space $(\mathsf{C}(X),\tau^s_{\mathfrak{B}},\tau_{\mathfrak{B}})$, where $\mathfrak{B}$ is a bornology on X.

For a bornology $\mathfrak{B}$ on a space (X,d) an open cover $\mathcal{U}$ of X is said to be a $\mathfrak{B}$-cover if each element in $\mathfrak{B}$ is contained in a member of $\mathcal{U}$ and $X\notin\mathcal{U}$. The collection of all $\mathfrak{B}$-covers of a space is denoted by $\mathcal{O}_{\mathfrak{B}}$.

The following notion was introduced in [3]. An open cover $\mathcal{U}$ of a metric space (X,d) with a bornology $\mathfrak{B}$ is said to be a *strong* $\mathfrak{B}$-*cover* of X (or shortly a $\mathfrak{B}^s$-*cover* of X) if $X\notin\mathcal{U}$ and for each $B\in\mathfrak{B}$ there exist $U\in\mathcal{U}$ and $\delta>0$ such that $B^{\delta}\subset U$. The collection of all strong $\mathfrak{B}$-covers of a space is denoted by $\mathcal{O}_{\mathfrak{B}^s}$.

Let us define also the following. A countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of a space X is called a $\gamma_{\mathfrak{B}}$-*cover* if each member B of $\mathfrak{B}$ belongs to U_n all but finitely many n. A countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of X is said to be a $\gamma_{\mathfrak{B}^s}$-*cover* (called a $\mathfrak{B}^s$-*sequence* in [3]) if for each $B \in \mathfrak{B}$ there are $n_0 \in \mathbb{N}$ and a sequence $(\delta_n : n \geq n_0)$ of positive real numbers such that $B^{\delta_n} \subset U_n$ for all $n \geq n_0$.

For a given bornology $\mathfrak{B}$ in a space X we denote by $\Gamma_{\mathfrak{B}}$ ($\Gamma_{\mathfrak{B}^s}$) the collection of all countable $\gamma_{\mathfrak{B}}$-covers ($\gamma_{\mathfrak{B}^s}$-covers) of X.

Definition 2.1. Let X be a space and $\mathfrak{B}$ a bornology on X. X is said to be a *($\mathfrak{B}^s, \mathfrak{B}$)-Lindelöf* if each $\mathfrak{B}^s$-cover contains a countable $\mathfrak{B}$-subcover.

Definition 2.2. ([12]) A bispace (X, τ_1, τ_2) has countable (τ_i, τ_j)-*tightness* ($i \neq j$; $i, j = 1, 2$) if for each $A \subset X$ and each $x \in Cl_{\tau_i}(A)$ there is a countable $C \subset A$ such that $x \in Cl_{\tau_j}(C)$.

The following three lemmas will be used in what follows.

Lemma 2.3. ([4]) *For every B in a bornology $\mathfrak{B}$ and every $\delta > 0$ it holds $\overline{B^{\delta}} \subset B^{2\delta}$.*

Lemma 2.4. ([4]) (*a*) *Let $\mathcal{U}$ be a $\mathfrak{B}^s$-cover of X. Set $A = \{f \in \mathsf{C}(X) : \exists U \in \mathcal{U}, f(x) = 1$ for all $x \in X \setminus U\}$. Then $\underline{0} \in \mathrm{Cl}_{\tau^s_{\mathfrak{B}}}(A)$.*

(*b*) *Let $A \subset (\mathsf{C}(X), \tau^s_{\mathfrak{B}})$, and let $\mathcal{U} = \{f^{\leftarrow}(-1/n, 1/n) : f \in A\}, n \in \mathbb{N}$. If $\underline{0} \in \overline{A}$ and $X \notin \mathcal{U}$, then $\mathcal{U}$ is a $\mathfrak{B}^s$-cover of X.*

Lemma 2.5. (*a*) *Let $\mathcal{U}$ be a $\mathfrak{B}$-cover of X. Set $A = \{f \in \mathsf{C}(X) : \exists U \in \mathcal{U}, f(x) = 1$ for all $x \in X \setminus U\}$. Then $\underline{0} \in \mathrm{Cl}_{\tau_{\mathfrak{B}}}(A)$.*

(*b*) *Let $A \subset (\mathsf{C}(X), \tau_{\mathfrak{B}})$, and let $\mathcal{U} = \{f^{\leftarrow}(-1/n, 1/n) : f \in A, n \in \mathbb{N}\}$. If $\underline{0} \in \overline{A}$ and $X \notin \mathcal{U}$, then $\mathcal{U}$ is a $\mathfrak{B}$-cover of X.*

Proof. (a) Since $X \notin \mathcal{U}$, $\underline{0} \notin A$, it is clear that $\underline{0} \notin A$. Let $B \in \mathfrak{B}$ and $\varepsilon > 0$. It is easy to check that the fact that $\mathfrak{B}$ has countable base implies $\overline{B} \in \mathfrak{B}$. Therefore, there is $U \in \mathcal{U}$ with $\overline{B} \subset U$. Pick a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in \overline{B}$ and $f(x) = 1$ for all $x \in X \setminus U$. Then $f \in [B, \varepsilon](\underline{0}) \cap A$.

(b) Let $B \in \mathfrak{B}$. Then there is $f \in [B, 1/n](\underline{0}) \cap A$. Hence $|f(x)| < 1/n$ for all $x \in B$, i.e. we have $B \subset f^{\leftarrow}(-1/n, 1/n) \in \mathcal{U}$. □

Theorem 2.6. *Let (X, d) be a metric space and $\mathfrak{B}$ a bornology on X with closed base. The following are equivalent:*

(1) *$(\mathsf{C}(X), \tau^s_{\mathfrak{B}}, \tau_{\mathfrak{B}})$ has countable $(\tau^s_{\mathfrak{B}}, \tau_{\mathfrak{B}})$-tightness;*

(2) *X is a $(\mathfrak{B}^s, \mathfrak{B})$-Lindelöf space.*

Proof. (1) ⇒ (2) Let $\mathcal{U}$ be a $\mathfrak{B}^s$-cover. By Lemma 2.4, $\underline{0} \in \mathrm{Cl}_{\tau^s_{\mathfrak{B}}}(A)$, where $A = \{f \in \mathsf{C}(X) : \exists U \in \mathcal{U}, f(x) = 1$ for all $x \in X \setminus U\}$. Since $(\tau^s_{\mathfrak{B}}, \tau_{\mathfrak{B}})$-tightness of $\mathsf{C}(X)$ is countable, there is a countable set $S \subset A$ such that $\underline{0} \in \mathrm{Cl}_{\tau_{\mathfrak{B}}}(S)$. Then, by Lemma 2.5, the countable set $\mathcal{V} = \{f^{\leftarrow}(-1/n, 1/n) : f \in S, n \in \mathbb{N}\}$ is a $\mathfrak{B}$-cover of X and $\mathcal{V} \subset \mathcal{U}$.

(2) ⇒ (1) Let $A \subset (\mathsf{C}(X), \tau^s_{\mathfrak{B}})$ be such that $\underline{0} \in \mathrm{Cl}_{\tau^s_{\mathfrak{B}}}(A)$. Set $\mathcal{U} = \{f^{\leftarrow}(-1/n, 1/n) : f \in A, n \in \mathbb{N}\}$. By Lemma 2.4, $\mathcal{U}$ is a $\mathfrak{B}^s$-cover of X. As X is a $(\mathfrak{B}^s, \mathfrak{B})$-Lindelöf space, there is a countable set $\mathcal{V} \subset \mathcal{U}$ such that $\mathcal{V}$ is a $\mathfrak{B}$-cover of X. By Lemma 2.5, then the $\tau_{\mathfrak{B}}$-closure of $S = \{f \in \mathsf{C}(X) : \exists V \in \mathcal{V}, f(x) = 1$ for all $x \in X \setminus V\}$ contains $\underline{0}$ which means that (1) is true. □

Our next aim is to prove similar results for countable fan tightness [1] and countable strong fan tightness [13].

Let (X, τ_1, τ_2) be a bitopological space. Then:

(i) X has countable (τ_i, τ_j)-*fan tightness* ($i \neq j; i, j = 1, 2$) if for each $x \in X$ and each sequence $(A_n : n \in \mathbb{N})$ of subsets of X such that $x \in Cl_i(A_n)$ for each $n \in \mathbb{N}$, there are finite sets $F_n \subset A_n$, $n \in \mathbb{N}$, with $x \in Cl_j\left(\bigcup_{n\in\mathbb{N}} F_n\right)$ (see [9]).

(ii) X has countable (τ_i, τ_j)-*strong fan tightness* ($i \neq j; i, j = 1, 2$), if for each $x \in X$ and each sequence $(A_n : n \in \mathbb{N})$ of subsets of X such that $x \in Cl_i(A_n)$ for each $n \in \mathbb{N}$, there are points $x_n \in A_n$, $n \in \mathbb{N}$, with $x \in Cl_j(\{x_n : n \in \mathbb{N}\})$ (see [9]).

Theorem 2.7. *Let (X, d) be a metric space and $\mathfrak{B}$ a bornology on X. The following are equivalent:*

(1) $(\mathsf{C}(X), \tau^s_{\mathfrak{B}}, \tau_{\mathfrak{B}})$ *has countable $(\tau^s_{\mathfrak{B}}, \tau_{\mathfrak{B}})$-strong fan tightness;*

(2) *X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}})$.*

Proof. (1) ⇒ (2): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open strong $\mathfrak{B}$-covers of X. For each $n \in \mathbb{N}$ let

$$A_n = \{f \in \mathsf{C}(X) : \exists U \in \mathcal{U}_n, f(x) = 1 \text{ for all } x \in X \setminus U\}.$$

Then $\underline{0} \in \mathrm{Cl}_{\tau^s_{\mathfrak{B}}}(A_n) \setminus A_n$, $n \in \mathbb{N}$, by Lemma 2.4 (a). By (1) there is a sequence $(f_n : n \in \mathbb{N})$ such that for each n, $f_n \in A_n$ and $\underline{0} \in \mathrm{Cl}_{\tau_{\mathfrak{B}}}(\{f_n : n \in \mathbb{N}\})$. For each $n \in \mathbb{N}$ take $U_n \in \mathcal{U}_n$ such that $f_n(x) = 1$ for all $x \in X \setminus U_n$. By Lemma 2.5 (b), $\{f_n^{\leftarrow}(-1, 1) : n \in \mathbb{N}\}$ is a $\mathfrak{B}$-cover of X. Since for each n, $f_n^{\leftarrow}(-1, 1) \subset U_n$, we have that $\{U_n : n \in \mathbb{N}\}$ is also a $\mathfrak{B}$-cover of X.

(2) ⇒ (1): Let $(A_n : n \in \mathbb{N})$ be a sequence of subsets of $(\mathsf{C}(X), \tau^s_{\mathfrak{B}})$ such that $\underline{0} \in \mathrm{Cl}_{\tau^s_{\mathfrak{B}}}(A_n) \setminus A_n$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ set

$$\mathcal{U}_n = \{f^{\leftarrow}(-1/n, 1/n) : f \in A_n\}.$$

If $X \in \mathcal{U}_n$ for infinitely many n, our conclusion follows easily, so we may assume $X \notin \mathcal{U}_n$, $n \in \mathbb{N}$. Then $\mathcal{U}_n$ is a $\mathfrak{B}^s$-cover of X by Lemma 2.4 (b). By (2), take $f_n \in A_n$, $n \in \mathbb{N}$, such that $\mathcal{V} = \{f_n^{\leftarrow}(-1/n, 1/n) : n \in \mathbb{N}\}$ is a $\mathfrak{B}$-cover of X. We show $\underline{0} \in \mathrm{Cl}_{\tau_{\mathfrak{B}}}(\{f_n : n \in \mathbb{N}\})$. Let $B \in \mathfrak{B}$ and $\varepsilon > 0$. Take $n_0 \in \mathbb{N}$ with $1/n_0 < \varepsilon$. Since $\mathcal{V}$ is a $\mathfrak{B}^s$-cover of X, $\{f_n^{\leftarrow}(-1/n, 1/n) : n \geq n_0\}$ is also a $\mathfrak{B}^s$-cover of X. Therefore there are $n \geq n_0$ and $\delta > 0$ such that $B \subset f_n^{\leftarrow}(-1/n, 1/n)$, and thus $f_n \in [B, \varepsilon](\underline{0})$. □

The following theorem can be proved similarly.

Theorem 2.8. *Let (X, d) be a metric space and $\mathfrak{B}$ a bornology on X. The following are equivalent:*

(1) $(\mathsf{C}(X), \tau^s_{\mathfrak{B}}, \tau_{\mathfrak{B}})$ *has countable $(\tau^s_{\mathfrak{B}}, \tau_{\mathfrak{B}})$-fan tightness;*

(2) *X satisfies $\mathsf{S}_{fin}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}})$.*

Call a bitopological space (X, τ_1, τ_2) *strictly (τ_i, τ_j)-Fréchet-Urysohn*, $i \neq j$, $i, j = 1, 2$, if for each sequence $(A_n : n \in \mathbb{N})$ and each $x \in \bigcap_{n\in\mathbb{N}} Cl_{\tau_i}(A_n)$ there are $x_n \in A_n$, $n \in \mathbb{N}$, such that the sequence $(x_n : n \in \mathbb{N})$ τ_j-converges to x.

Theorem 2.9. *Let (X, d) be a metric space and $\mathfrak{B}$ be a bornology on X. The following are equivalent:*

(1) $(\mathsf{C}(X), \tau^s_{\mathfrak{B}}, \tau_{\mathfrak{B}})$ *is a strictly $(\tau^s_{\mathfrak{B}}, \tau_{\mathfrak{B}})$-Fréchet-Urysohn space;*

(2) *X satisfies $\mathsf{S}_1(\mathcal{O}^s_{\mathfrak{B}}, \Gamma_{\mathfrak{B}})$.*

Proof. (1) ⇒ (2) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $\mathfrak{B}^s$-covers of X. For each $n \in \mathbb{N}$ and each $B \in \mathfrak{B}$ pick an element $U_{B,n} \in \mathcal{U}_n$ and $\delta > 0$ satisfying $B^{2\delta} \subset U_{B,n}$. Set

$$\mathcal{U}_{B,n} := \{U \in \mathcal{U}_n : B^{2\delta} \subset U\}.$$

For each $U \in \mathcal{U}_{B,n}$ there is a continuous function $f_{B,U} : X \to [0,1]$ such that $f_{B,U}(B^\delta) = \{0\}$ and $f_{B,U}(X \setminus U) = \{1\}$. For each n, let

$$A_n = \{f_{B,U} : B \in \mathfrak{B}, U \in \mathcal{U}_{B,n}\}.$$

Obviously, the function $\underline{0}$ belongs to $\mathrm{Cl}_{\tau^s_{\mathfrak{B}}}(A_n)$ for each $n \in \mathbb{N}$. By (1), there exist $f_{B_n,U_n} \in A_n$, $n \in \mathbb{N}$, such that the sequence $(f_{B_n,U_n} : n \in \mathbb{N})$ $\tau_{\mathfrak{B}}$-converges to $\underline{0}$. It remains to prove that the set $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}}$-cover of X. Let $B \in \mathfrak{B}$. For the neighbourhood $[B,1](\underline{0})$ of $\underline{0}$ there is $n_0 \in \mathbb{N}$ such that $f_{B_n,U_n} \in [B,1](\underline{0})$ for all $n > n_0$. In other words, for each $n > n_0$ we have $B \subset f^{\leftarrow}_{B_n,U_n}(-1,1)$, hence $B \subset U_n$.

(2) ⇒ (1) Let $(A_n : n \in \mathbb{N})$ be a sequence of subsets of $\mathsf{C}(X)$ such that $\underline{0} \in \bigcap_{n\in\mathbb{N}} \mathrm{Cl}_{\tau^s_{\mathfrak{B}}}(A_n) \setminus A_n$. For each $n \in \mathbb{N}$ define

$$\mathcal{U}_n = \{f^{\leftarrow}(-1/n, 1/n) : f \in A_n\}.$$

We may assume $X \notin \mathcal{U}_n$, $n \in \mathbb{N}$, and thus each $\mathcal{U}_n$ is a $\mathfrak{B}^s$-cover of X. By (2), there are $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that the set $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}}$-cover of X. For each U_n take the function f_n such that $U_n = f_n^{\leftarrow}(-1/n, 1/n)$. To end the proof we have to prove that the sequence $(f_n : n \in \mathbb{N})$ $\tau_{\mathfrak{B}}$-converges to $\underline{0}$.

Let $[B,\varepsilon](\underline{0})$ be a $\tau_{\mathfrak{B}}$-neighbourhood of $\underline{0}$. The set $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}}$-cover of X so that there is $m \in \mathbb{N}$ such that $1/m < \varepsilon$ and for each $n > m$, $B \subset U_n$. Therefore, for all $n > m$, $f_n(B \subset (-1/n, 1/n) \subset (-\varepsilon, , \varepsilon)$, which means that $f_n \in [B,\varepsilon](\underline{0})$. □

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