Filomat 27:7 (2013), 1205–1218
DOI 10.2298/FIL1307205H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Non-Ultra Regular Digital Covering Spaces with Nontrivial Automorphism Groups

Sang-Eon Han

Faculty of Liberal Education, Institute of Pure and Applied Mathematics, Chonbuk National University, 567 Baekje-daero, deokjin-gu, Jeonju-si, Jeollabuk-do 561-756 Republic of Korea

Abstract. The study of digital covering transformation groups (or automorphism groups, discrete deck transformation groups) plays an important role in the classification of digital spaces (or digital images). In particular, the research into transitive or nontransitive actions of automorphism groups of digital covering spaces is one of the most important issues in digital covering and digital homotopy theory. The paper deals with the problem: Is there a digital covering space which is not ultra regular and has an automorphism group which is not trivial? To solve the problem, let us consider a digital wedge of two simple closed k_i-curves with a compatible adjacency, $i \in \{1, 2\}$, denoted by (X, k). Since the digital wedge (X, k) has both infinite or finite fold digital covering spaces, in the present paper some of these infinite fold digital covering spaces were found not to be ultra regular and further, their automorphism groups are not trivial, which answers the problem posed above. These findings can be substantially used in classifying digital covering spaces and digital images so that the paper improves on the research in Section 4 of [3] (compare Figure 2 of the present paper with Figure 2 of [3]), which corrects an error that appears in the Boxer and Karaca's paper [3] (see the points $(0, 0), (0, 8), (6, -1)$ and $(6, 7)$ in Figure 2 of [3]).

1. Introduction

The study of covering spaces plays an important role in many areas of mathematics such as algebraic topology, Riemannian geometry, harmonic analysis, differential topology and so forth. Research into covering transformation groups (or automorphism groups) of covering spaces is also very important in covering and homotopy theory. In particular, given a covering $(\tilde{X}, p, X)$ we can consider a transitive or a nontransitive action of the automorphism group (briefly, $Aut(\tilde{X}, p)$) on the fiber $p^{-1}(x_0)$ [27]. This approach is often used for classifying covering spaces over (X, x_0). For instance, let $(\tilde{X}, p)$ be a covering space of X. The automorphism group $Aut(\tilde{X}, p)$ operates transitively on $p^{-1}(x), x \in X$ if and only if $(\tilde{X}, p)$ is a regular covering space of X [27].

2010 *Mathematics Subject Classification*. Primary 55Q70; Secondary 55Q20,68R10,68U05

Keywords. digital topology, digital isomorphism, local isomorphism, strong k-deformation retract, ultra regular covering space, digital covering space, simply k-connected, digital covering transformation group, discrete deck transformation group, compatible adjacency, automorphism group.

Received: 25 February 2013; Accepted: 05 May 2013

Communicated by Dragan S. Djordjević

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2010002325).

This paper was supported by the selection of research-oriented professor of Chonbuk National University in 2011.

Email address: sehan@jbnu.ac.kr (Sang-Eon Han)

Let us consider a graph on the nD lattice space such as $\mathbf{Z}^n$, where $\mathbf{Z}$ is the set of integers. Then we can call the graph a digital graph (or digital image). In the classification of digital images, we often use methods involving a digital fundamental group [1, 24, 26], a digital covering transformation group (or a discrete deck transformation group) [12] and so forth [3, 8, 13–15, 23]. The calculation of both an automorphism group and a digital fundamental group of a digital space is one of the most important issues in digital covering and digital homotopy theory. Thus the paper [5] (see also [4, 17, 18]) establishes the notion of digital covering space, studies a digital wedge of simple closed k-curves (see also [2]) and calculates its digital fundamental group. The paper [12] develops the notion of a digital covering transformation group. The paper [11] introduces the action of $Aut(\tilde{X}, p)$ on the fiber $p^{-1}(x_0)$ (see also (3.2) and Definition 3.11 in the present paper). In relation to the study of a transitive action of an automorphism group of a digital covering space, the recent paper [17] develops the notion of an ultra regular covering space (see Definition 3.14 of the present paper) and proposes several kinds of digital covering spaces over a digital wedge. Further, the papers [3, 17] study a nontransitive action of an automorphism of a digital covering over a digital wedge that satisfies a radius 2 local isomorphism. More precisely, the paper [3] treats a nontransitive action of $Aut(\tilde{X}, p)$ on the fiber $p^{-1}(x_0)$, where $(\tilde{X}, k')$ is an infinite fold digital covering space over a digital wedge of two simple closed 8-curves. Indeed, the authors of [3] use an unusual adjacency of $\tilde{X} \subset \mathbf{Z}^2$ which is not a standard adjacency relations from (2.1). Instead of the approach, the present paper uses a traditional approach so that the paper improves on the research in [3] (see Sections 4 and 5).

We may raise the problem: Is there a digital covering (E, p, B) which is not ultra regular and has an automorphism group $Aut(E\,|\,p)$ that is not trivial?

In relation to the study of this problem, the present paper deals with three associated problems below. Let (X, k) be a digital wedge of two simple closed k_i-curves with a compatible k-adjacency (see Definition 3.6), $i \in \{1, 2\}$, where each simple closed k_i-curve need not be k_i-contractible.

(Q1) How many digital covering spaces are there over (X, k)?

(Q2) Assume that $p : (\tilde{X}, k') \to (X, k)$ is a (k', k)-covering map. In relation to the study of a transitive or a nontransitive action of $Aut(\tilde{X}, p)$ on the fiber $p^{-1}(x_0)$, what types of digital covering spaces $(\tilde{X}, k')$ exist?

(Q3) What are the automorphism groups of the digital covering spaces $(\tilde{X}, k')$ over (X, k)?

Given a digital wedge (X, k), the present paper proves that it has countably many and infinite fold digital covering spaces $(\tilde{X}, k')$. Some of them are not ultra regular and the others are ultra regular. Further, their automorphism groups are not trivial. This research improves on the results in [3] and answers the problem posed earlier in this paper.

The paper has main results in Sections 4 and 5 and is organized as follows: Section 2 provides some basic notions which underpin our work. Section 3 reviews some properties of digital covering spaces and their automorphism groups. Section 4 shows that the digital space (X, k) of Questions 1-3 above has countably many and infinite fold digital covering spaces $(\tilde{X}, k')$ which are not necessarily ultra regular spaces and further, proves that their associated automorphism groups are not trivial. Section 5 proposes an infinite fold digital covering space (E, k') over a digital wedge of (X, k) of Questions 1-3 above whose digital covering transformation group is transitive on the fiber and calculates its automorphism group. Section 6 concludes the paper with a summary.

2. Preliminaries

Let **N** denote the sets of natural numbers. Let $\mathbf{Z}^n$ denote the set of points in the Euclidean nD space with integer coordinates. Useful tools from algebraic topology and geometric topology for studying the digital topological properties of a (binary) digital space include a digital covering space [4, 5], a (digital) k-fundamental group [1], a digital k-surface [8, 9] and so forth [6, 7, 10, 12, 14, 15].

To study a multidimensional space $X \subset \mathbf{Z}^n$ (or a digital space), let us now recall the k-adjacency relations of $\mathbf{Z}^n$ as well as some essential terminology such as a digital isomorphism [1, 6, 19], a digital homotopy [1], a strong k-deformation retract [10] and so forth. As a generalization of the k-adjacency relations of $2D$ and $3D$ digital spaces [25, 28], the k-adjacency relations of $\mathbf{Z}^n$ were established in [4] (see also [5, 14, 16]):
For a natural number m where $1 \leq m \leq n$, two distinct points $p = (p_1, p_2, \cdots, p_n)$ and $q = (q_1, q_2, \cdots, q_n) \in \mathbf{Z}^n$ are called $k(m, n)$- (briefly, k-) adjacent if
• there are at most m indices i such that $|p_i - q_i| = 1$ and
• for all other indices i such that $|p_i - q_i| \neq 1, p_i = q_i$.
Concretely, according to the two numbers $m, n \in \mathbf{N}$, the $k(m, n)$ (or k)-adjacency relations of $\mathbf{Z}^n$ were represented in [5], as follows (for more details, see also [14, 16]):

$$k := k(m, n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n, \text{where } C_i^n = \frac{n!}{(n-i)!\, i!}. \tag{2.1}$$

For $\{a, b\} \subset \mathbf{Z}$ with $a \lneq b$, $[a, b]_{\mathbf{Z}} = \{a \leq n \leq b \,|\, n \in \mathbf{Z}\}$ is considered with 2-adjacency [1]. In this paper we are not concerned with $\bar{k}$-adjacency between two points in $\mathbf{Z}^n \setminus X$.

We say that two subsets (A, k) and (B, k) of (X, k) are k-adjacent to each other if $A \cap B = \emptyset$ and there are points $a \in A$ and $b \in B$ such that a and b are k-adjacent to each other [25]. We say that a set $X \subset \mathbf{Z}^n$ is k-connected if it is not a union of two disjoint non-empty sets which are not k-adjacent to each other [25]. For an adjacency relation k of $\mathbf{Z}^n$, a simple k-path with $l + 1$ elements in $\mathbf{Z}^n$ is assumed to be an injective sequence $(x_i)_{i \in [0,l]_{\mathbf{Z}}} \subset \mathbf{Z}^n$ such that x_i and x_j are k-adjacent if and only if either $j = i + 1$ or $i = j + 1$ [25]. If $x_0 = x$ and $x_l = y$, then we say that the length of the simple k-path, denoted by $l_k(x, y)$, is the number l. A simple closed k-curve with l elements in $\mathbf{Z}^n$, denoted by $SC_k^{n,l}$ [5], is the simple k-path $(x_i)_{i \in [0,l-1]_{\mathbf{Z}}}$, where x_i and x_j are k-adjacent if and only if $j = i + 1 (mod\, l)$ or $i = j + 1 (mod\, l)$ [25].

In the study of digital continuity and the various properties of a digital space [5], we have often used the following digital k-neighborhood of a point $x \in X$ with radius $\varepsilon \in \mathbf{N}$ [4] (see also [5]): For a digital space (X, k) in $\mathbf{Z}^n$, the digital k-neighborhood of $x_0 \in X$ with radius ε is defined in X to be the following subset of X:

$$N_k(x_0, \varepsilon) = \{x \in X \,|\, l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\}, \tag{2.2}$$

where $l_k(x_0, x)$ is the length of a shortest simple k-path from x_0 to x and $\varepsilon \in \mathbf{N}$.

To map every k_0-connected subset of (X, k_0) into a k_0-connected subset of (Y, k_1), the papers [1, 28] established the notion of digital continuity. Motivated from both the digital continuity in [28] and the (k_0, k_1)-continuity in [1], we present the digital continuity of maps between digital spaces, which can be widely used for studying digital spaces in $\mathbf{Z}^n, n \in \mathbf{N}$, as follows:

Proposition 2.1. ([13]) Let (X, k_0) and (Y, k_1) be digital spaces in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. A function $f : X \to Y$ is (k_0, k_1)-continuous if and only if for every $x \in X$ $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1)$.

In Proposition 2.1 if $n_0 = n_1$ and $k_0 = k_1$, then we can call it a k_0-*continuous*.

Since the digital space (X, k) can be considered to be a digital k-graph, we use the term a (k_0, k_1)-isomorphism as in [6, 19] rather than a (k_0, k_1)-homeomorphism as in [1], as follows:

Definition 2.2. ([6] see also [1, 13, 19]) For two digital spaces (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, a map $h : X \to Y$ is called a (k_0, k_1)-isomorphism if h is a (k_0, k_1)-continuous bijection and further, $h^{-1} : Y \to X$ is (k_1, k_0)-continuous. Then we use the notation $X \approx_{(k_0,k_1)} Y$. If $n_0 = n_1$ and $k_0 = k_1$, then we speak out a k_0-isomorphism and use the notation $X \approx_{k_0} Y$.

For a digital space (X, k) and $A \subset X$, (X, A) is called a digital space pair with a k-adjacency [7]. Furthermore, if A is a singleton set $\{x_0\}$, then (X, x_0) is called a pointed digital space [25]. Based on the pointed digital homotopy in [1], the following notion of k-homotopy relative to a subset $A \subset X$ is often used in studying a k-homotopic thinning and a strong k-deformation retract of a digital space (X, k) in $\mathbf{Z}^n$ [9, 14].

Definition 2.3. ([7] see also [8, 10]) Let $((X, A), k_0)$ and (Y, k_1) be a digital space pair and a digital space, respectively. Let $f, g : X \to Y$ be (k_0, k_1)-continuous functions. Suppose there exist $m \in \mathbf{N}$ and a function $F : X \times [0, m]_{\mathbf{Z}} \to Y$ such that

• for all $x \in X, F(x, 0) = f(x)$ and $F(x, m) = g(x)$;

• for all $x \in X$, the induced function $F_x : [0, m]_{\mathbf{Z}} \to Y$ given by $F_x(t) = F(x, t)$ for all $t \in [0, m]_{\mathbf{Z}}$ is $(2, k_1)$-continuous;

• for all $t \in [0, m]_{\mathbf{Z}}$, the induced function $F_t : X \to Y$ given by $F_t(x) = F(x, t)$ for all $x \in X$ is (k_0, k_1)-continuous.

Then we say that F is a (k_0, k_1)-homotopy between f and g [1].

• Furthermore, for all $t \in [0, m]_{\mathbf{Z}}$, assume that the induced map F_t on A is a constant which follows the prescribed function from A to Y. In other words, $F_t(x) = f(x) = g(x)$ for all $x \in A$ and for all $t \in [0, m]_{\mathbf{Z}}$.

Then we call F a (k_0, k_1)-homotopy relative to A between f and g, and we say that f and g are (k_0, k_1)-homotopic relative to A in Y, $f \simeq_{(k_0,k_1) relA} g$ in symbols.

In Definition 2.3, if $A = \{x_0\} \subset X$, then we say that F is a pointed (k_0, k_1)-homotopy at $\{x_0\}$ [1]. When f and g are pointed (k_0, k_1)-homotopic in Y, we use the notation that $f \simeq_{(k_0,k_1)} g$. In addition, if $k_0 = k_1$ and $n_0 = n_1$, then we say that f and g are pointed k_0-homotopic in Y and we use the notation that $f \simeq_{k_0} g$ and $f \in [g]$ which denotes the k_0-homotopy class of g. If, for some $x_0 \in X$, 1_X is k-homotopic to the constant map in the space x_0 relative to $\{x_0\}$, then we say that (X, x_0) is pointed k-contractible [1].

Definition 2.4. ([8] see also [10]) For a digital space pair $((X, A), k)$, we say that A is a strong k-deformation retract of X if there is a digital k-continuous map r from X onto A such that $F : i \circ r \simeq_{k \cdot rel.A} 1_X$ and $r \circ i = 1_A$. Then a point $x \in X \setminus A$ is called strong k-deformation retractable.

Since the *Khalimsky operation* [22] is essentially used in establishing a digital fundamental group, we need to recall it as follows: assume $F^k(X, x_0) = \{f \,|\, f$ is a k-loop based at $x_0\}$. For members $f : [0, m_f]_{\mathbf{Z}} \to X$ and $g : [0, m_g]_{\mathbf{Z}} \to X$ of $F^k(X, x_0)$, in [22] the map

$$f * g : [0, m_f + m_g]_{\mathbf{Z}} \to X$$

is given by

$$f * g(t) = \begin{cases} f(t) \text{ if } 0 \le t \le m_f; \\ g(t - m_f) \text{ if } m_f \le t \le m_f + m_g. \end{cases}$$

Further, using the trivial extension [1] and the *Khalimsky operation* [22], the paper[1] establishes the following k-fundamental group: For a digital space (X, k), consider a k-loop f with a base point x_0, we denote $[f]_X$ (briefly, $[f]$) as the k-homotopy class of f in X. Then for a k-loop f_1 with the same base point $x_0 \in X$, $f_0 \in [f]$ means that the two k-loops f and f_0 have trivial extensions that can be joined by a k-homotopy keeping the base point fixed [1]. Furthermore, if $f_1, f_2, g_1, g_2 \in F^k(X, x_0)$, $f_1 \in [f_2]$, and $g_1 \in [g_2]$, then $f_1 * g_1 \in [f_2 * g_2]$, *i.e.* $[f_1 * g_1] = [f_2 * g_2]$ [1, 22]. Then we use the notation $\pi^k(X, x_0) = \{[f] \,|\, f \in F^k(X, x_0)\}$ which is a group [1] with the operation $[f] \cdot [g] = [f * g]$ and called the (digital) k-fundamental group of (X, x_0) [1], where the base point is assumed to be a point which cannot be deleted by a strong deformation retract [12] if the given space (X, k) is not k-contractible. If X is pointed k-contractible, then $\pi^k(X, x_0)$ is trivial [1].

Let $((X, A), k)$ be a digital space pair with k-adjacency. A map $f : ((X, A), k_0) \to ((Y, B), k_1)$ is called (k_0, k_1)-continuous if f is (k_0, k_1)-continuous and $f(A) \subset B$ [7]. If $A = \{a\}$, $B = \{b\}$, we write $(X, A) = (X, a)$, $(Y, B) = (Y, b)$, and we say that f is a pointed (k_0, k_1)-continuous map [25]. A (k_0, k_1)-continuous map $f : ((X, x_0), k_0) \to ((Y, y_0), k_1)$ induces a group homomorphism [1]

$$f_* : \pi^{k_0}(X, x_0) \to \pi^{k_1}(Y, y_0) \text{ given by } f_*([\alpha]) = [f \circ \alpha]. \tag{2.3}$$

The following notion of "simply k-connected" in [5] has been often used in digital k-homotopy and digital covering theory: A pointed k-connected digital space (X, x_0) is called *simply k-connected* if $\pi^k(X, x_0)$ is a trivial group.

Since the k-contractibility requires a digital space (X, k) to shrink k-continuously to a point over a finite time interval, we cannot say that $\mathbf{Z}^n$ is $2n$-contractible, $n \in \mathbf{N}$. However, using the simply 2-connectedness of $\mathbf{Z}$ [5], we can show that $(\mathbf{Z}^n, 0_n)$ is simply k-connected, where the k-adjacency is assumed to be anyone of the k-adjacency relations of $\mathbf{Z}^n$.

Using the non-8-contractibility of $SC_8^{2,6}$ [5], the paper [5] (see also [8, 10]) proved that $\pi^k(SC_k^{n,l})$ is an infinite cyclic group, where $SC_k^{n,l}$ is not k-contractible. More precisely, we obtain the following:

Theorem 2.5. ([5] see also, [8, 10]) $\pi^k(SC_k^{n,l}, x_0) \simeq (l\mathbf{Z}, +)$, where $SC_k^{n,l}$ is not k-contractible, " $\simeq$" means a group isomorphism and $x_0 \in SC_k^{n,l}$. In addition, $\pi^k(SC_k^{n,4})$ is trivial if $k = 3^n - 1, n \in \mathbf{N} - \{1\}$.

3. Some Properties of Digital Covering Spaces

Let (X, k) be a digital space in $\mathbf{Z}^n$. In relation to the calculation of $\pi^k(X, x_0)$ and the classification of digital spaces in terms of a digital k-homotopy, we have often used some properties of digital coverings [2–5, 14, 15]. In digital covering theory each digital space (X, k) is assumed to be k-connected. Therefore, in the rest of this paper every (X, k) is assumed to be k-connected, unless stated otherwise. In this section we review some properties of regular covering spaces and ultra regular covering spaces. Let us now recall the typical axioms of a digital covering space.

Definition 3.1. ([5] see also [13]) Let (E, k_0) and (B, k_1) be digital spaces in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. Let $p : E \to B$ be a (k_0, k_1)-continuous surjection. Suppose, for any $b \in B$ there exists $\varepsilon \in \mathbf{N}$ such that
(1) for some index set M, $p^{-1}(N_{k_1}(b, \varepsilon)) = \cup_{i \in M} N_{k_0}(e_i, \varepsilon)$ with $e_i \in p^{-1}(b)$;
(2) if $i, j \in M$ and $i \neq j$, then $N_{k_0}(e_i, \varepsilon) \cap N_{k_0}(e_j, \varepsilon)$ is an empty set; and
(3) the restriction map p on $N_{k_0}(e_i, \varepsilon)$ is a (k_0, k_1)-isomorphism for all $i \in M$.
Then the map p is called a (k_0, k_1)-covering map, (E, p, B) is said to be a (k_0, k_1)-covering and (E, k_0) is called a digital (k_0, k_1)-covering space over (B, k_1).

The k_1-neighborhood $N_{k_1}(b, \varepsilon)$ of Definition 3.1 is called an elementary k_1-neighborhood of b with some radius ε. While in Definition 3.1 we may take $\varepsilon = 1$ [2, 8], the paper [17] established a simpler form of the axioms of a digital covering space, as follows:

Proposition 3.2. ([17]) For the (k_0, k_1)-covering of Definition 3.1 we can replace "(k_0, k_1)-continuous surjection" by "surjection".

Definition 3.3. ([8] see also [13]) We say that a (k_0, k_1)-covering map $p : (E, e_0) \to (B, b_0)$ is an m-fold (k_0, k_1)-covering map if the cardinality of the index set M is m.

Definition 3.3 can be restated as follows: For a (k_0, k_1)-covering map $p : (E, e_0) \to (B, b_0)$, if the set $p^{-1}(b_0)$ has n elements (or the number n can also be called the sheets of the digital covering (see [27]), then the map p is called an m-fold (k_0, k_1)-covering map because any points $b_1, b_2 \in B$ satisfy the following identity in terms of the digital version of the corresponding properties of a covering in [27]: $\#\{p^{-1}(b_1)\} = \#\{p^{-1}(b_2)\} = m$, where "#" means the cardinality of the given set.

For pointed digital spaces $((E, e_0), k_0)$ and $((B, b_0), k_1)$, if $p : (E, e_0) \to (B, b_0)$ is a (k_0, k_1)-covering map such that $p(e_0) = b_0$, then p is called a pointed (k_0, k_1)-covering map [5]. Hereafter, we assume that each digital covering map is a pointed one, unless stated otherwise.

Definition 3.4. ([4]) For $n \in \mathbf{N}$, a (k_0, k_1)-covering (E, p, B) is a radius n local isomorphism if the restriction map $p|_{N_{k_0}(e_i,n)} : N_{k_0}(e_i, n) \to N_{k_1}(b, n)$ is a (k_0, k_1)-isomorphism for all i, where $e_i \in p^{-1}(b)$.

According to Definition 3.4, we can say that a (k_0, k_1)-covering (E, p, B) is a radius n-(k_0, k_1)-covering if $\varepsilon \geq n$, where the number ε is the same as the ε of Definition 3.1 [4] (see also [13]).

In view of Definitions 3.1 and 3.4, we observe that a (k_0, k_1)-covering which satisfies a radius n local isomorphism is equivalent to a radius n-(k_0, k_1)-covering [8].

Let us recall the notion of a digital covering transformation group. For three digital spaces (B, k), (E_1, k_1) and (E_2, k_2), let (E_1, p_1, B) and (E_2, p_2, B) be (k_1, k)- and (k_2, k)-coverings, respectively. Then we say that a (k_1, k_2)-continuous map $\phi : E_1 \to E_2$ such that $p_2 \circ \phi = p_1$ is a (k_1, k_2)-covering homomorphism from (E_1, p_1, B) into (E_2, p_2, B) [10], where " $\circ$ " means composition. As a special case of this (k_1, k_2)-covering homomorphism, we obtain the digital version of a covering transformation group of a covering space in algebraic topology [29], as follows:

Definition 3.5. ([12]) Consider a (k_0, k_1)-covering map $p : (E, e_0) \to (B, b_0)$. A self k_0-isomorphism of the (k_0, k_1)-covering map p, denoted by $h : (E, k_0) \to (E, k_0)$, is called a digital k_0-covering transformation or an automorphism of a digital covering map p if $p = p \circ h$, where " $\circ$ " means composition.

Note that the set of the automorphisms of a digital covering map under composition is a group which is denoted by $Aut(E\,|\,B)$ (or $Aut(E, p)$) [12].

Since a digital wedge has often been used in the study of an automorphism groups of digital coverings, let us now recall the digital wedge in [5] (see also [2]). For digital spaces (X_i, k_i) in $\mathbf{Z}^{n_i}$, $i \in \{0, 1\}$ the notion of digital wedge of (X_i, k_i) was introduced in [5]. Using the version of a digital wedge developed in [5], we construct a notion of *compatible k-adjacency* of the digital wedge, as follows:

Definition 3.6. ([17]) For pointed digital spaces $((X, x_0), k_0)$ in $\mathbf{Z}^{n_0}$ and $((Y, y_0), k_1)$ in $\mathbf{Z}^{n_1}$, the wedge of (X, k_0) and (Y, k_1), written $(X \vee Y, (x_0, y_0))$, is the digital space in $\mathbf{Z}^n$

$$\{(x, y) \in X \times Y \mid x = x_0 \text{ or } y = y_0\} \tag{3.1}$$

with the following compatible $k(m, n)$(or k)-adjacency relative to both (X, k_0) and (Y, k_1), and the only one point (x_0, y_0) in common such that

(W1) the $k(m, n)$ (or k)-adjacency is determined by the numbers m and n with $n = n_0 + n_1$, $m = m_0 + m_1$ which satisfies $(W1 - 1)$ below, where the numbers m_i are taken from the k_i(or $k(m_i, n_i)$)-adjacency relations of the given digital spaces $((X, x_0), k_0)$ and $((Y, y_0), k_1)$, $i \in \{0, 1\}$.

(W 1-1) In view of (3.1), we can consider the projection maps from $X \vee Y$ onto X and Y, respectively denoted by

$$W_X : (X \vee Y, (x_0, y_0)) \to (X, x_0) \text{ and } W_Y : (X \vee Y, (x_0, y_0)) \to (Y, y_0),$$

where $W_X(x, y) = x$ and $W_Y(x, y) = y$.

In relation to the establishment of a compatible k-adjacency of the digital wedge $(X \vee Y, (x_0, y_0))$, the following restriction maps of W_X and W_Y on $(X \times \{y_0\}, (x_0, y_0)) \subset (X \vee Y, (x_0, y_0))$ and $(\{x_0\} \times Y, (x_0, y_0)) \subset (X \vee Y, (x_0, y_0))$ satisfy the following properties, respectively:

$$\begin{cases} (1)\ W_X\,|_{X\times\{y_0\}} : (X \times \{y_0\}, k) \to (X, k_0) \text{ is a } (k, k_0)\text{-isomorphism; and} \\ (2)\ W_Y\,|_{\{x_0\}\times Y} : (\{x_0\} \times Y, k) \to (Y, k_1) \text{ is a } (k, k_1)\text{-isomorphism.} \end{cases}$$

(W2) Any two distinct elements $(x, y_0) \in X \times \{y_0\}$ and $(x_0, y) \in \{x_0\} \times Y$ of $X \vee Y$, such that $x \neq x_0$ and $y \neq y_0$, are not $k(m, n)$ (or k)-adjacent to each other.

Remark 3.7. In relation to the choice of a compatible $k := k(m, n)$-adjacency of the digital wedge $X \vee Y$ in (W1) of Definition 3.6, we referred to $n = n_0 + n_1$ and $m = m_0 + m_1$. However, the wedge $X \vee Y \subset \mathbf{Z}^n$ with the $k := k(m, n)$-adjacency can be always (k, k')-isomorphically transformed into $\mathbf{Z}^{n'}$ with $k' := k'(m', n')$-adjacency [3, 5], where $m' = \max\{m_0, m_1\}$ and $n' = \max\{n_0, n_1\}$ satisfy the property (W1) and (W2) above. For instance, see the digital wedges in Figures 1, 2, and 3 of the present paper, i.e. $SC_8^{2,6} \vee SC_8^{2,6} \subset \mathbf{Z}^2$ with an 8 adjacency of $\mathbf{Z}^2$ instead of $\mathbf{Z}^4$. Thus for convenience, when studying a digital wedge, we may consider the digital wedge $X \vee Y$ in definition 3.6 with $k := k(m', n')$-adjacency, where $m' = \max\{m_0, m_1\}$ and $n' = \max\{n_0, n_1\}$.

The following notion has often been used for calculating the k-fundamental group of a digital space (X, k) and classifying digital spaces [11, 12].

Definition 3.8. ([8])(see also [12]) A (k_0, k_1)-covering $((E, e_0), p, (B, b_0))$ is called regular if $p_*\pi^{k_0}(E, e_0)$ is a normal subgroup of $\pi^{k_1}(B, b_0)$, where $p_* : \pi^{k_0}(E, e_0) \to \pi^{k_1}(B, b_0)$ is a group homomorphism of (2.3).

Using Massey's program of an automorphism group [27], we obtain a connection between $Aut(E\,|\,B)$ and the action of $\pi^{k_1}(B, b_0)$ on $p^{-1}(b_0)$ which represents digital topological versions of Proposition 7.1, Theorem 7.2 and Corollary 7.3 in [27], as follows: Let $((E, e_0), p, (B, b_0)))$ be a radius 2-(k_0, k_1)-covering. For any automorphism $\phi \in Aut(E\,|\,B)$, any point $\tilde{e} \in p^{-1}(b_0)$ and any $\alpha \in \pi^{k_1}(B, b_0)$, we obtain that [11]

$$\phi(\tilde{e} \cdot \alpha) = (\phi\tilde{e}) \cdot \alpha, \tag{3.2}$$

where the operation "$\cdot$" in (3.2) is easily induced from [11] as follows: Take $\alpha = [f] \in \pi^{k_1}(B, b_0)$, where $f : [0, m_f]_{\mathbf{Z}} \to (B, b_0)$ represents α. Consider the digital lifting $\tilde{f}$ of f [5, 18] and $\phi\tilde{f} : [0, m_f]_{\mathbf{Z}} \to E$ such that $\phi\tilde{f}(0) = \phi(\tilde{e})$, $\phi\tilde{f}(m_f) = \phi(\tilde{e} \cdot \alpha)$ and $p\phi\tilde{f} = p\tilde{f} = f$. Thus $\phi\tilde{f}$ is a digital lifting of f. Therefore, we obtain $\phi(\tilde{e}) \cdot \alpha = \phi\tilde{f}(m_f) = \phi(\tilde{e} \cdot \alpha)$.
In other words, each element $\phi \in Aut(E\,|\,B)$ induces an automorphism of the set $p^{-1}(b_0)$ which is considered as a right $\pi^{k_1}(B, b_0)$-space [12] (see also [15]). Further, we can state the following:

Theorem 3.9. ([15]) Let $((E, e_0), p, (B, b_0))$ be a radius 2-(k_0, k_1)-covering. Then, $Aut(E\,|\,B)$ is isomorphic to the group of automorphisms of the set $p^{-1}(b_0)$, which is considered as a right $\pi^{k_1}(B, b_0)$-space.

In relation to the study of an automorphism group of a digital covering space, this kind of approach used in Theorem 3.9 has some limitations because Theorem 3.9 are only valid under the hypothesis regarding the radius 2-(k_0, k_1)-covering. However, if a (k_0, k_1)-covering does not satisfy a radius 2 local isomorphism, then we have an obstacle to the study of the digital homotopic properties of a digital covering as well as its automorphism group (see [2–4, 8]) because we cannot use the developed digital homotopic tools (more precisely, the digital homotopy lifting theorem in [4]). For instance, let us now consider the map $p_1 : E_1 \to SC_8^{2,4} \vee SC_8^{2,6}$ in Figure 1(a) given by

$$p_1(x) = v_i \text{ if } x \text{ is labeled } i, \text{ for all } x \in E_1 \text{ and } i \in [0, 8]_{\mathbf{Z}}. \tag{3.3}$$

Then it is an $(8, 8)$-covering map that cannot satisfy a radius 2 local isomorphism.
Meanwhile, consider the map $p_2 : E_2 \to SC_8^{2,4} \vee SC_8^{2,6}$ in Figure 1(b) given by

$$p_2(y) = v_j \text{ if } y \text{ is labeled } j, \text{ for all } y \in E_2 \text{ and } j \in [0, 8]_{\mathbf{Z}}. \tag{3.4}$$

The map p_2 is an $(8, 8)$-covering map satisfying a radius 2 local isomorphism condition. Using the same method as that shown with the map p_1 of (3.3), we observe that the digital covering $(E_3, p_3, SC_8^{2,4} \vee SC_8^{2,6})$ in Figure 1(c) cannot satisfy a radius 2 local isomorphism either. Thus Theorem 3.9 cannot support the derivation of both $Aut(E_1\,|\,SC_8^{2,4} \vee SC_8^{2,6})$ and $Aut(E_3\,|\,SC_8^{2,4} \vee SC_8^{2,6})$. However, we can clearly obtain the following:

Remark 3.10. In view of (3.3) and (3.4), for the space $B := SC_8^{2,4} \vee SC_8^{2,6}$ we observe that $Aut(E_1\,|\,B)$ is trivial, and both $Aut(E_2\,|\,B)$ and $Aut(E_3\,|\,B)$ are isomorphic to the infinite cyclic group $(\mathbf{Z}, +)$.

Motivated from the transitive action of an automorphism of a covering space on the fiber of the covering map p [27], we can define the following:

Definition 3.11. ([17]) For a (k_0, k_1)-covering $((E, e_0), p, (B, b_0))$ we say that $Aut(E\,|\,B)$ acts transitively on $p^{-1}(b_0)$ if for any two distinct points e_0 and e_1 in $p^{-1}(b_0)$ there is $\phi \in Aut(E\,|\,B)$ such that $\phi(e_0) = e_1$.

In general, for a (k_0, k_1)-covering $((E, e_0), p, (B, b_0))$ $Aut(E\,|\,B)$ need not act transitively on $p^{-1}(b_0)$ (see the digital covering space in Example 3.12), we can show this as follows:

Example 3.12. Consider the $(8,8)$-covering $(E_1, p_1, SC_8^{2,4} \vee SC_8^{2,6})$ with the map of (3.3) as given in Figure 1(a). Then $Aut(E_1 \,|\, SC_8^{2,4} \vee SC_8^{2,6})$ cannot act transitively on $p_1^{-1}(v_0)$ for the point $v_0 \in SC_8^{2,4} \vee SC_8^{2,6}$ [17]. More precisely, for two distinct points e_i and e_j in $p_1^{-1}(v_0)$(e.g. the points $(0,0), (6,-1)$ in $p_1^{-1}(v_0)$) there is no $\phi \in Aut(E_1 \,|\, SC_8^{2,4} \vee SC_8^{2,6})$ such that $\phi(e_i) = e_j$.

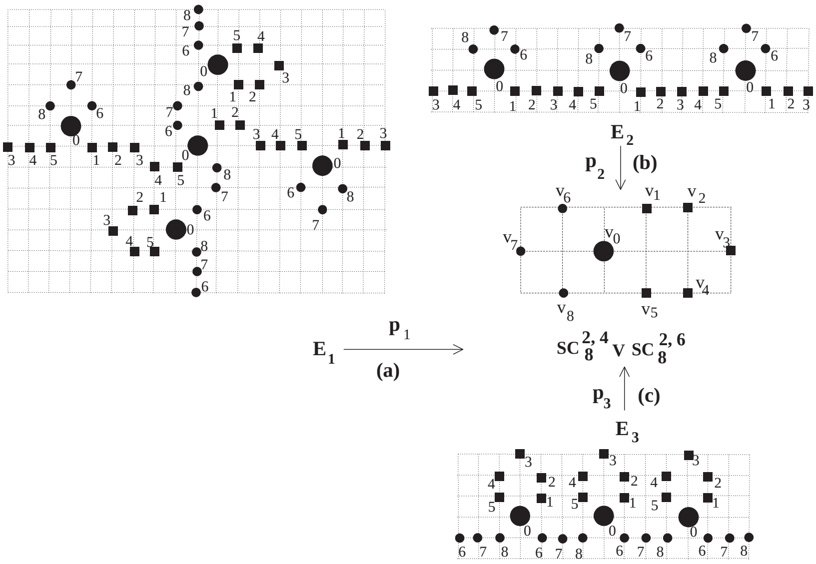

Figure 1: (a) Portion of an infinite fold $(8,8)$-covering $(E_1, p_1, SC_8^{2,4} \vee SC_8^{2,6})$ [5] which does not allow an automorphism which performs a transitive action on the fiber $p_1^{-1}(b), b \in SC_8^{2,4} \vee SC_8^{2,6}$, $E_1 \subset \mathbf{Z}^2$; (b) Portion of an infinite fold $(8,8)$-covering $(E_2, p_2, SC_8^{2,4} \vee SC_8^{2,6})$ from [2] which is both regular and ultra regular, $E_2 \subset \mathbf{Z}^2$; (c) Portion of an infinite fold $(8,8)$-covering $(E_3, p_3, SC_8^{2,4} \vee SC_8^{2,6})$ from [2] which is ultra regular, $E_3 \subset \mathbf{Z}^2$.

Unlike in Example 3.12, we obtain the following:

Lemma 3.13. ([17]) If a radius 2-(k_0, k_1)-covering map $p : (E, e_0) \to (B, b_0)$ is regular, then $Aut(E \,|\, B)$ acts transitively on $p^{-1}(b_0)$.

In view of Remark 3.10, for a (k_0, k_1)-covering map $p : (E, e_0) \to (B, b_0)$ which does not satisfy a radius 2 local isomorphism, we observe that $Aut(E \,|\, B)$ depends on the situation. In order to deal with this problem, we need to establish the following notion which is different from the notion of a regular (k_0, k_1)-covering.

Definition 3.14. ([17]) A (k_0, k_1)-covering $((E, e_0), p, (B, b_0))$ is called an ultra regular (briefly, UR-) (k_0, k_1)-covering if $Aut(E \,|\, B)$ acts transitively on $p^{-1}(b_0)$.

Let us now recall the following property of a UR-(k_0, k_1)-covering which characterizes a UR-(k_0, k_1)-covering.

Theorem 3.15. ([17]) The following are equivalent.

(1) A (k', k)-covering $((E, e_0), p, (B, b_0))$ is ultra regular.

(2) For a (k', k)-covering $((E, e_0), p, (B, b_0))$ we assume a closed k-curve $\alpha : [0, m]_{\mathbf{Z}} \to (B, k)$ with $\alpha(0) = b_0 \in B$. Either each of all the liftings of α on (E, k') is a k'-closed curve or none of them is a k'-closed curve.

Due to Theorem 3.15, hereafter, regardless of the requirement of a radius 2 local isomorphism of a (k', k)-covering, we now have a very convenient method of determining if a digital covering is UR-(k', k)-regular and further, we can study $Aut(E \,|\, B)$ without using the digital homotopic tools of a digital covering (E, p, B).

We now pose the following question: For a digital space (X, k_1) how can we describe a difference between a UR-(k_0, k_1)-covering and a regular (k_0, k_1)-covering over (X, k_1)? In the light of Theorem 3.15, we can mention some merits of a UR-(k_0, k_1)-covering (for more details, see [17]). Namely, a UR-(k_0, k_1)-covering need not require a radius 2 local isomorphism. In view of this difference, a UR-(k_0, k_1)-covering has strong merits when used in the classification of digital covering spaces. Comparing an ultra regular covering space with a regular covering space, we obtain the following:

Theorem 3.16. ([17]) (1) A regular (k_0, k_1)-covering space does not imply a UR-(k_0, k_1)-covering space.

(2) For a digital space (X, k_1) let $R_2(X)$ denote the set of all radius 2-(k_0, k_1)-coverings over (X, k_1). Then we obtain the following: In $R_2(X)$ a UR-(k_0, k_1)-covering is equivalent to a regular (k_0, k_1)-covering (see also Corollary 4.13 of [3]).

In view of Theorem 3.16, if a (k_0, k_1)-covering is not a radius 2-(k_0, k_1)-covering, then whether a comparison can be made between an ultra regular covering space and a regular covering space depends on the situation.

4. Existence of Non-Ultra Regular Digital Covering Spaces with Nontrivial Automorphism Groups

In relation to the solution of the three questions posed in Section 1, recently the paper [3] studied a nontransitive action of an automorphism group of a digital covering space over $SC_8^{2,8} \vee SC_8^{2,6}$. To be specific, the authors of [3] use an extraordinary digital covering space (E, k') in $\mathbf{Z}^2$ which is not ultra regular (see Figure 2 of [3]) and further, the k'-adjacency of $E \subset \mathbf{Z}^2$ is not the usual adjacency relation of $\mathbf{Z}^2$ in [20, 28], *i.e.* $k' \notin \{4, 6, 8\}$.

Instead of the approach, to improve on the results in [3](see the points $(0,0), (0,8), (6,-1)$ and $(6,7)$ in Figure 2 of [3] which cannot be a $(k, 8)$-covering space, where $k \in \{4, 8\}$) and to address the three problems of Section 1, this section proposes that a digital wedge of two simple closed k_i-curves with a compatible k-adjacency, $i \in \{1, 2\}$ has countably many and infinite fold digital covering spaces with the usual adjacency relations of $\mathbf{Z}^n$ as in (2.1). Further, we prove that some of them are not ultra regular, while the others are ultra regular, and their automorphism groups are not trivial.

In this section, in particular, we prove that $(SC_8^{2,6} \vee SC_8^{2,6}, 8)$ (see Remark 3.7) has countably many and infinite fold $(k, 8)$-covering spaces (D_i, k) in $\mathbf{Z}^n, n \geq 2, i \in \mathbf{N}$ (see Theorem 4.1), where the k-(or $k(m,n)$-)adjacency is one of the usual adjacency relations of $\mathbf{Z}^n$ of (2.1) and $m = 2$. Further, we prove that for each $i \in \mathbf{N}$ $Aut(D_i \,|\, SC_8^{2,6} \vee SC_8^{2,6})$ is isomorphic to the infinite cyclic group $(\mathbf{Z}, +)$ (see Corollary 4.2). Similarly, we can prove that the digital wedge $(SC_8^{2,4} \vee SC_8^{2,6}, 8)$ has countably many and infinite fold $(k, 8)$-covering spaces in $\mathbf{Z}^n, n \geq 3$ which are not ultra regular, where the k-adjacency is equal to the $k(2,n)$-adjacency of (2.1).

In general, let us consider the digital wedge $(SC_{k_1}^{n_1,l_1} \vee SC_{k_2}^{n_2,l_2}, k)$ with a compatible k-adjacency, where $n = \max\{n_1, n_2\}$ (see Remark 3.7) and $SC_{k_i}^{n_i,l_i}$ need not be k_i-contractible, $i \in \{1, 2\}$. Then we can prove that it also has countably many and infinite fold digital covering spaces which are not ultra regular and their automorphism groups are not trivial. This answers the open problem in Section 1.

Theorem 4.1. The digital wedge $(SC_8^{2,6} \vee SC_8^{2,6}, 8)$ has countably many and infinite fold $(8, 8)$-covering spaces $(D_n, 8)$ which are not ultra regular, where $D_n \subset \mathbf{Z}^2$ and $n \in \mathbf{N}$.

Proof. (Case 1) Let $X := SC_8^{2,6} \vee SC_8^{2,6} = \{v_i\}_{i \in [0,10]_\mathbf{Z}} \subset \mathbf{Z}^2$, where $v_0 = (0,0), v_1 = (1,-1), v_2 = (1,-2), v_3 = (0,-3), v_4 = (-1,-2), v_5 = (-1,-1), v_6 = (1,1), v_7 = (1,2), v_8 = (0,3), v_9 = (-1,2), v_{10} = (-1,1)$. Then $(X, 8)$ is a digital wedge with compatible 8-adjacency (see Remark 3.7).

Note that $X = UX_6 \cup LX_6$, where $UX_6 = \{v_0\} \cup \{v_i\}_{i \in [6,10]_\mathbf{Z}}, LX_6 = \{v_i\}_{i \in [0,5]_\mathbf{Z}}$ (see Figure 2) and each of UX_6 and LX_6 is 8-isomorphic to $SC_8^{2,6}$.

Let $D_1 \subset \mathbf{Z}^2$ be the infinite set partially described in Figure 2(a). More precisely, for $j \in \mathbf{Z}$ let $h_j : \mathbf{Z}^2 \to \mathbf{Z}^2$ be the translation defined by $h_j(x, y) = (x, y + j)$. Assume that the space $(D_1, 8)$ contains the set

$$\bigcup_{j \in \mathbf{Z}} h_{6j}(UX_6) := T_1$$

consisting of countably many simple closed 8-curves of which each of them is 8-isomorphic to $SC_8^{2,6} := UX_6$ and are indexed by the numbers $i \in \{0\} \cup [6, 10]_{\mathbf{Z}}$ (see Figure 2(a)), *i.e.* for each $j \in \mathbf{Z}$ $h_{6j}(UX_6)$ is 8-isomorphic to $SC_8^{2,6} := UX_6$. Further, the space $(D_1, 8)$ also has countably many points that are indexed by the numbers $i \in [0, 10]_{\mathbf{Z}}$ and is partially described in Figure 2(a).
Let $d_0 = (0, 0) \in D_1$. Let $q_1 : (D_1, d_0) \to (X, v_0)$ be the map given by

$$q_1(x) = v_i \text{ if } x \text{ is labeled } i, \text{ for all } x \in D_1 \text{ and } i \in [0, 10]_{\mathbf{Z}}. \tag{4.1}$$

Namely, the map q_1 wraps the points in D_1 around X. To be specific,

$$q_1(T_1) = UX_6 \subset X$$

and the other countable set of points in $D_1 \setminus T_1$ is mapped onto X according to the map in (4.1), *i.e.*

$$\begin{cases} q_1(x) = LX_6, \ x \text{ is labeled } i, \text{ for all } x \in D_1 \setminus T_1 \text{ and } i \in [0, 5]_{\mathbf{Z}}; \\ q_1(x) = UX_6, \ x \text{ is labeled } i, \text{ for all } x \in D_1 \setminus T_1 \text{ and } i \in \{0\} \cup [6, 10]_{\mathbf{Z}}. \end{cases}$$

Then, by (4.1), q_1 is clearly an infinite fold $(8, 8)$-covering map.

(Case 2) Consider the digital space $(X, 8)$ of Case 1. Let $D_2 \subset \mathbf{Z}^2$ be the infinite set partially described in Figure 2(b). More precisely, for each $i \in \mathbf{Z}$ let $s_i : \mathbf{Z}^2 \to \mathbf{Z}^2$ be the translation defined by $s_i(x, y) = (x + i, y)$. Now assume that the space $(D_2, 8)$ contains the infinite set

$$(\bigcup_{j \in \mathbf{Z}} h_{6j}(UX_6)) \cup (\bigcup_{j \in \mathbf{Z}} h_{6j}(s_6(UX_6))) := T_2$$

consisting of countably many simple closed 8-curves that are 8-isomorphic to $SC_8^{2,6} := UX_6$, *i.e.* for each $j \in \mathbf{Z}$ each of $h_{6j}(UX_6))$ and $h_{6j}(s_6(UX_6)$ is 8-isomorphic to $SC_8^{2,6}$. Further, the space $(D_2, 8)$ also has countably many points indexed by the numbers $i \in [0, 10]_{\mathbf{Z}}$ and is partially described in Figure 2(b).
Note that $d_0 = (0, 0) \in D_2$. Let $q_2 : (D_2, d_0) \to (X, v_0)$ be the map given by

$$q_2(y) = v_i \text{ if } y \text{ is labeled } i, \text{ for all } y \in D_2 \text{ and } i \in [0, 10]_{\mathbf{Z}}. \tag{4.2}$$

Namely, the map q_2 wraps the points in D_2 around X. To be specific,

$$q_2(T_2) = UX_6$$

and further, the other countable set of points in $D_2 \setminus T_2$ is indexed by the numbers $i \in [0, 10]_{\mathbf{Z}}$ in D_2 and are mapped onto X according to the map of (4.2), *i.e.*

$$\begin{cases} q_2(y) = LX_6, \ y \text{ is labeled } i, \text{ for all } y \in D_2 \setminus T_2 \text{ and } i \in [0, 5]_{\mathbf{Z}}; \\ q_2(y) = UX_6, \ y \text{ is labeled } i, \text{ for all } y \in D_2 \setminus T_2 \text{ and } i \in \{0\} \cup [6, 10]_{\mathbf{Z}}. \end{cases}$$

Then, by (4.2), q_2 is clearly an infinite fold $(8, 8)$-covering map.
Here, we need to point out that the two digital covering spaces $(D_1, 8)$ and $(D_2, 8)$ are not 8-isomorphic to each other, as follows: Note that if $f : (X, k_0) \to (Y, k_1)$ is a (k_0, k_1)-isomorphism then for any nonempty set $(X', k_0) \subset (X, k_0)$ the restriction map $f|_{X'} : X' \to f(X')$ is also a (k_0, k_1)-isomorphism [7]. Let us now suppose that $(D_1, 8)$ and $(D_2, 8)$ are 8-isomorphic to each other with the 8-isomorphism f. Further, put the sets $X' \subset D_1$ and $Y' \subset D_2$ described with bands in Figure 2. Besides, in view of the construction of D_1 and D_2 we can see

that the two digital images $D_1 \setminus X'$ and $D_2 \setminus Y'$ are 8-isomorphic to each other. Thus, reminding the restriction property of an digital isomorphism above mentioned, let us consider the following restriction map on the set X', $f|_{X'} : X' \to Y'$ given by $f|_{X'}(x) = x \in Y'$, x is labeled i, for all $x \in X'$, $x \in Y'$ and $i \in \{0\} \cup [6, 10]_{\mathbf{Z}}$. Then, while $f|_{X'}$ is a bijection, it cannot be an 8-isomorphism, which contradict the hypothesis that $(D_1, 8)$ and $(D_2, 8)$ are 8-isomorphic to each other.

(Case 3) In general, using the same method as that shown in the establishment of $(D_1, 8)$ and $(D_2, 8)$ in Cases 1 and 2, for $n \in \mathbf{N}$, let us consider

$$\bigcup_{j\in\mathbf{Z}} h_{6j}([\cup_{i\in[0,n-1]_{\mathbf{Z}}} s_{6i}(UX_6)]) := T_n, \tag{4.3}$$

with 8-adjacency. Assume that the space $(D_n, 8)$ contains the set T_n in (4.3), and the other countable set of points which is established by the methods similar to those used with $D_1 \setminus T_1$ and $D_2 \setminus T_2$ is indexed by the numbers $i \in [0, 10]_{\mathbf{Z}}$.
Assume a map $q_n : (D_n, 8) \to (X, 8)$ to be the generalization of the maps of (4.1) and (4.2), as follows:

$$q_n(T_n) = UX_6$$

and further, the other countable set of points in $D_n \setminus T_n$ is mapped onto X in such a way:

$$\begin{cases} q_n(z) = LX_6, \ z \text{ is labeled } i, \text{ for all } z \in D_n \setminus T_n \text{ and } i \in [0,5]_{\mathbf{Z}}; \\ q_n(z) = UX_6, \ z \text{ is labeled } i, \text{ for all } z \in D_n \setminus T_n \text{ and } i \in \{0\} \cup [6,10]_{\mathbf{Z}}. \end{cases}$$

Consequently, $(D_n, 8)$ is an infinite fold $(8, 8)$-covering space over the digital wedge $(X, 8)$.
Using the same method as that shown in the comparison between $(D_1, 8)$ and $(D_2, 8)$ above, we observe that each pair $(D_i, 8)$ and $(D_j, 8)$ are not 8-isomorphic to each other if $i \neq j$, where $i, j \in \mathbf{N}$.
Therefore, we conclude that the digital wedge $(X, 8)$ has countably many and infinite fold $(8, 8)$-covering spaces $(D_n, 8), n \in \mathbf{N}$.

In view of the establishment of $(D_n, 8), n \in \mathbf{N}$ above and Theorem 3.15, each $(D_n, 8)$ is not ultra regular covering space over the digital wedge $(X, 8)$. For instance, let us show that $(D_1, q_1, SC_8^{2,6} \vee SC_8^{2,6})$ is not an ultra regular $(8, 8)$-covering. Consider the two distinct points $d_0 := (0, 0), d_1 := (6, -1)$ in $q_1^{-1}(v_0) \subset D_1$. Then there is no $\phi \in Aut(D_1 \,|\, SC_8^{2,6} \vee SC_8^{2,6})$ such that $\phi(d_0) = d_1$, which means that $(D_1, q_1, SC_8^{2,6} \vee SC_8^{2,6})$ is not an ultra regular $(8, 8)$-covering.
Similarly, for each $n \in \mathbf{N}$ $(D_n, q_n, SC_8^{2,6} \vee SC_8^{2,6})$ is proven not to be an ultra regular $(8, 8)$-covering. □

While the digital coverings $(D_i, q_i, SC_8^{2,6} \vee SC_8^{2,6}), i \in \mathbf{N}$ in Theorem 4.1 are not ultra regular, their automorphism groups are not trivial, as follows:

Corollary 4.2. For each of the $(8, 8)$-covering spaces $(D_i, q_i, SC_8^{2,6} \vee SC_8^{2,6}), i \in \mathbf{N}$ in Theorem 4.1, $Aut(D_i, q_i)$ is isomorphic to the infinite cyclic group $(\mathbf{Z}, +)$.

Proof. It is easily seen that $Aut(D_i, q_i) = \bigcup_{j\in\mathbf{Z}}\{h_{6j}\}$. The assertion follows. □

5. Existence of an Infinite Fold Ultra Regular Covering Space over a Digital Wedge whose Automorphism Group is not Trivial

Unlike the study of nontransitive action of an automorphism group of a digital covering space over a digital wedge, to solve the problems mentioned in Section 1, this section studies a transitive action of an automorphism group of a digital covering space over the digital wedge $(SC_{k_1}^{n_1,l_1} \vee SC_{k_2}^{n_2,l_2}, k)$ with a compatible k-adjacency. In particular, we prove that $(SC_8^{2,6} \vee SC_8^{2,6}, 8)$ has an infinite fold and ultra regular $(k, 8)$-covering space (E, k) in $\mathbf{Z}^n, n \geq 2, i \in \mathbf{N}$ (see Theorem 5.1 and Corollary 5.2), where the k-(or $k(m, n)$-)adjacency is the adjacency relations of $\mathbf{Z}^n$ in (2.1) and $m = 2$. Further, we prove that $Aut(E \,|\, SC_8^{2,6} \vee SC_8^{2,6})$ is isomorphic to the

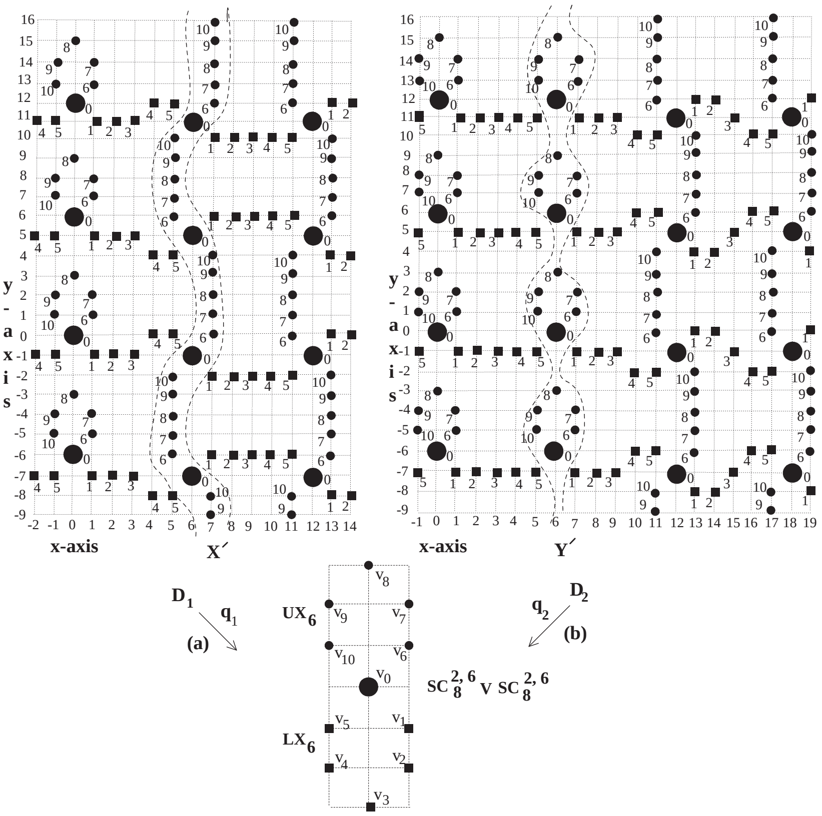

Figure 2: (a) Portion of an infinite fold (8, 8)-covering $(D_1, q_1, SC_8^{2,6} \vee SC_8^{2,6})$ which is not UR-regular, $D_1 \subset \mathbf{Z}^2$;
(b) Portion of an infinite fold (8, 8)-covering $(D_2, q_2, SC_8^{2,6} \vee SC_8^{2,6})$ which is not UR-regular and $(D_1, 8)$ is not 8-isomorphic to $(D_2, 8)$, $D_2 \subset \mathbf{Z}^2$.

infinite group $(\mathbf{Z} \times \mathbf{Z}, +)$ (see Corollary 5.2). In general, let us consider the digital wedge $(SC_{k_1}^{n_1,l_1} \vee SC_{k_2}^{n_2,l_2}, k)$ with a compatible k-adjacency, denoted by (X, k), where $n = \max\{n_1, n_2\}$ and SC_{k_i} need not be k_i-contractible. Then this section proves that the digital wedge (X, k) has an infinite fold (k, k)-covering space in $\mathbf{Z}^n$ which is ultra regular (see Theorem 5.3). Further, we prove that its automorphism group is not trivial. This approach answers the open problem posed in Section 1. Reminding Remark 3.7, we obtain the following:

Theorem 5.1. $SC_8^{2,6} \vee SC_8^{2,6}$ has an infinite fold and ultra regular (8, 8)-covering space in $\mathbf{Z}^2$.

Proof. Let $X := SC_8^{2,6} \vee SC_8^{2,6} = \{v_i\}_{i \in [0,10]_\mathbf{Z}} \subset \mathbf{Z}^2$ be the set in Theorem 4.1 which is a digital wedge with a compatible 8-adjacency.
Note $X = UX_6 \cup LX_6$, where $UX_6 = \{v_0\} \cup \{v_i\}_{i \in [6,10]_\mathbf{Z}}$, $LX_6 = \{v_i\}_{i \in [0,5]_\mathbf{Z}}$ (see Figure 3) and each of UX_6 and LX_6 is 8-isomorphic to $SC_8^{2,6}$.
Let $E_4 \subset \mathbf{Z}^2$ be the infinite set, partially described in Figure 3, which is (8, 4)-isomorphic to $6\mathbf{Z} \times 6\mathbf{Z} \subset \mathbf{Z}^2$ (see Figure 3). More precisely, $(E_4, 8)$ contains countably many points indexed by the numbers $i \in [0, 10]_\mathbf{Z}$.
Let $e_0 = (0, 0) \in E_4$. Let $p_4 : (E_4, e_0) \to (X, v_0)$ be the map given by

$$p_4(x) = v_i \text{ if } x \text{ is labeled } i, \text{ for all } x \in E_4 \text{ and } i \in [0, 10]_\mathbf{Z}. \tag{5.1}$$

Then the map p_4 wraps the points in E_4 around X. To be specific,

$$\begin{cases} p_4(x) = LX_6,\ x \text{ is labeled } i, \text{ for all } x \in E_4 \text{ and } i \in [0,5]_{\mathbf{Z}}; \\ p_4(x) = UX_6,\ x \text{ is labeled } i, \text{ for all } x \in E_4 \text{ and } i \in \{0\} \cup [6,10]_{\mathbf{Z}}. \end{cases}$$

Then p_4 is an infinite fold $(8,8)$-covering map.

In view of the establishment of $(E_4, 8)$ above and Theorem 3.15, it is an ultra regular $(8,8)$-covering space over the digital wedge $(X, 8)$. To be specific, for the $(8,8)$-covering (E_4, p_4, X) consider any two points e_i, e_j in $p_4^{-1}(v_0) \subset E_4$. Then there is a $\phi \in Aut(E_4 \,|\, X)$ such that $\phi(e_i) = e_j$, which means that (E_4, p_4, X) is an ultra regular $(8,8)$-covering. □

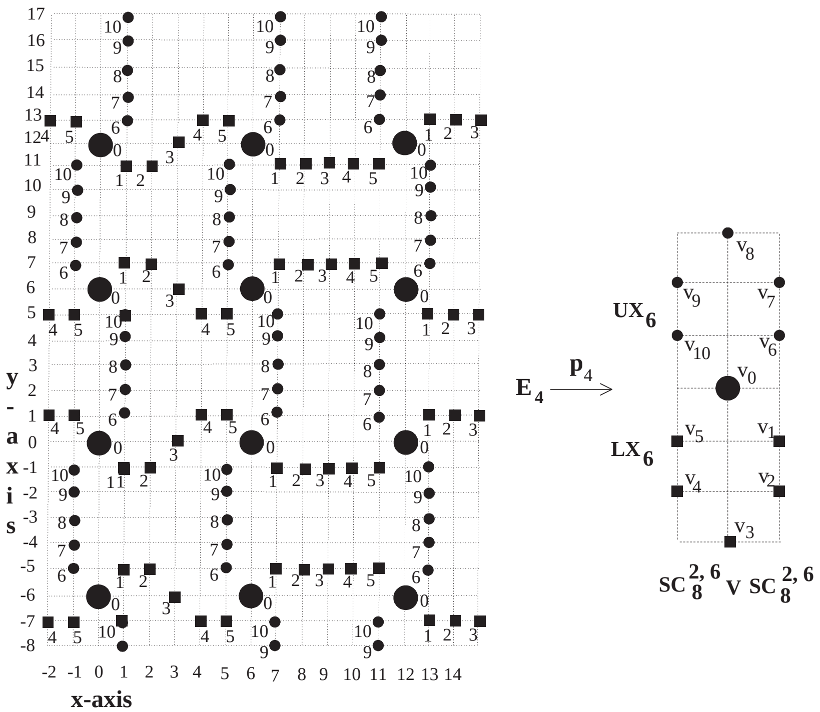

Figure 3: Portion of an infinite fold $(8,8)$-covering $(E_4, p_4, SC_8^{2,6} \vee SC_8^{2,6})$, $E_4 \subset \mathbf{Z}^2$ that is ultra regular.

For the digital covering in Theorem 5.1, we obtain the following:

Corollary 5.2. $Aut(E_4, p_4)$ is an infinite group that is isomorphic to $(6\mathbf{Z} \times 6\mathbf{Z}, +)$.

Proof. We observe that $p_4^{-1}(v_0)$ is the set $6\mathbf{Z} \times 6\mathbf{Z}$. Further, using vertical parallel and horizontal parallel transformations of $(E_4, 8)$, we prove this assertion. □

By analogy to Theorem 5.1 and Corollary 5.2, we obtain the following:

Theorem 5.3. Consider $(SC_{k_1}^{n_1,l_1} \vee SC_{k_2}^{n_2,l_2}, k) := (X, k)$ in $\mathbf{Z}^n, n = \max\{n_1, n_2\}$ with a compatible k-adjacency. Then there is an infinite fold and ultra regular covering space $(E, 8)$ up to 8-isomorphism, where $E \subset \mathbf{Z}^2$. $Aut(E \,|\, X)$ is an infinite group that is isomorphic to $(l_1\mathbf{Z} \times l_2\mathbf{Z}, +)$.

6. Concluding Remark

In relation to the study of an automorphism group of a digital covering and the classification of digital spaces, this paper has addressed the following issues:
We answered the problem of whether there is a digital covering (E, p, B) which is not ultra regular but its automorphism group $Aut(E\,|\,p)$ is not trivial, we found the following: Let (X, k) be a digital wedge of two simple closed k_i-curves with a compatible k-adjacency (see Definition 3.6), $i \in \{1, 2\}$, where each simple closed k_i-curve need not be k_i-contractible. Then we have proved that the digital space (X, k) has countably many and infinite fold digital covering spaces $(\tilde{X}, k')$ up to k'-isomorphism. Some of them are not ultra regular and the others are ultra regular and further, their automorphism groups are not trivial. Using this approach, we can substantially classify digital covering spaces and digital images and further, we can develop a digital covering space from the viewpoint of *CTC* in [21].

Acknowledgement We are grateful to anonymous referees for their many suggestions.

References

[1] L. Boxer, A classical construction for the digital fundamental group, *Jour. of Mathematical Imaging and Vision* **10** (1999) 51-62.
[2] L. Boxer, Digital Products, Wedge; and Covering Spaces, *Jour. of Mathematical Imaging and Vision* **25** (2006) 159-171.
[3] L. Boxer and I. Karaca, Some Properties of digital covering spaces, *Jour. of Mathematical Imaging and Vision* **37** (2010) 17-26.
[4] S.E. Han, Digital coverings and their applications, *Jour. of Applied Mathematics and Computing* **18**(1-2) (2005) 487-495.
[5] S.E. Han, Non-product property of the digital fundamental group, *Information Sciences* **171** (1-3) (2005) 73-91.
[6] S.E. Han, On the simplicial complex stemmed from a digital graph, *Honam Mathematical Journal* **27** (1) (2005) 115-129.
[7] S.E. Han, Connected sum of digital closed surfaces, *Information Sciences* **176**(3)(2006)332-348.
[8] S.E. Han, Discrete Homotopy of a Closed k-Surface, *LNCS 4040*, Springer-Verlag, Berlin, pp.214-225 (2006).
[9] S.E. Han, Erratum to "Non-product property of the digital fundamental group", *Information Sciences* **176**(1) (2006) 215-216.
[10] S.E. Han, Strong k-deformation retract and its applications, *Journal of the Korean Mathematical Society* **44**(6) (2007) 1479-1503.
[11] S.E. Han, The fundamental group of a closed k-surface, *Information Sciences* **177**(18) (2007) 3731-3748.
[12] S.E. Han, Comparison among digital k-fundamental groups and its applications, *Information Sciences* **178** (2008) 2091-2104.
[13] S.E. Han, Equivalent (k_0, k_1)-covering and generalized digital lifting, *Information Sciences* **178**(2) (2008)550-561.
[14] S.E. Han, The k-homotopic thinning and a torus-like digital image in $\mathbf{Z}^n$, *Journal of Mathematical Imaging and Vision* **31**(1) (2008) 1-16.
[15] S.E. Han, Regular covering space in digital covering theory and its applications, *Honam Mathematical Journal* **31**(3) (2009) 279-292.
[16] S.E. Han, KD-(k_0, k_1)-homotopy equivalence and its applications, *Journal of Korean Mathematical Society* **47**(5)(2010) 1031-1054.
[17] S.E. Han, Ultra regular covering space and its automorphism group, *International Jour. of Applied Mathematics and Computer Science* **20**(4) (2010) 699-710.
[18] S.E. Han, Unique pseudolifting property in digital topology, *Filomat* **26**(4) (2012) 739-746.
[19] S.E. Han and B.G. Park, Digital graph (k_0, k_1)-isomorphism and its applications, *http://atlas-conferences.com/c/a/k/b/36.htm*(2003).
[20] G. T. Herman, Oriented surfaces in digital spaces, *CVGIP: Graphical Models and Image Processing* **55** (1993) 381-396.
[21] Jeongmin Kang and S.E. Han, Compression of Khalimsky topological spaces, *Filomat* **26**(6) (2012) 1101-1114.
[22] E. Khalimsky, Motion, deformation, and homotopy in finite spaces, *Proceedings IEEE International Conferences on Systems, Man, and Cybernetics* (1987) 227-234.
[23] In-Soo Kim and S.E. Han, Digital covering therory and its applications, *Honam Mathematical Journal* **30**(4) (2008) 589-602.
[24] T.Y. Kong, A digital fundamental group, *Computers and Graphics* **13** (1989) 159-166.
[25] T.Y. Kong, A. Rosenfeld, Topological Algorithms for the Digital Image Processing, Elsevier Science, Amsterdam, (1996).
[26] R. Malgouyres and A. Lenoir, Topology preservation within digital surfaces, *Graphical Models* **62** (2000) 71-84.
[27] W.S. Massey, Algebraic Topology, Springer-Verlag, New York, 1977.
[28] A. Rosenfeld, Digital topology, *Am. Math. Mon.* **86** (1979) 76-87.
[29] E.H. Spanier, Algebraic Topology, McGraw-Hill Inc., New York, 1966.