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Matrix Transforms of A-statistically Convergent Sequences with Speed

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Abstract. Let X, Y be sequence spaces and (X, Y) the set of all matrices mapping X into Y. Let A be a non-negative regular matrix and λ a speed, i.e. a positive monotonically increasing sequence. In this paper the notion of A-statistical convergence with speed λ is introduced and the class of matrices $(st_A^\lambda \cap X, Y)$, where st_A^λ is the set of all A-statistically convergent sequences with speed λ, is described.

1. Introduction

The notion of statistical convergence of a sequence was defined by Fast [6] and Steinhaus [20] independently in 1951. Further this subject have been studied in several works (see, for example, [1], [8], [13], [17]-[19]). In 1988 Kolk [14] (see also [15]-[16]) introduced the concept of A-statistical convergence for a non-negative regular matrix A and studied matrix transforms of A-statistically convergent sequences.

The speed (or the rate) of convergence of statistically and A-statistically convergent sequences were introduced in various ways [3]-[5], [9]. In the present paper we choose another way to define the speed of convergence of A-statistically convergent sequences, applying the notion of convergence with speed λ (λ is a monotonically increasing positive sequence), introduced by Kangro in 1969 [11]. Also we investigate the matrix transforms of A-statistically convergent sequences with speed. We show that in the special case, if λ is bounded, from our results follow some results of Kolk [15]-[16]. The author wish to thank professor Kolk for valuable advices for preparing this paper.

2. Notation and preliminaries

Let, as usual, s, m, c, c_0 be respectively the spaces of all sequences $x := (x_k)$ (with real or complex entries), of all bounded sequences, of all convergent sequences, of all sequences converging to 0. Throughout this paper we assume that indices and summation indices run from 0 to ∞ unless otherwise specified.

In the following by an index set we mean an infinite subset $\{k_i\}$ of $\mathbf{N}$ with $k_i < k_{i+1}$. For an arbitrary index set $K = \{k_i\}$ the sequence $x^{[K]} = (y_k)$, where

$$y_k = \begin{cases} x_k & \text{if } k \in K, \\ 0 & \text{otherwise,} \end{cases}$$

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is called the K-section of x. A sequence space X is called section-c1osed if $x^{[K]} \in X$ for every $x \in X$ and for every index set K. A sequence space X is called subsequence-c1osed if every subsequence of X belongs to X. For example, the spaces s, m, c_0 are section-closed and the space c is subsequence-c1osed, but not section-closed.

Let X, Y be two sequence spaces and $A = (a_{nk})$ be a matrix with real or complex entries. If for each $x = (x_k) \in X$ the series

$$A_n x = \sum_k a_{nk} x_k$$

converge and the sequence $Ax = (A_n x)$ belongs to Y, we say that the matrix A transforms X into Y. By (X, Y) we denote the set of all matrices which transform X into Y. A matrix A is said to be regular if $A \in (c, c)$ and

$$\lim_n A_n x = \lim_k x_k$$

for each $x \in c$. It is well-known (see [2]) that $A = (a_{nk})$ is regular if and only if

$$\lim_n a_{nk} = 0, \tag{2.1}$$

$$\lim_n \sum_k a_{nk} = 1, \tag{2.2}$$

$$\sum_k |a_{nk}| = O(1). \tag{2.3}$$

A matrix A is called uniformly regular if it satisfies the conditions (2.2)-(2.3) and

$$\lim_n \sup_k |a_{nk}| = 0. \tag{2.4}$$

For example, the Cesàro matrix $C^1 = (a_{nk})$, where $a_{nk} = 1/n$ for $k \leq n$ and $a_{nk} = 0$ otherwise, is uniformly regular.

Let K be a fixed index set. By $A^{[K]}$ we denote the K-column-section of the matrix A, i.e.$A^{[K]} = (d_{nk})$, where

$$d_{nk} = \begin{cases} a_{nk} & \text{if } k \in K, \\ 0 & \text{otherwise.} \end{cases}$$

By $\varphi^{[K]}$ we denote the characteristic sequence of K, i.e. $\varphi^{[K]} = \left(\varphi_j^{[K]}\right)$, where

$$\varphi_j^{[K]} = \begin{cases} 1 & \text{if } j \in K, \\ 0 & \text{otherwise.} \end{cases}$$

If C^1 transforms $\varphi^{[K]}$ into c, then the limit

$$\lim_n \frac{1}{n} \sum_{j=0}^{n} \varphi_j^{[K]}$$

is said to be the asymptotic density of K and is denoted by $\delta(K)$. Let further throughout the paper A is a regular non-negative matrix. Following Freedman and Sember [7], for A an index set K is said to have A-density

$$\delta_A(K) := \lim_n A_n \varphi^{[K]}$$

if $A\varphi^{[K]} \in c$. So

$$\delta_A(K) = \lim_n \sum_{k \in K} a_{nk}.$$

By Fast [6] (see also [20]), a sequence $x = (x_k)$ is called statistically convergent to a number ξ if $\delta(K_\epsilon) = 0$ for every $\epsilon > 0$, where

$$K_\epsilon := \{k : |x_k - \xi| \geq \epsilon\}.$$

A sequence x is said to be A-statistically convergent to ξ if $\delta_A(K_\epsilon) = 0$ for every $\epsilon > 0$ (see [14], [16]). In this case we write

$$st_A - \lim x_k = \xi.$$

We denote the set of all A-statistically convergent sequences by st_A, and the set of all sequences, converging A-statistically to 0, by st_A^0.

As for every $x = (x_k) \in c$ the set K_ϵ is finite for each $\epsilon > 0$, then $\delta_A(K_\epsilon) = 0$ for every $\epsilon > 0$ by condition (2.1). Thus every convergent sequence is A-statistically convergent, i.e. $c \in st_A$. It was shown in [16] that for a uniformly regular non-negative matrix A the A-statistical convergence is strictly stronger than ordinary convergence.

Observe that for $A = I$ (where I is the identity matrix) the concept of A-statistical convergence coincides with ordinary convergence, and for $A = C^1$ with the statistical convergence.

Let throughout this paper $\lambda = (\lambda_k)$ be a positive monotonically increasing sequence. Following Kangro [11], [12] a convergent sequence $x = (x_k)$ with

$$\lim_k x_k := \xi \text{ and } l_k = \lambda_k (x_k - \xi) \tag{2.5}$$

is called bounded with the speed λ (shortly, λ-bounded) if $l_k = O(1)$, and convergent with the speed λ (shortly, λ-convergent) if

$$\exists \lim_k l_k := b. \tag{2.6}$$

We denote the set of all λ-bounded sequences by m^λ, and the set of all λ-convergent sequences by c^λ. Let

$$c_0^\lambda = \{x = (x_k) : x \in c^\lambda \text{ and } b = 0\},$$

$$m_0^\lambda = \{x = (x_k) : x \in m^\lambda \text{ and } \xi = 0\},$$

$$z^\lambda = \{x = (x_k) : x \in c^\lambda \text{ and } \xi = 0\},$$

$$n^\lambda = \{x = (x_k) : x \in c^\lambda \text{ and } \xi = b = 0\}.$$

It is not difficult to see that $n^\lambda \subset z^\lambda \subset c^\lambda \subset m^\lambda \subset c$ and $n^\lambda \subset c_0^\lambda \subset c^\lambda$. In addition to it, for unbounded sequence λ these inclusions are strict. For $\lambda_k = O(1)$ we get $c_0^\lambda = c^\lambda = m^\lambda = c$ and $z^\lambda = n^\lambda = c_0$.

For a non-negative regular matrix A we say that a sequence $x = (x_k)$ is A-statistically convergent with speed λ (shortly, A^λ-statistically convergent) if $x \in st_A$ and $l := (l_k) \in st_A$. For the particular case $A = C^1$ we say that a sequence x is statistically convergent to ξ with speed λ (shortly, λ-statistically convergent to ξ). The set of all A^λ-statistically convergent sequences we denote by st_A^λ.

Obviously $st_A^\lambda \subset st_A$. Also we can assert that $c^\lambda \subset st_A^\lambda$. Indeed, for every $x \in c^\lambda$ we have $x \in c \subset st_A$ and $l \in c \subset st_A$. Consequently $x \in st_A^\lambda$.

It is easy to see that for $A = I$ the concept of A^λ-statistical convergence coincides with the concept of λ-convergence, and for $\lambda_k = O(1)$ with the concept of ordinary A-statistical convergence.

For $x = (x_k) \in st_A^\lambda$ we denote

$$st_A - \lim x_k = \eta \text{ and } st_A - \lim t_k = \beta; \ t_k = \lambda_k (x_k - \eta). \tag{2.7}$$

Further we also need the following subsets of st_A^λ:

$$st_{A,0}^\lambda = \{x = (x_k) : x \in st_A^\lambda \text{ and } \beta = 0\},$$

$$z\text{-}st_A^\lambda = \{x = (x_k) : x \in st_A^\lambda \text{ and } \eta = 0\},$$

$$n\text{-}st_A^\lambda = \{x = (x_k) : x \in st_A^\lambda \text{ and } \eta = 0, \ \beta = 0\}.$$

We note that $n\text{-}st_A^\lambda \subset z\text{-}st_A^\lambda \subset st_A^\lambda$ and $n\text{-}st_A^\lambda \subset st_{A,0}^\lambda$.

3. Matrix transforms of A-statistically convergent sequences with speed

At first we introduce some notations and formulate some results, which we need further. Let $e = (1, 1, ...)$, $e^k = (0, ..., 0, 1, 0, ...)$, where 1 is in the k-th position, and $\lambda^{-1} = (1/\lambda_k)$.

Lemma 3.1 ([14], [15], see also [8]). *Let A be a non-negative regular matrix. A sequence $x = (x_k)$ converges A-statistically to ξ if and only if there exists an infinite set of indices $K = \{k_i\}$ so that the subsequence (x_{k_i}) converges to ξ and $\delta_A(\mathbf{N}\backslash K) = 0$ (and hence $\delta_A(K) = 1$).*

Lemma 3.2 (see [16]). *Let $Y \neq s$ be a subsequence-closed sequence space. If a matrix A is uniformly regular, the following statements about a matrix $B = (b_{nk})$ (with real or complex entries) are equivalent:*
(I) $B \in (s, Y)$;
(II) $B^{[K]} \in (s, Y)$ *for every index set K with $\delta_A(K) = 0$;*
(III) $Be^k \in Y$ *and there exists a number k_0 such that*

$$b_{nk} = 0 \text{ for every } k > k_0 \text{ and } n \in \mathbf{N}.$$

Now we are able to prove the main results of the paper.

Theorem 3.3. *Let λ be a monotonically increasing positive sequence and A a non-negative regular matrix. A sequence $x = (x_k)$ converges A^λ-statistically to η if and only if there exists an infinite set of indices $K = \{k_i\}$ so that the subsequence (x_{k_i}) is λ-convergent to η and $\delta_A(\mathbf{N}\backslash K) = 0$ (and hence $\delta_A(K) = 1$).*
Proof. As for a bounded sequence λ we have $st_A^\lambda = st_A$, then in this case Theorem 3.3 coincides with Lemma 3.1. Therefore suppose that λ is unbounded. By the definition, $x = (x_k) \in st_A^\lambda$ if and only if relation (2.7) holds. With the help of Lemma 3.1 we get that $st_A - \lim t_k = \beta$ if and only if there exists an infinite set of indices $K = \{k_i\}$ so that the ordinary limit

$$\lim_i t_{k_i} = \beta \tag{3.1}$$

exists and $\delta_A(K) = 1$. As the sequence λ is monotonically increasing, then the subsequence (λ_{k_i}) of λ is unbounded. Consequently it follows from (3.1) that there exists the ordinary limit $\lim_i x_{k_i} = \eta$. This relation together with $\delta_A(K) = 1$ implies $st_A - \lim x_k = \eta$. Thus the proof is complete.

Theorem 3.4. *Let $\lambda = (\lambda_k)$ be a monotonically increasing positive sequence, X a section-closed sequence space containing e and λ^{-1}, Y an arbitrary sequence space and A a regular non-negative matrix. Then a matrix $B \in \left(st_A^\lambda \bigcap X, Y\right)$ if and only if $B \in \left(c^\lambda \bigcap X, Y\right)$ and*

$$B^{[K]} \in (X, Y) \ \ (\delta_A(K) = 0). \tag{3.2}$$

Proof. Necessity. Let $B \in \left(st_A^\lambda \bigcap X, Y\right)$. As $c^\lambda \subset st_A^\lambda$, we immediately have $B \in \left(c^\lambda \bigcap X, Y\right)$. Let K be a set of indices with $\delta_A(K) = 0$, and $x = (x_k) \in X$. Then the sequence $y^\lambda := \left(y_k^\lambda\right)$, where

$$y_k^\lambda = \begin{cases} \lambda_k x_k & \text{if } k \in K, \\ 0 & \text{otherwise,} \end{cases}$$

converges A-statistically to 0. Consequently the sequence $y = x^{[K]}$ converges A^λ-statistically to 0. Moreover, $y \in X$, since X is section-closed. Hence $y \in st_A^\lambda \bigcap X$, and therefore $By \in Y$. As

$$B^{[K]}x = \left(B_n^{[K]}x\right) = (B_n y) = By,$$

then $B^{[K]}x \in Y$. Thus condition (3.2) is valid.
Sufficiency. We show that $B \in \left(st_A^\lambda \bigcap X, Y\right)$ if $B \in \left(c^\lambda \bigcap X, Y\right)$ and condition (3.2) is fulfilled. Let $x = (x_k) \in st_A^\lambda \bigcap X$, i.e. relation (2.7) holds. We prove that $Bx \in Y$. It follows from (2.7) that

$$x_k = \eta + \frac{\beta}{\lambda_k} + \frac{t_k - \beta}{\lambda_k},$$

i.e. every $x \in st_A^\lambda$ we can present in the form

$$x = \eta e + \beta \lambda^{-1} + x^0, \text{ where } x^0 := \left(\frac{t_k - \beta}{\lambda_k} \right) \in n\text{-}st_A^\lambda.$$

As $e \in c^\lambda \bigcap X$ and $\lambda^{-1} \in c^\lambda \bigcap X$, then $Be \in Y$ and $B\lambda^{-1} \in Y$. Therefore it is sufficient to show that $Bx \in Y$ for every $x \in n\text{-}st_A^\lambda$. Thus suppose $x \in n\text{-}st_A^\lambda$, i.e. relation (2.7) holds with $\eta = \beta = 0$. Then by Theorem 3.3 there exist a set of indices K with $\delta_A(K) = 0$ (then $\delta_A(\mathbf{N}\backslash K) = 1$) so that $\lim_k u_k = 0$, where $u := (u_k)$ is the $\mathbf{N}\backslash K$-section of $t := (t_k)$, i.e.

$$u_k = \begin{cases} \lambda_k x_k & \text{if } k \in \mathbf{N}\backslash K, \\ 0 & \text{otherwise.} \end{cases}$$

Hence for $y = (y_k) := x^{[\mathbf{N}\backslash K]}$ we get $\lim_k y_k = 0$. Thus $y \in n^\lambda \subset c^\lambda$. Also $y \in X$, since X is section-closed. This implies $By \in Y$. Moreover, $B^{[K]}x \in Y$ by (3.2), and

$$Bx = By + B^{[K]}x.$$

Therefore $Bx \in Y$. The proof is complete.

Similarly to Theorem 3.4 we can prove the analogous results for the sets c_0^λ, z^λ and n^λ.

Theorem 3.5. *Let λ be a monotonically increasing positive sequence, X a section-closed sequence space containing e, Y an arbitrary sequence space and A a regular non-negative matrix. Then a matrix $B \in \left(st_{A,0}^\lambda \bigcap X, Y\right)$ if and only if $B \in \left(c_0^\lambda \bigcap X, Y\right)$ and relation* (3.2) holds.

Theorem 3.6. *Let λ be a monotonically increasing positive sequence, X a section-closed sequence space containing λ^{-1}, Y an arbitrary sequence space and A a regular non-negative matrix. Then a matrix $B \in \left(z\text{-}st_A^\lambda \bigcap X, Y\right)$ if and only if $B \in \left(z^\lambda \bigcap X, Y\right)$ and relation* (3.2) holds.

Theorem 3.7. *Let λ be a monotonically increasing positive sequence, X a section-closed sequence space, Y an arbitrary sequence space and A a regular non-negative matrix. Then a matrix $B \in \left(n\text{-}st_A^\lambda \bigcap X, Y\right)$ if and only if $B \in \left(n^\lambda \bigcap X, Y\right)$ and relation* (3.2) holds.

For a bounded sequence λ from Theorem 3.4 it follows Theorem 4.1 of Kolk [16] and from Theorem 3.6 (or Theorem 3.7) we get Theorem 4.2 of Kolk [16]. For a uniformly regular matrix A we get the following result.

Theorem 3.8. *Let λ be a monotonically increasing positive sequence, $Y \neq s$ a subsequence-closed sequence space and A a uniformly regular non-negative matrix. Then $\left(st_A^\lambda, Y\right) = \left(st_{A,0}^\lambda, Y\right) = \left(z\text{-}st_A^\lambda, Y\right) = \left(n\text{-}st_A^\lambda, Y\right) = (s, Y)$.*

Proof. As

$$(s, Y) \subset \left(st_A^\lambda, Y\right) \subset \left(st_{A,0}^\lambda, Y\right) \subset \left(n\text{-}st_A^\lambda, Y\right)$$

and

$$(s, Y) \subset \left(st_{A,0}^\lambda, Y\right) \subset \left(n\text{-}st_A^\lambda, Y\right),$$

then it is sufficient to prove that $\left(n\text{-}st_A^\lambda, Y\right) \subset (s, Y)$. If $B \in \left(n\text{-}st_A^\lambda, Y\right)$, then by Theorem 3.7 (for the case $X = s$) $B^{[K]} \in (s, Y)$ for every set of indices K with $\delta_A(K) = 0$. Hence $B \in (s, Y)$ by Lemma 3.2. The proof is complete.

For a bounded sequence λ from Theorem 3.8 we get Theorem 4.3 of [16]. Using the equivalence of conditions (I) and (III) of Lemma 3.2, from Theorem 3.8 we immediately get the extension of Corollary 4.4 of [16].

Corollary 3.9. *By the assumptions of Theorem 3.8 the equivalent inclusions $B \in \left(st_A^\lambda, Y\right)$, $B \in \left(st_{A,0}^\lambda, Y\right)$, $B \in \left(z\text{-}st_A^\lambda, Y\right)$ and $B \in \left(n\text{-}st_A^\lambda, Y\right)$* hold if and only if *$B$ has at most finitely many non-zero columns belonging to Y.*

As s, m, c_0 are section- and subsequence-closed, and c is only subsequence-closed, then Theorems 3.4-3.7 it is possible to apply for the cases $X = s$, $X = m$ and $X = c_0$, and Theorem 3.8 for the cases $Y = m$, $Y = c_0$ and $Y = c_0$. We prove that similarly we can apply Theorems 3.4-3.7 for $X = m_0^\lambda$ and $X = n^\lambda$, and Theorem 3.8 for $Y = c_0^\lambda$, $Y = n^\lambda$ and $Y = m^\lambda$. Also we show that Theorem 3.8 is applicable for $Y = c^\lambda$ and $Y = z^\lambda$ only in the special case.

Proposition 3.9. *Let $\lambda = (\lambda_k)$ be a monotonically increasing positive unbounded sequence. Then the following statements hold:*

a) *m_0^λ and n^λ are section-closed, but m^λ, c^λ, c_0^λ and z^λ are not;*

b) *m_0^λ, m^λ, n^λ and c_0^λ are subsequence-closed;*

c) *c^λ and z^λ are subsection-closed if and only if there exists the finite limit*

$$\lim_k \frac{\lambda_k}{\lambda_{i_k}} \tag{3.3}$$

for every subsequence (λ_{i_k}) of λ.

Proof.

a) The assertion immediately follows from the definitions of these sets.

b) Let $x = (x_k) \in m^\lambda$, i.e. relation (2.5) with $l_k = O(1)$ is satisfied. Then for every subsequence (x_{i_k}) of x we get $\lim_k x_{i_k} = \xi$, since $m^\lambda \subset c$. For every subsequence (x_{i_k}) of x we define the sequence (u_k) with

$$u_k := \lambda_k (x_{i_k} - \xi).$$

As λ is monotonically increasing, then

$$|u_k| \leq \lambda_{i_k} \left| x_{i_k} - \xi \right| \leq \lambda_k |(x_k - \xi|.$$

This implies $u_k = O(1)$ for each $x \in m^\lambda$ and each subsequence (x_{i_k}) of x, and $\lim_k u_k = 0$ for each $x \in c_0^\lambda$ and each subsequence (x_{i_k}) of x. Moreover, $m_0^\lambda \subset m^\lambda$ and $n^\lambda \subset c_0^\lambda$. Hence m_0^λ, m^λ, n^λ and c_0^λ are subsequence-closed.

c) Let $x = (x_k) \in c^\lambda$, i.e. relations (2.5) and (2.6) are valid. As in this case $l := (l_k) \in c$, then l can be represented in the form

$$l = l^0 + be, \ l^0 := \left(l_k^0\right) \in c_0.$$

Thus

$$l_k^0 + b = \lambda_k (x_k - \xi),$$

and consequently

$$x_k = \frac{l_k^0}{\lambda_k} + \frac{b}{\lambda_k} + \xi. \tag{3.4}$$

Obviously for every subsequence (x_{i_k}) of x we get $\lim_k x_{i_k} = \xi$. Using (3.4) we can write

$$v_k := \lambda_k (x_{i_k} - \xi) = \frac{\lambda_k}{\lambda_{i_k}} \left(l_{i_k}^0 + b\right).$$

If $b = 0$, then $(v_k) \in c$, because $\lambda_k / \lambda_{i_k} = O(1)$ by the monotonicity of λ. If $b \neq 0$, then $(v_k) \in c$ for every subsequence (x_{i_k}) of x if and only if there exists finite limit (3.3) for every subsequence (λ_{i_k}) of λ. Thus in this case c^λ is subsequence-closed. The proof for z^λ is similar.

Remark 3.1. *We note that the finite limit (3.3) exists for every subsequence (λ_{i_k}) of λ if, for example, λ is defined by the equality $\lambda_k := (k + 1)^k$.*

4. Matrix transforms of A^λ-statistically convergent λ-bounded null sequences

Let throughout this section $\lambda = (\lambda_k)$ and $\mu = (\mu_k)$ are monotonically increasing positive unbounded sequences. We consider some applications of the results of previous section. Taking $X = m_0^\lambda$ and $Y = z^\mu$ in theorem 3.6, we get that $B \in \left(z\text{-}st_A^\lambda \bigcap m_0^\lambda, z^\mu\right)$ if and only if $B \in \left(z^\lambda, z^\mu\right)$ and $B^{[K]} \in \left(m_0^\lambda, z^\mu\right)$ for every index set K with $\delta_A(K) = 0$ (since $z^\lambda \subset m_0^\lambda$). Similarly, if we take $X = m_0^\lambda$ and $Y = n^\mu$ in theorem 3.7, we get that $B \in \left(n\text{-}st_A^\lambda \bigcap m_0^\lambda, n^\mu\right)$ if and only if $B \in \left(n^\lambda, n^\mu\right)$ and $B^{[K]} \in \left(m_0^\lambda, n^\mu\right)$ for every index set K with $\delta_A(K) = 0$ (since $n^\lambda \subset m_0^\lambda$). The characterizations of matrix classes $\left(c^\lambda, c^\mu\right)$ and $\left(m^\lambda, m^\mu\right)$ are given by Kangro (Theorem 1 of [11] and Theorem 1 of [12]). Similarly, using the known characterizations of matrix classes (c, c), (c_0, c_0), (m, c) and (m, c_0) (see [2], [10]), it can be easily show that the following results hold.

Lemma 4.1. *A matrix $B = (b_{nk}) \in \left(z^\lambda, z^\mu\right)$ if and only if $Be^k \in z^\mu$, $B\lambda^{-1} \in z^\mu$ and*

(IV) $\mu_n \sum\limits_k \frac{|b_{nk}|}{\lambda_k} = O(1)$.

Lemma 4.2. *A matrix $B = (b_{nk}) \in \left(m_0^\lambda, z^\mu\right)$ if and only if condition* (IV) *holds and*

(V) $\exists \lim\limits_n \mu_n b_{nk} := L_k$,

(VI) $\lim\limits_n \sum\limits_k \frac{|\mu_n b_{nk} - L_k|}{\lambda_k} = 0.$

Lemma 4.3. *A matrix $B = (b_{nk}) \in \left(n^\lambda, n^\mu\right)$ if and only if $Be^k \in n^\mu$ and condition* (IV) *is satisfied.*

Lemma 4.4. *A matrix $B = (b_{nk}) \in \left(m_0^\lambda, n^\mu\right)$ if and only if*

(VII) $\lim\limits_n \mu_n \sum\limits_k \frac{|b_{nk}|}{\lambda_k} = 0.$

Using Lemmas 4.1 and 4.2, from Theorem 3.6 we immediately get

Corollary 4.5. *A matrix $B = (b_{nk}) \in \left(z\text{-}st_A^\lambda \bigcap m_0^\lambda, z^\mu\right)$ if and only if $Be^k \in z^\mu$, $B\lambda^{-1} \in z^\mu$, condition* (IV) *holds and there exists an index set K with $\delta_A(K) = 0$ so that*

(VIII) $\lim\limits_n \sum\limits_{k \in K} \frac{|\mu_n b_{nk} - L_k|}{\lambda_k} = 0.$

Using Lemmas 4.3 and 4.4, from Theorem 3.7 we immediately get

Corollary 4.6. *A matrix $B = (b_{nk}) \in \left(n\text{-}st_A^\lambda \bigcap m_0^\lambda, n^\mu\right)$ if and only if $Be^k \in n^\mu$, condition* (IV) *holds and there exists an index set K with $\delta_A(K) = 0$ so that*

(IX) $\lim\limits_n \mu_n \sum\limits_{k \in K} \frac{|b_{nk}|}{\lambda_k} = 0.$

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