

Some Hermite-Hadamard type inequalities for functions whose n -th derivatives are (α, m) -convex

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Abstract. In the paper, the authors establish some new integral inequalities of Hermite-Hadamard type for functions whose n -th derivatives are of (α, m) -convexity and, from these, deduce some known results.

1. Introduction

The following definition is well known in the literature that a function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex on $I \neq \emptyset$ if the inequality

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \quad (1.1)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$; If the inequality (1.1) reverses, then f is said to be concave on I .

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on an interval I and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}. \quad (1.2)$$

This double inequality is well known in the literature as Hermite-Hadamard integral inequality for convex functions. If f is concave, both inequalities in (1.2) hold in the reversed direction. For more information, see [4, 8] and closely related references therein.

The concept of usually used convexity has been generalized by a number of mathematicians. Some of them can be recited as follows.

Definition 1.1 ([14]). Let $f : [0, b] \rightarrow \mathbb{R}$ be a function and $m \in (0, 1]$. If the inequality

$$f(\lambda x + m(1 - \lambda)y) \leq \lambda f(x) + m(1 - \lambda)f(y)$$

holds for all $x, y \in [0, b]$ and $\lambda \in [0, 1]$, then we say that the function $f(x)$ is m -convex on $[0, b]$.

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Definition 1.2 ([7]). Let $f : [0, b] \rightarrow \mathbb{R}$ be a function and $(\alpha, m) \in (0, 1] \times (0, 1]$. If the inequality

$$f(\lambda x + m(1 - \lambda)y) \leq \lambda^\alpha f(x) + m(1 - \lambda^\alpha)f(y)$$

is valid for all $x, y \in [0, b]$ and $\lambda \in [0, 1]$, then we say that $f(x)$ is an (α, m) -convex function on $[0, b]$.

For $(\alpha, m) \in \{(0, 0), (\alpha, 0), (1, 0), (1, m), (1, 1), (\alpha, 1)\}$, we can obtain the following classes of functions: increasing, α -star-shaped, star-shaped, m -convex, convex, and α -convex functions.

In past recent years, a few of inequalities of Hermite-hadamard type for (α, m) -convex functions were presented, some of them can be recited as the following theorems.

Theorem 1.1 ([3, Theorem 2]). Let $f : I^\circ \subset [0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable function such that $f'' \in L([a, b])$ for $a, b \in I$ with $a < b$. If $|f''(x)|^q$ is m -convex on $[a, b]$ for some fixed $q \geq 1$ and $m \in (0, 1]$, then

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx \right| \leq \frac{(b - a)^2}{12} \left[\frac{|f''(a)|^q + m|f''(b/m)|^q}{2} \right]^{1/q}. \tag{1.3}$$

Theorem 1.2 ([12, Theorem 4]). Let $I \subseteq \mathbb{R}$ be an open interval and $a, b \in I$ with $a < b$, and let $f : I \rightarrow \mathbb{R}$ be a twice differentiable mapping such that $f''(x)$ is integrable. If $0 \leq \lambda \leq 1$ and $|f''(x)|$ is convex on $[a, b]$, then

$$\begin{aligned} & \left| (\lambda - 1)f\left(\frac{a+b}{2}\right) - \lambda \frac{f(a) + f(b)}{2} + \int_a^b f(x) dx \right| \\ & \leq \begin{cases} \frac{(b - a)^2}{24} \left\{ \left[\lambda^4 + (1 + \lambda)(1 - \lambda)^3 + \frac{5\lambda - 3}{4} \right] |f''(a)| + \left[\lambda^4 + (2 - \lambda)\lambda^3 + \frac{1 - 3\lambda}{4} \right] |f''(b)| \right\}, & 0 \leq \lambda \leq \frac{1}{2}; \\ \frac{(b - a)^2}{48} (3\lambda - 1) (|f''(a)| + |f''(b)|), & \frac{1}{2} \leq \lambda \leq 1. \end{cases} \end{aligned}$$

Remark 1.1. In Theorem 1.2, when $\lambda = 1$, we have

$$\left| \frac{1}{b - a} \int_a^b f(x) dx - \frac{f(a) + f(b)}{2} \right| \leq \frac{(b - a)^2}{24} [|f''(a)| + |f''(b)|]. \tag{1.4}$$

Theorem 1.3 ([10, Theorem 3]). Let $b^* > 0$ and $f : [0, b^*] \rightarrow \mathbb{R}$ be a twice differentiable function such that $f'' \in L([a, b])$ for $a, b \in [0, b^*]$ with $a < b$. If $|f''(x)|^q$ is (α, m) -convex on $[a, b]$ for $(\alpha, m) \in (0, 1] \times (0, 1]$ and $q \geq 1$, then

$$\begin{aligned} & \left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb - a} \int_a^{mb} f(x) dx \right| \\ & \leq \frac{(mb - a)^2}{2} \left(\frac{1}{6}\right)^{1-1/q} \left\{ \frac{1}{(\alpha + 2)(\alpha + 3)} |f''(a)|^q + m \left[\frac{1}{6} - \frac{1}{(\alpha + 2)(\alpha + 3)} \right] |f''(b)|^q \right\}^{1/q}. \tag{1.5} \end{aligned}$$

For more information on this topic, please refer to [1, 2, 5, 9, 11, 13, 17–23] and plenty of references cited therein.

In this paper, we will establish some new inequalities of Hermite-Hadamard type for functions whose n -th derivatives are of (α, m) -convexity and deduce some known results in the form of corollaries.

2. A Lemma

For establishing new integral inequalities of Hermite-Hadamard type for (α, m) -convex functions, we need the following lemma.

Lemma 2.1 ([16, Lemma 2.1]). For $n \in \mathbb{N}$, let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be n -time differentiable. If $a, b \in I$ with $a < b$ and $f^{(n)}(x) \in L([a, b])$, then

$$\begin{aligned} \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx - \frac{1}{2} \sum_{k=1}^{n-1} \frac{k-1}{(k+1)!} (a-b)^k f^{(k)}(b) \\ = \frac{(-1)^n (b-a)^n}{2 n!} \int_0^1 (2t+n-2)(1-t)^{n-1} f^{(n)}(ta + (1-t)b) \, dt, \end{aligned} \quad (2.1)$$

where the sum takes 0 when $n = 1$.

Remark 2.1. Similar integral identities to (2.1), produced by replacing $f^{(k)}(b)$ in (2.1) by $f^{(k)}(a)$ or by $f^{(k)}(\frac{a+b}{2})$, and corresponding integral inequalities of Hermite-Hadamard type have been established in [6, 15].

3. Inequalities of Hermite-Hadamard type

Now we are in a position to establish some new integral inequalities of Hermite-Hadamard type for functions which are n -time differentiable and (α, m) -convex.

Theorem 3.1. Let $(\alpha, m) \in (0, 1] \times (0, 1]$ and $f : \mathbb{R}_0 = [0, \infty) \rightarrow \mathbb{R}$ be a n -time differentiable function such that $f^{(n)}(x) \in L[\frac{a}{m}, \frac{b}{m}]$, where $0 < a < b < \infty$ and $n \geq 2$. If $|f^{(n)}(x)|^p$ is (α, m) -convex on $[\frac{a}{m}, \frac{b}{m}]$ for $p \geq 1$, then

$$\begin{aligned} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx - \frac{1}{2} \sum_{k=1}^{n-1} \frac{k-1}{(k+1)!} (a-b)^k f^{(k)}(b) \right| \leq \frac{1}{2} \frac{(b-a)^n}{n!} \left(\frac{n-1}{n+1} \right)^{1-1/p} \\ \times \left\{ (n+\alpha-1)B(\alpha+1, n+1) |f^{(n)}(a)|^p + m \left[\frac{n-1}{n+1} - (n+\alpha-1)B(\alpha+1, n+1) \right] \left| f^{(n)}\left(\frac{b}{m}\right) \right|^p \right\}^{1/p}, \end{aligned} \quad (3.1)$$

where B is the classical Beta function which may be defined for $\Re(x) > 0$ and $\Re(y) > 0$ by

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (3.2)$$

and $\Gamma(z)$ the classical Euler gamma function which may be defined for $\Re(z) > 0$ by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt. \quad (3.3)$$

Proof. It follows from Lemma 2.1 that

$$\begin{aligned} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx - \frac{1}{2} \sum_{k=1}^{n-1} \frac{k-1}{(k+1)!} (a-b)^k f^{(k)}(b) \right| \\ \leq \frac{1}{2} \frac{(b-a)^n}{n!} \int_0^1 (2t+n-2)(1-t)^{n-1} \left| f^{(n)}\left(ta + \frac{m(1-t)b}{m} \right) \right| \, dt. \end{aligned}$$

When $p = 1$, since $|f^{(n)}(x)|$ is (α, m) -convex on $[\frac{a}{m}, \frac{b}{m}]$, we have

$$\left| f^{(n)}\left(ta + \frac{m(1-t)b}{m} \right) \right| \leq t^\alpha |f^{(n)}(a)| + m(1-t)^\alpha \left| f^{(n)}\left(\frac{b}{m}\right) \right|.$$

Multiplying by the factor $(2t + n - 2)(1 - t)^{n-1}$ on both sides of the above inequality and integrating with respect to $t \in [0, 1]$ lead to

$$\begin{aligned} & \int_0^1 (2t + n - 2)(1 - t)^{n-1} \left| f^{(n)} \left(ta + \frac{m(1-t)b}{m} \right) \right| dt \\ & \leq \int_0^1 (2t + n - 2)(1 - t)^{n-1} \left[t^\alpha |f^{(n)}(a)| + m(1 - t^\alpha) \left| f^{(n)} \left(\frac{b}{m} \right) \right| \right] dt \\ & = |f^{(n)}(a)| \int_0^1 (1 - t)^{n-1} t^\alpha (2t + n - 2) dt + m \left| f^{(n)} \left(\frac{b}{m} \right) \right| \int_0^1 (2t + n - 2)(1 - t)^{n-1} (1 - t^\alpha) dt \\ & = [(n - 2)B(\alpha + 1, n) + 2B(\alpha + 2, n)] |f^{(n)}(a)| + m[(n - 2)B(1, n) + 2B(2, n) \\ & \quad - (n - 2)B(\alpha + 1, n) - 2B(\alpha + 2, n)] \left| f^{(n)} \left(\frac{b}{m} \right) \right|. \end{aligned}$$

The proof for the case $p = 1$ is complete.

When $p > 1$, by the well-known Hölder integral inequality, we have

$$\begin{aligned} & \int_0^1 (2t + n - 2)(1 - t)^{n-1} \left| f^{(n)} \left(ta + \frac{m(1-t)b}{m} \right) \right| dt \\ & \leq \left[\int_0^1 (2t + n - 2)(1 - t)^{n-1} dt \right]^{1-1/p} \left[\int_0^1 (2t + n - 2)(1 - t)^{n-1} \left| f^{(n)} \left(ta + \frac{m(1-t)b}{m} \right) \right|^p dt \right]^{1/p}. \end{aligned} \tag{3.4}$$

Utilizing the (α, m) -convexity of $|f^{(n)}(x)|^p$ reveals

$$\begin{aligned} & \int_0^1 (2t + n - 2)(1 - t)^{n-1} \left| f^{(n)} \left(ta + \frac{m(1-t)b}{m} \right) \right|^p dt \\ & \leq \int_0^1 (2t + n - 2)(1 - t)^{n-1} \left[t^\alpha |f^{(n)}(a)|^p + m(1 - t^\alpha) \left| f^{(n)} \left(\frac{b}{m} \right) \right|^p \right] dt \\ & = [(n - 2)B(\alpha + 1, n) + 2B(\alpha + 2, n)] |f^{(n)}(a)|^p + m[(n - 2)B(1, n) + 2B(2, n) \\ & \quad - (n - 2)B(\alpha + 1, n) - 2B(\alpha + 2, n)] \left| f^{(n)} \left(\frac{b}{m} \right) \right|^p. \end{aligned} \tag{3.5}$$

Making use of the identity (3.2) and the property

$$B(x, y + 1) = \frac{y}{x} B(x + 1, y) = \frac{y}{x + y} B(x, y) \tag{3.6}$$

yields

$$(n - 2)B(1, n) + 2B(2, n) = \frac{n - 1}{n + 1} \quad \text{and} \quad (n - 2)B(\alpha + 1, n) + 2B(\alpha + 2, n) = (n + \alpha - 1)B(\alpha + 1, n + 1). \tag{3.7}$$

Substituting the identities in (3.7) into (3.5), and then combining (3.5) with (3.4), yield the inequality (3.1). This completes the proof of Theorem 3.1. \square

Corollary 3.1. Under the conditions of Theorem 3.1,

1. when $m = 1$,

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx - \frac{1}{2} \sum_{k=1}^{n-1} \frac{k - 1}{(k + 1)!} (a - b)^k f^{(k)}(b) \right| \leq \frac{1}{2} \frac{(b - a)^n}{n!} \left(\frac{n - 1}{n + 1} \right)^{1-1/p} \\ & \quad \times \left\{ (n + \alpha - 1)B(\alpha + 1, n + 1) |f^{(n)}(a)|^p + \left[\frac{n - 1}{n + 1} - (n + \alpha - 1)B(\alpha + 1, n + 1) \right] |f^{(n)}(b)|^p \right\}^{1/p}; \end{aligned}$$

2. when $n = 2$,

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\ & \leq \frac{(b-a)^2}{2} \left(\frac{1}{6}\right)^{1-1/p} \left\{ \frac{1}{(\alpha+2)(\alpha+3)} |f''(a)|^p + m \left[\frac{1}{6} - \frac{1}{(\alpha+2)(\alpha+3)} \right] \left| f''\left(\frac{b}{m}\right) \right|^p \right\}^{1/p}; \end{aligned}$$

- 3. when $\alpha = 1$ and $n = 2$, the inequality (1.3) is valid;
- 4. when $m = \alpha = p = 1$ and $n = 2$, the inequality (1.4) holds;
- 5. when $m = \alpha = 1$ and $p = n = 2$,

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{(b-a)^2}{12} \left[\frac{|f''(a)|^2 + |f''(b)|^2}{2} \right]^{1/2}.$$

Theorem 3.2. Let $(\alpha, m) \in (0, 1] \times (0, 1]$ and $f : \mathbb{R}_0 \rightarrow \mathbb{R}$ be a n -time differentiable function such that $f^{(n)}(x) \in L\left[\left[a, \frac{b}{m}\right]\right]$, where $0 < a < b < \infty$ and $n \geq 2$. If $|f^{(n)}(x)|^p$ is (α, m) -convex on $\left[a, \frac{b}{m}\right]$ for $p > 1$, then

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx - \frac{1}{2} \sum_{k=1}^{n-1} \frac{k-1}{(k+1)!} (a-b)^k f^{(k)}(b) \right| \leq \frac{1}{2} \frac{(b-a)^n}{n!} \left[\frac{n^{q+1} - (n-2)^{q+1}}{2(q+1)} \right]^{1/q} \\ & \quad \times \left\{ B(\alpha+1, p(n-1)+1) |f^{(n)}(a)|^p + m \left[\frac{1}{p(n-1)+1} - B(\alpha+1, p(n-1)+1) \right] \left| f^{(n)}\left(\frac{b}{m}\right) \right|^p \right\}^{1/p}, \quad (3.8) \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and B is defined by (3.2).

Proof. It follows from Lemma 2.1 that

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx - \frac{1}{2} \sum_{k=1}^{n-1} \frac{k-1}{(k+1)!} (a-b)^k f^{(k)}(b) \right| \\ & \leq \frac{1}{2} \frac{(b-a)^n}{n!} \int_0^1 (2t+n-2)(1-t)^{n-1} \left| f^{(n)}\left(ta + \frac{m(1-t)b}{m} \right) \right| dt. \quad (3.9) \end{aligned}$$

By Hölder integral inequality, we have

$$\begin{aligned} & \int_0^1 (2t+n-2)(1-t)^{n-1} \left| f^{(n)}\left(ta + \frac{m(1-t)b}{m} \right) \right| dt \\ & \leq \left[\int_0^1 (2t+n-2)^q dt \right]^{1/q} \left[\int_0^1 (1-t)^{p(n-1)} \left| f^{(n)}\left(ta + \frac{m(1-t)b}{m} \right) \right|^p dt \right]^{1/p} \\ & = \left[\frac{n^{q+1} - (n-2)^{q+1}}{2(q+1)} \right]^{1/q} \left[\int_0^1 (1-t)^{p(n-1)} \left| f^{(n)}\left(ta + \frac{m(1-t)b}{m} \right) \right|^p dt \right]^{1/p}. \quad (3.10) \end{aligned}$$

Using the (α, m) -convexity of $|f^{(n)}(x)|^p$, we have

$$\begin{aligned} & \int_0^1 (1-t)^{p(n-1)} \left| f^{(n)}\left(ta + \frac{m(1-t)b}{m} \right) \right|^p dt \\ & \leq \int_0^1 (1-t)^{p(n-1)} \left[t^\alpha |f^{(n)}(a)|^p + m(1-t^\alpha) \left| f^{(n)}\left(\frac{b}{m}\right) \right|^p \right] dt \\ & = |f^{(n)}(a)|^p \int_0^1 t^\alpha (1-t)^{p(n-1)} dt + m \left| f^{(n)}\left(\frac{b}{m}\right) \right|^p \int_0^1 (1-t^\alpha)(1-t)^{p(n-1)} dt \end{aligned}$$

$$\begin{aligned}
 &= B(\alpha + 1, p(n - 1) + 1) |f^{(n)}(a)|^p + m [B(1, p(n - 1) + 1) - B(\alpha + 1, p(n - 1) + 1)] \left| f^{(n)}\left(\frac{b}{m}\right) \right|^p \\
 &= B(\alpha + 1, p(n - 1) + 1) |f^{(n)}(a)|^p + m \left[\frac{1}{p(n - 1) + 1} - B(\alpha + 1, p(n - 1) + 1) \right] \left| f^{(n)}\left(\frac{b}{m}\right) \right|^p.
 \end{aligned}$$

Substituting the above inequality into (3.10), and then substituting (3.10) into (3.9), result in the inequality (3.8). This completes the proof of Theorem 3.2. \square

Corollary 3.2. *Under the conditions of Theorem 3.2,*

1. *when $m = 1$, we have*

$$\begin{aligned}
 &\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx - \frac{1}{2} \sum_{k=1}^{n-1} \frac{k-1}{(k+1)!} (a-b)^k f^{(k)}(b) \right| \leq \frac{1}{2} \frac{(b-a)^n}{n!} \left[\frac{n^{q+1} - (n-2)^{q+1}}{2(q+1)} \right]^{1/q} \\
 &\quad \times \left\{ B(\alpha + 1, p(n - 1) + 1) |f^{(n)}(a)|^p + \left[\frac{1}{p(n - 1) + 1} - B(\alpha + 1, p(n - 1) + 1) \right] |f^{(n)}(b)|^p \right\}^{1/p};
 \end{aligned}$$

2. *when $n = 2$, we have*

$$\begin{aligned}
 &\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \\
 &\quad \leq \frac{(b-a)^2}{2} \left(\frac{1}{q+1} \right)^{1/q} \left\{ B(\alpha + 1, p + 1) |f''(a)|^p + m \left[\frac{1}{p+1} - B(\alpha + 1, p + 1) \right] \left| f''\left(\frac{b}{m}\right) \right|^p \right\}^{1/p};
 \end{aligned}$$

3. *when $\alpha = 1$ and $n = 2$, we have*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{(b-a)^2}{2} \left(\frac{1}{q+1} \right)^{1/q} \left[\frac{|f''(a)|^p + m(p+1) |f''(b/m)|^p}{(p+1)(p+2)} \right]^{1/p};$$

4. *when $m = \alpha = 1$ and $p = n = 2$, we have*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{(b-a)^2}{12} \left[|f''(a)|^2 + 3|f''(b)|^2 \right]^{1/2}.$$

Theorem 3.3. *Let $(\alpha, m) \in (0, 1] \times (0, 1]$ and $f : \mathbb{R}_0 \rightarrow \mathbb{R}$ be a n -time differentiable function such that $f^{(n)}(x) \in L\left[\left[a, \frac{b}{m}\right]\right)$, where $0 < a < b < \infty$ and $n \geq 2$. If $|f^{(n)}(x)|^p$ is (α, m) -convex on $\left[a, \frac{b}{m}\right]$ for $p \geq 1$, then*

$$\begin{aligned}
 &\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx - \frac{1}{2} \sum_{k=1}^{n-1} \frac{k-1}{(k+1)!} (a-b)^k f^{(k)}(b) \right| \\
 &\quad \leq \frac{(n-1)^{1-1/p}}{2} \frac{(b-a)^n}{n!} \left\{ L(p, \alpha) |f^{(n)}(a)|^p + m [Q(p, \alpha) - L(p, \alpha)] \left| f^{(n)}\left(\frac{b}{m}\right) \right|^p \right\}^{1/p}, \quad (3.11)
 \end{aligned}$$

where B is defined by (3.2),

$$L(p, \alpha) = \left[\frac{p(n-2)(n-1) + n(\alpha+2) - 2}{p(n-1) + \alpha + 2} \right] B(\alpha + 1, p(n - 1) + 1), \quad (3.12)$$

and

$$Q(p, \alpha) = \frac{(n-1)[p(n-2) + 2]}{[p(n-1) + 1][p(n-1) + 2]}. \quad (3.13)$$

Proof. It follows from Lemma 2.1 that

$$\begin{aligned} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{2} \sum_{k=1}^{n-1} \frac{k-1}{(k+1)!} (a-b)^k f^{(k)}(b) \right| \\ \leq \frac{1}{2} \frac{(b-a)^n}{n!} \int_0^1 (2t+n-2)(1-t)^{n-1} \left| f^{(n)} \left(ta + \frac{m(1-t)b}{m} \right) \right| dt. \end{aligned} \quad (3.14)$$

By the well-known Hölder integral inequality, we have

$$\begin{aligned} \int_0^1 (2t+n-2)(1-t)^{n-1} \left| f^{(n)} \left(ta + \frac{m(1-t)b}{m} \right) \right| dt \\ \leq \left[\int_0^1 (2t+n-2) dt \right]^{1-1/p} \left[\int_0^1 (2t+n-2)(1-t)^{p(n-1)} \left| f^{(n)} \left(ta + \frac{m(1-t)b}{m} \right) \right|^p dt \right]^{1/p}. \end{aligned} \quad (3.15)$$

Using the (α, m) -convexity of $|f^{(n)}(x)|^p$ shows

$$\begin{aligned} & \int_0^1 (2t+n-2)(1-t)^{p(n-1)} \left| f^{(n)} \left(ta + \frac{m(1-t)b}{m} \right) \right|^p dt \\ & \leq \int_0^1 (2t+n-2)(1-t)^{p(n-1)} \left[t^\alpha |f^{(n)}(a)|^p + m(1-t^\alpha) \left| f^{(n)} \left(\frac{b}{m} \right) \right|^p \right] dt \\ & = |f^{(n)}(a)|^p \int_0^1 (2t+n-2)t^\alpha (1-t)^{p(n-1)} dt + m \left| f^{(n)} \left(\frac{b}{m} \right) \right|^p \int_0^1 (2t+n-2)(1-t^\alpha)(1-t)^{p(n-1)} dt \\ & = [(n-2)B(\alpha+1, p(n-1)+1) + 2B(\alpha+2, p(n-1)+1)] |f^{(n)}(a)|^p + m[(n-2)B(1, p(n-1)+1) \\ & \quad + 2B(2, p(n-1)+1) - (n-2)B(\alpha+1, p(n-1)+1) - 2B(\alpha+2, p(n-1)+1)] \left| f^{(n)} \left(\frac{b}{m} \right) \right|^p. \end{aligned} \quad (3.16)$$

Employing the identities (3.2) and (3.6) give

$$(n-2)B(1, p(n-1)+1) + 2B(2, p(n-1)+1) = \frac{(n-1)[p(n-2)+2]}{[p(n-1)+1][p(n-1)+2]} \quad (3.17)$$

and

$$\begin{aligned} (n-2)B(\alpha+1, p(n-1)+1) + 2B(\alpha+2, p(n-1)+1) \\ = \frac{p(n-2)(n-1) + n(\alpha+2) - 2}{p(n-1) + \alpha + 2} B(\alpha+1, p(n-1)+1). \end{aligned} \quad (3.18)$$

Substituting (3.17) and (3.18) into (3.16), and then combining (3.16) with (3.15) and (3.14) in sequence, yield the inequality (3.11). This completes the proof of Theorem 3.3. \square

Corollary 3.3. *Under the conditions of Theorem 3.3,*

1. *when $n = 2$, we have*

$$\begin{aligned} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq \frac{(b-a)^2}{4} \left\{ 2B(\alpha+2, p+1) |f''(a)|^p + m \left[\frac{2}{(p+1)(p+2)} - 2B(\alpha+2, p+1) \right] \left| f'' \left(\frac{b}{m} \right) \right|^p \right\}^{1/p}; \end{aligned}$$

2. when $\alpha = 1$ and $n = 2$, we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{4} \left[\frac{4|f''(a)|^p}{(p+1)(p+2)(p+3)} + \frac{2m|f''(b/m)|^p}{(p+2)(p+3)} \right]^{1/p};$$

3. when $m = \alpha = p = 1$ and $n = 2$, we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{24} [|f''(a)| + |f''(b)|].$$

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