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Symbolic computation of the Moore-Penrose inverse using the LDL* decomposition of the polynomial matrix

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Abstract. The full-rank LDL* decomposition of a polynomial Hermitian matrix is examined. Explicit formulae are given evaluating the coefficients of matrices l_{ij} and d_{jj}. Also, a new method is developed, based on the LDL^* factorization of the matrix product A^*A, for symbolic computation of the Moore-Penrose inverse matrix. The paper follows the results of [I.P. Stanimirović, M.B. Tasić, *Computation of generalized inverses by using the LDL* decomposition*, Appl. Math. Lett., **25** (2012), 526–531], where the matrix products A^*A, AA^* and the corresponding LDL^* factorizations are considered in order to compute the generalized inverse of A.

1. Introduction

A well known technique from numerical analysis is to replace LL^* factorization by LDL^*, such that the computation of square root entries can be avoided (therefore, the LDL^* decomposition is often called square-root free Cholesky decomposition). Obviously, this method (explained by Golub and Van Loan in [4]) is of great importance when working with polynomial and rational matrices. Therefore, a motivation is to modify the method for the Cholesky decomposition of a polynomial matrix $A(x)$, by providing an additional diagonal matrix D which ensures no entries containing square roots. Obviously, the LDL^* decomposition is much more appropriate to manipulate with polynomial matrices, and can be later used to find generalized inverses of the decomposed matrix.

For the case of a complex Hermitian matrix A, its LDL^* decomposition can be obtained, using the following recursive relations for the entries of D and L:

$$d_{jj} = a_{jj} - \sum_{k=1}^{j-1} l_{jk} l_{jk}^* d_{kk}, \quad l_{ij} = \frac{1}{d_j}(a_{ij} - \sum_{k=1}^{j-1} l_{ik} l_{jk}^* d_{kk}), \ for\ i > j. \tag{1}$$

We propose that these calculations only have to be performed for $j = \overline{1, r}$, where $r = rank(A)$. In that case, the iterative procedure (1) generates the full-rank factorization of A, where the matrix L is without the zero columns, and the matrix D is without zero rows and zero columns (since the matrices L and D are of the rank r).

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Denote the set of $m \times n$ matrices with elements in $\mathbf{C}(x)$ by $\mathbf{C}(x)^{m\times n}$, and the conjugate transpose matrix of L by L^*. Let us observe the polynomial Hermitian matrix $A \in \mathbf{C}(x)_r^{n\times n}$ of rank r with the following entries:

$$a_{ij}(x) = \sum_{k=0}^{a_q} a_{k,i,j}x^k, \quad 1 \le i,\ j \le n, \tag{2}$$

where the maximal exponent of $A(x)$ is denoted by a_q.

Full-rank square-root free Cholesky decomposition of the matrix A is $A = LDL^*$, where $L \in \mathbf{C}(x)^{n\times r}$, $l_{ij} = 0$ for $i < j$, and $D \in \mathbf{C}(x)^{r\times r}$ is the diagonal rational matrix. Non-zero elements of the rational matrices $L(x)$ and $D(x)$ are of the form:

$$d_{jj}(x) = \frac{\sum\limits_{k=0}^{\overline{d}_q} \overline{d}_{k,j,j}x^k}{\sum\limits_{k=0}^{\overline{\overline{d}}_q} \overline{\overline{d}}_{k,j,j}x^k}, \quad l_{jj}(x) = 1, \quad 1 \le j \le r; \quad l_{ij}(x) \ = \frac{\sum\limits_{k=0}^{\overline{l}_q} \overline{l}_{k,i,j}x^k}{\sum\limits_{k=0}^{\overline{\overline{l}}_q} \overline{\overline{l}}_{k,i,j}x^k}, \quad 1 \le j \le n, \quad 1 \le i \le r, \quad j < i. \tag{3}$$

For an arbitrary matrix $A \in \mathbf{C}(x)^{m\times n}$, the matrix $A^\dagger$ is said to be the Moore-Penrose inverse matrix of A if it satisfies equations (1)–(4) of the unknown X, where $*$ denotes conjugate transpose.

$$(1)\ AXA = A, \quad (2)\ XAX = X, \quad (3)\ (AX)^* = AX, \quad (4)\ (XA)^* = XA.$$

So far, the Greville's partitioning method [5] and the Leverrier–Faddeev algorithm are used in the symbolic implementation of generalized inverses. Various extensions of the partitioning algorithm to rational and polynomial matrices have been established. The first generalization is the extension on Greville's algorithm to the set of one-variable polynomial and/or rational matrices, introduced in [10]. For the further uses of this problem and a modification of the algorithm for computations of the weighted Moore-Penrose inverse see [7, 12]. For more information about the practical computation of Drazin inverse, the Moore-Penrose inverse or the weighted Moore-Penrose inverse, see [1–3, 8, 9, 11].

Our main motivation is to develop an efficient method for the symbolic computation of the Moore-Penrose inverse matrix of polynomial matrices.By using the LDL^* factorization instead of Cholesky decomposition, the computations of square root entries are avoided, and this is of essential importance in symbolic polynomial computation.

The paper is organized as follows. In the second section, the full-rank LDL^* decomposition of a polynomial matrix is observed. Therefore, a theorem is developed providing a practical method for the evaluation of the coefficients occurring in the entries of rational matrices L and D. In the third section, we use the method for the symbolic computation of the full-rank LDL^* decomposition to generate a theorem providing an effective way of computing the Moore-Penrose inverse of a polynomial matrix. Thereat, we give an algorithm for the symbolic computation of the Moore-Penrose inverse, which summarizes the previous results. In the fourth section, some numerical experiments are carried out to illustrate the presented methods. Some conclusion remarks are provided in the last section.

2. Full-rank LDL* decomposition of a polynomial matrix

The mentioned iterative procedure (1) can be modified in order to produce the full-rank decomposition of a polynomial matrix $A(x)$. Therefore, the following relations for the rational entries of matrices $D(x)$ and $L(x)$ are satisfied for each $j = \overline{1, r}$:

$$f_{ij}(x) = \sum_{k=1}^{j-1} l_{ik}(x) l^*_{jk}(x) d_{kk}(x), \ \text{for } i = \overline{j, n}, \tag{4}$$

$$d_{jj}(x) = a_{jj}(x) - f_{jj}(x), \tag{5}$$

$$l_{ij}(x) = \frac{1}{d_{jj}(x)}(a_{ij}(x) - f_{ij}(x)), \ \text{for } i = \overline{j+1, n}. \tag{6}$$

To determine the coefficients of the rational matrices L and D, relations (4)-(6) are used in the following theorem. In the sequel of the paper, variables with one bar will be used for numerator coefficients, and variables with two bars will be used to denote denominator coefficients.

Theorem 2.1. *Full-rank LDL^* decomposition of a Hermitian polynomial matrix $A(x) \in \mathbf{C}(x)_r^{n\times n}$ with entries of the form* (2) *is $A(x) = L(x)D(x)L(x)^*$, where $L(x)$ and $D(x)$ are rational matrices of the forms* (3), *and the coefficients of $d_{jj}(x)$ and $l_{ij}(x)$ are as follows*

$$\begin{aligned}
\overline{d}_{k,j} &= \sum_{k_1=0}^{k} a_{k-k_1,j,j}\overline{\overline{f}}_{k_1,j,j} - \overline{f}_{k,j,j},\ 0\le k\le \overline{d}_q = \max(a_q+\overline{\overline{f}}_q, \overline{f}_q), &(7)\\
\overline{\overline{d}}_{k,j} &= \overline{\overline{f}}_{k,j,j},\ 0\le k\le \overline{\overline{d}}_q = \overline{\overline{f}}_q, &(8)\\
\overline{l}_{k,i,j} &= \sum_{k_1=0}^{k} \overline{\overline{d}}_{k-k_1,j}\left(\sum_{k_2=0}^{k_1} a_{k_1-k_2,i,j}\overline{\overline{f}}_{k_2,i,j} - \overline{f}_{k_1,i,j}\right),\\
&\quad 0\le k\le \overline{l}_q = \overline{\overline{d}}_q + \max(a_q+\overline{\overline{f}}_q, \overline{f}_q), &(9)\\
\overline{\overline{l}}_{k,i,j} &= \sum_{k_1=0}^{k} \overline{d}_{k-k_1,j}\overline{\overline{f}}_{k_1,i,j},\ 0\le k\le \overline{\overline{l}}_q = \overline{d}_q+\overline{\overline{f}}_q, &(10)
\end{aligned}$$

whereat the coefficients $\overline{f}_{k,i,j}$ are evaluated as

$$\overline{f}_{k,i,j} = \sum_{k_2=0}^{k}\sum_{k_3=0}^{j-1} \overline{p}_{k-k_2,i,j,k_3} q_{k_2,i,j,k_3},\ 0\le k\le \overline{f}_q = 2\overline{l}_q+\overline{d}_q+\overline{\overline{f}}_q - 2\overline{\overline{l}}_q - \overline{\overline{d}}_q, \tag{11}$$

and $\overline{\overline{f}}_{k,i,j}$, $0\le k\le \overline{\overline{f}}_q$, are the coefficients of the following polynomial:

$$PolynomialLCM\left(\sum_{k=0}^{2\overline{\overline{l}}_q+\overline{\overline{d}}_q} \overline{\overline{p}}_{k,i,j,1}x^k, \sum_{k=0}^{2\overline{\overline{l}}_q+\overline{\overline{d}}_q} \overline{\overline{p}}_{k,i,j,2}x^k, \dots, \sum_{k=0}^{2\overline{\overline{l}}_q+\overline{\overline{d}}_q} \overline{\overline{p}}_{k,i,j,j-1}x^k\right), \tag{12}$$

where

$$\begin{aligned}
\overline{p}_{t_1,i,j,k} &= \sum_{t_2=0}^{t_1}\sum_{t_3=0}^{t_1-t_2} \overline{l}_{t_3,i,k}\overline{l}^*_{t_1-t_2-t_3,j,k}\overline{d}_{t_2,k},\ 0\le t_1\le 2\overline{l}_q+\overline{d}_q,\\
\overline{\overline{p}}_{t_1,i,j,k} &= \sum_{t_2=0}^{t_1}\sum_{t_3=0}^{t_1-t_2} \overline{\overline{l}}_{t_3,i,k}\overline{\overline{l}}^*_{t_1-t_2-t_3,j,k}\overline{\overline{d}}_{t_2,k},\ 0\le t_1\le 2\overline{\overline{l}}_q+\overline{\overline{d}}_q.
\end{aligned}$$

and the values $q_{k,i,j,t}$ are the coefficients of polynomial $q_{i,j,t}(x) = \dfrac{\sum_{k=0}^{\overline{\overline{f}}_q}\overline{\overline{f}}_{k,i,j}x^k}{\sum_{k=0}^{2\overline{\overline{l}}_q+\overline{\overline{d}}_q}\overline{\overline{p}}_{k,i,j,t}x^k}$.

Proof. Since the entries of $L(x)$ and $D(x)$ are rational functions, the equality (4) becomes:

$$
\begin{aligned}
f_{ij}(x) &= \sum_{k=1}^{j-1} \frac{\sum_{t=0}^{\bar{l}_q} \bar{l}_{t,i,k} x^t}{\sum_{t=0}^{\bar{\bar{l}}_q} \bar{\bar{l}}_{t,i,k} x^t} \frac{\sum_{t=0}^{\bar{l}_q} \bar{l}^*_{t,j,k} x^t}{\sum_{t=0}^{\bar{\bar{l}}_q} \bar{\bar{l}}^*_{t,j,k} x^t} \frac{\sum_{t=0}^{\bar{d}_q} \bar{d}_{t,k} x^t}{\sum_{t=0}^{\bar{\bar{d}}_q} \bar{\bar{d}}_{t,k} x^t} = \sum_{k=1}^{j-1} \frac{\sum_{t_1=0}^{2\bar{l}_q+\bar{d}_q} \left(\sum_{t_2=0}^{t_1} \sum_{t_3=0}^{t_1-t_2} \bar{l}_{t_3,i,k} \bar{l}^*_{t_1-t_2-t_3,j,k} \bar{d}_{t_2,k} \right) x^{t_1}}{\sum_{t_1=0}^{2\bar{\bar{l}}_q+\bar{\bar{d}}_q} \left(\sum_{t_2=0}^{t_1} \sum_{t_3=0}^{t_1-t_2} \bar{\bar{l}}_{t_3,i,k} \bar{\bar{l}}^*_{t_1-t_2-t_3,j,k} \bar{\bar{d}}_{t_2,k} \right) x^{t_1}} \\
&= \sum_{k=1}^{j-1} \frac{\sum_{t_1=0}^{2\bar{l}_q+\bar{d}_q} \bar{p}_{t_1,i,j,k} x^{t_1}}{\sum_{t_1=0}^{2\bar{\bar{l}}_q+\bar{\bar{d}}_q} \bar{\bar{p}}_{t_1,i,j,k} x^{t_1}} = \frac{\sum_{t_1=0}^{2\bar{l}_q+\bar{d}_q} \bar{p}_{t_1,i,j,1} x^{t_1}}{\sum_{t_1=0}^{2\bar{\bar{l}}_q+\bar{\bar{d}}_q} \bar{\bar{p}}_{t_1,i,j,1} x^{t_1}} + \frac{\sum_{t_1=0}^{2\bar{l}_q+\bar{d}_q} \bar{p}_{t_1,i,j,2} x^{t_1}}{\sum_{t_1=0}^{2\bar{\bar{l}}_q+\bar{\bar{d}}_q} \bar{\bar{p}}_{t_1,i,j,2} x^{t_1}} + \ldots + \frac{\sum_{t_1=0}^{2\bar{l}_q+\bar{d}_q} \bar{p}_{t_1,i,j,j-1} x^{t_1}}{\sum_{t_1=0}^{2\bar{\bar{l}}_q+\bar{\bar{d}}_q} \bar{\bar{p}}_{t_1,i,j,j-1} x^{t_1}}.
\end{aligned}
$$

Since the least common multiple (LCM) of the denominator polynomials is denoted by:

$$
PolynomialLCM\left(\sum_{k=0}^{2\bar{\bar{l}}_q+\bar{\bar{d}}_q} \bar{\bar{p}}_{k,i,j,1} x^k, \sum_{k=0}^{2\bar{\bar{l}}_q+\bar{\bar{d}}_q} \bar{\bar{p}}_{k,i,j,2} x^k, \ldots, \sum_{k=0}^{2\bar{\bar{l}}_q+\bar{\bar{d}}_q} \bar{\bar{p}}_{k,i,j,j-1} x^k \right) = \sum_{k=0}^{\bar{\bar{f}}_q} \bar{\bar{f}}_{k,i,j} x^k,
$$

and the following identities hold

$$
q_{i,j,t}(x) = \frac{\sum_{k=0}^{\bar{\bar{f}}_q} \bar{\bar{f}}_{k,i,j} x^k}{\sum_{k=0}^{2\bar{\bar{l}}_q+\bar{\bar{d}}_q} \bar{\bar{p}}_{k,i,j,t} x^k} = \sum_{k=0}^{\bar{\bar{f}}_q-2\bar{\bar{l}}_q-\bar{\bar{d}}_q} q_{k,i,j,t} x^k, \quad 1 \le t < j < i,
$$

the polynomials $f_{ij}(x)$ can be expressed by

$$
f_{ij}(x) = \frac{\sum_{k=1}^{j-1} \sum_{k_1=0}^{\bar{f}_q} \left(\sum_{k_2=0}^{k_1} \bar{p}_{k_1-k_2,i,j,k} q_{k_2,i,j,k} \right) x^{k_1}}{\sum_{k=0}^{\bar{\bar{f}}_q} \bar{\bar{f}}_{k,i,j} x^k} = \frac{\sum_{k_1=0}^{\bar{f}_q} \sum_{k_2=0}^{k_1} \sum_{k_3=0}^{j-1} \bar{p}_{k_1-k_2,i,j,k_3} q_{k_2,i,j,k_3} x^{k_1}}{\sum_{k=0}^{\bar{\bar{f}}_q} \bar{\bar{f}}_{k,i,j} x^k} = \frac{\sum_{k=0}^{\bar{f}_q} \bar{f}_{k,i,j} x^k}{\sum_{k=0}^{\bar{\bar{f}}_q} \bar{\bar{f}}_{k,i,j} x^k}.
$$

Observe that from the equation (5) proceeds the following equation:

$$
\begin{aligned}
d_{jj}(x) &= \sum_{k=0}^{a_q} a_{k,j,j} x^k - \frac{\sum_{k=0}^{\bar{f}_q} \bar{f}_{k,j,j} x^k}{\sum_{k=0}^{\bar{\bar{f}}_q} \bar{\bar{f}}_{k,j,j} x^k} = \frac{\sum_{k=0}^{a_q} a_{k,j,j} x^k \sum_{k=0}^{\bar{\bar{f}}_q} \bar{\bar{f}}_{k,j,j} x^k - \sum_{k=0}^{\bar{f}_q} \bar{f}_{k,j,j} x^k}{\sum_{k=0}^{\bar{\bar{f}}_q} \bar{\bar{f}}_{k,j,j} x^k} \\
&= \frac{\sum_{k_1=0}^{\max(a_q+\bar{\bar{f}}_q, \bar{f}_q)} \left(\sum_{k_2=0}^{k_1} a_{k_1-k_2,j,j} \bar{\bar{f}}_{k_2,j,j} - \bar{f}_{k_1,j,j} \right) x^{k_1}}{\sum_{k=0}^{\bar{\bar{f}}_q} \bar{\bar{f}}_{k,j,j} x^k} = \frac{\sum_{k=0}^{\bar{d}_q} \bar{d}_{k,j} x^k}{\sum_{k=0}^{\bar{\bar{d}}_q} \bar{\bar{d}}_{k,j} x^k}.
\end{aligned}
$$

Finally, according to the equality (6), the next equation is valid:

$$
\begin{aligned}
l_{ij}(x) &= \frac{\sum\limits_{k=0}^{\bar{\bar{d}}_q} \bar{\bar{d}}_{k,j}x^k}{\sum\limits_{k=0}^{\bar{d}_q} \bar{d}_{k,j}x^k} \left(\sum_{k=0}^{a_q} a_{k,i,j}x^k - \frac{\sum\limits_{k=0}^{\bar{f}_q} \bar{f}_{k,i,j}x^k}{\sum\limits_{k=0}^{\bar{\bar{f}}_q} \bar{\bar{f}}_{k,i,j}x^k} \right) = \frac{\sum\limits_{k=0}^{\bar{\bar{d}}_q} \bar{\bar{d}}_{k,j}x^k}{\sum\limits_{k=0}^{\bar{d}_q} \bar{d}_{k,j}x^k} \frac{\sum\limits_{k_1=0}^{\max(a_q+\bar{\bar{f}}_q,\bar{f}_q)} \left(\sum\limits_{k_2=0}^{k_1} a_{k_1-k_2,i,j}\bar{\bar{f}}_{k_2,i,j} - \bar{f}_{k_1,i,j} \right) x^{k_1}}{\sum\limits_{k=0}^{\bar{\bar{f}}_q} \bar{\bar{f}}_{k,j,j}x^k} \\
&= \frac{\sum\limits_{k=0}^{\bar{\bar{d}}_q+\max(a_q+\bar{\bar{f}}_q,\bar{f}_q)} \left(\sum\limits_{k_1=0}^{k} \bar{\bar{d}}_{k-k_1,j} \left(\sum\limits_{k_2=0}^{k_1} a_{k_1-k_2,i,j}\bar{\bar{f}}_{k_2,i,j} - \bar{f}_{k_1,i,j} \right) \right) x^k}{\sum\limits_{k=0}^{\bar{d}_q+\bar{\bar{f}}_q} \left(\sum\limits_{k_1=0}^{k} \bar{d}_{k-k_1,j}\bar{\bar{f}}_{k_1,i,j} \right) x^k} = \frac{\sum\limits_{k=0}^{\bar{l}_q} \bar{l}_{k,i,j}x^k}{\sum\limits_{k=0}^{\bar{\bar{l}}_q} \bar{\bar{l}}_{k,i,j}x^k}.
\end{aligned}
$$

□

Theorem 2.1 provides the practical method for the calculation of the coefficients of $l_{ij}(x)$ and $d_{jj}(x)$ from the previously evaluated $l_{ik}(x)$, $l_{jk}(x)$ and $d_{kk}(x)$, for $k < j$.

3. Evaluation of the Moore-Penrose inverse of a polynomial matrix

For the sake of completeness we restate the result from [6].

Theorem 3.1. [6] *Consider the rational matrix $A \in \mathbf{C}(x)_r^{m\times n}$ and the full-rank LDL^* factorization of the matrix $(A^*A)^*(A^*A)$, where $L \in \mathbf{C}(x)^{n\times r}$ and $D \in \mathbf{C}(x)^{r\times r}$. Then it is satisfied:*

$$A^\dagger = L(L^*LDL^*L)^{-1}L^*(A^*A)^*A^*. \tag{13}$$

By observing the full-rank LDL^* factorization of the matrix $(AA^*)(AA^*)^*$, where $L \in \mathbf{C}(x)^{m\times r}$ and $D \in \mathbf{C}(x)^{r\times r}$, it is satisfied:

$$A^\dagger = A^*(AA^*)^*L(L^*LDL^*L)^{-1}L^*. \tag{14}$$

According to the previous result we introduce the next Theorem.

Theorem 3.2. *Let $A \in \mathbf{C}(x)_r^{m\times n}$ be a polynomial matrix with entries of the form* (2). *Consider the full-rank LDL^* factorization of the matrix $(A^*A)^*(A^*A)$, where $L \in \mathbf{C}(x)^{n\times r}$ and $D \in \mathbf{C}(x)^{r\times r}$ are matrices with entries of the form* (3). *We will denote the entries of the inverse matrix $N = (L^*LDL^*L)^{-1} \in \mathbf{C}(x)^{r\times r}$ by*

$$n_{ij}(x) = \frac{\sum\limits_{t=0}^{\bar{n}_q} \bar{n}_{t,k,l}x^t}{\sum\limits_{t=0}^{\bar{\bar{n}}_q} \bar{\bar{n}}_{t,k,l}x^t}.$$

Then an arbitrary element of the generalized inverse of A can be evaluated as $A^\dagger_{ij}(x) = \frac{\bar{\Gamma}_{ij}(x)}{\bar{\bar{\Gamma}}_i(x)}$, where

$$\bar{\Gamma}_{ij}(x) = \sum_{t=0}^{\bar{\bar{\Gamma}}_q-\bar{\bar{b}}_q+\bar{b}_q} \left(\sum_{\mu=1}^{n} \sum_{l=1}^{\min\{\mu,r\}} \sum_{k=1}^{\min\{i,r\}} \sum_{\lambda=1}^{n} \sum_{\kappa=1}^{m} \sum_{t_1=0}^{t} \bar{\beta}_{t_1,i,j,k,l,\mu,\kappa,\lambda}\gamma_{t-t_1,i,k,l,\mu} \right) x^t, \tag{15}$$

$$\bar{\bar{\Gamma}}_i(x) = PolynomialLCM \left\{ \sum_{t=0}^{\bar{\bar{b}}_q} \bar{\bar{\beta}}_{t,i,k,l,\mu}x^t \middle| \mu = \overline{1,n},\ k = \overline{1,\min\{i,r\}},\ l = \overline{1,\min\{\mu,r\}} \right\} \tag{16}$$

and where $\overline{\overline{\Gamma}}_q$ is the maximal exponent of polynomials $\overline{\overline{\Gamma}}_i(x)$, $1 \le i \le m$, the values $\gamma_{t,i,k,l,\mu}$, $0 \le t \le \overline{\overline{\Gamma}}_q - \overline{\overline{b}}_q$, are coefficients of the polynomial $\Gamma_{i,k,l,\mu}(x) = \frac{\overline{\overline{\Gamma}}_i(x)}{\sum_{t=0}^{\overline{\overline{b}}_q} \overline{\overline{\beta}}_{t,i,k,l,\mu} x^t}$, for each $\mu = \overline{1,n}$, $k = \overline{1,\min\{i,r\}}$, $l = \overline{1,\min\{\mu,r\}}$, and the following notations are used, for $\kappa = \overline{1,m}$, $\lambda = \overline{1,n}$:

$$\overline{\beta}_{t,i,j,k,l,\mu,\kappa,\lambda} = \sum_{t_1=0}^{t} \overline{p}_{t_1,i,k,l,\mu} \alpha_{t-t_1,j,\mu,\kappa,\lambda}, \qquad 0 \le t \le \overline{b}_q = 2\overline{l}_q + \overline{n}_q + 3\overline{a}_q, \tag{17}$$

$$\overline{\overline{\beta}}_{t,i,k,l,\mu} = \sum_{t_1=0}^{t} \sum_{t_2=0}^{t-t_1} \overline{\overline{l}}_{t_1,i,k} \overline{\overline{n}}_{t-t_1-t_2,k,l} \overline{\overline{l}}^*_{t_2,\mu,l}, \qquad 0 \le t \le \overline{\overline{b}}_q = 2\overline{\overline{l}}_q + \overline{\overline{n}}_q, \tag{18}$$

$$\overline{p}_{t,i,k,l,\mu} = \sum_{t_1=0}^{t} \sum_{t_2=0}^{t-t_1} \overline{l}_{t_1,i,k} \overline{n}_{t-t_1-t_2,k,l} \overline{l}^*_{t_2,\mu,l}, \qquad 0 \le t \le 2\overline{l}_q + \overline{n}_q \tag{19}$$

$$\alpha_{t,j,\mu,\kappa,\lambda} = \sum_{t_1=0}^{t} \sum_{t_2=0}^{t-t_1} a_{t_1,\kappa,\lambda} a^*_{t-t_1-t_2,\kappa,\mu} a^*_{t_2,j,\lambda}, \qquad 0 \le t \le 3a_q. \tag{20}$$

Proof. Notice that the following statements are valid:

$$\begin{aligned}
(LNL^*)_{ij} &= \sum_{l=1}^{r} \left(\sum_{k=1}^{r} l_{ik} n_{kl} \right) l^*_{jl} = \sum_{l=1}^{r} \sum_{k=1}^{r} l_{ik} n_{kl} l^*_{jl}, \\
((A^*A)^*A^*)_{ij} &= \sum_{l=1}^{n} \left(\sum_{k=1}^{m} a_{kl} a^*_{ki} \right) a^*_{jl} = \sum_{l=1}^{n} \sum_{k=1}^{m} a_{kl} a^*_{ki} a^*_{jl}.
\end{aligned}$$

According to the first statement of the Theorem 3.1, an arbitrary (i, j)-th element of the Moore-Penrose inverse of A can be calculated as:

$$A^\dagger_{ij} = \sum_{\mu=1}^{n} \left(\sum_{l=1}^{\min\{\mu,r\}} \sum_{k=1}^{\min\{i,r\}} l_{ik} n_{kl} l^*_{\mu l} \right) \left(\sum_{\lambda=1}^{n} \sum_{\kappa=1}^{m} a_{\kappa\lambda} a^*_{\kappa\mu} a^*_{j\lambda} \right) = \sum_{\mu=1}^{n} \sum_{l=1}^{\min\{\mu,r\}} \sum_{k=1}^{\min\{i,r\}} \sum_{\lambda=1}^{n} \sum_{\kappa=1}^{m} l_{ik} n_{kl} l^*_{\mu l} a_{\kappa\lambda} a^*_{\kappa\mu} a^*_{j\lambda}$$

Therefore, by working with polynomial entries, the (i, j)-th element of $A^\dagger$ is evaluated as:

$$\begin{aligned}
A^\dagger_{ij}(x) &= \sum_{\mu=1}^{n} \sum_{l=1}^{\min\{\mu,r\}} \sum_{k=1}^{\min\{i,r\}} \sum_{\lambda=1}^{n} \sum_{\kappa=1}^{m} \frac{\sum_{t=0}^{\overline{l}_q} \overline{l}_{t,i,k} x^t}{\sum_{t=0}^{\overline{\overline{l}}_q} \overline{\overline{l}}_{t,i,k} x^t} \frac{\sum_{t=0}^{\overline{n}_q} \overline{n}_{t,k,l} x^t}{\sum_{t=0}^{\overline{\overline{n}}_q} \overline{\overline{n}}_{t,k,l} x^t} \frac{\sum_{t=0}^{\overline{l}_q} \overline{l}^*_{t,\mu,l} x^t}{\sum_{t=0}^{\overline{\overline{l}}_q} \overline{\overline{l}}^*_{t,\mu,l} x^t} \sum_{t=0}^{a_q} a_{t,\kappa,\lambda} x^t \sum_{t=0}^{a_q} a^*_{t,\kappa,\mu} x^t \sum_{t=0}^{a_q} a^*_{t,j,\lambda} x^t \\
&= \sum_{\mu=1}^{n} \sum_{l=1}^{\min\{\mu,r\}} \sum_{k=1}^{\min\{i,r\}} \sum_{\lambda=1}^{n} \sum_{\kappa=1}^{m} \frac{\sum_{t=0}^{2\overline{l}_q+\overline{n}_q} \overline{p}_{t,i,k,l,\mu} x^t}{\sum_{t=0}^{2\overline{\overline{l}}_q+\overline{\overline{n}}_q} \overline{\overline{\beta}}_{t,i,k,l,\mu} x^t} \sum_{t=0}^{3a_q} \alpha_{t,j,\mu,\kappa,\lambda} x^t \\
&= \sum_{\mu=1}^{n} \sum_{l=1}^{\min\{\mu,r\}} \sum_{k=1}^{\min\{i,r\}} \sum_{\lambda=1}^{n} \sum_{\kappa=1}^{m} \frac{\sum_{t=0}^{\overline{b}_q} \overline{\beta}_{t,i,j,k,l,\mu,\kappa,\lambda} x^t}{\sum_{t=0}^{\overline{\overline{b}}_q} \overline{\overline{\beta}}_{t,i,k,l,\mu} x^t}.
\end{aligned}$$

Therefore, we have $A^{\dagger}_{ij}(x) = \frac{\overline{\Gamma}_{ij}(x)}{\overline{\overline{\Gamma}}_i(x)}$, where

$$\overline{\overline{\Gamma}}_i(x) = PolynomialLCM\left\{\sum_{t=0}^{\overline{\overline{b}}_q} \overline{\overline{\beta}}_{t,i,k,l,\mu} x^t \middle| \mu = \overline{1,n},\ k = \overline{1,\min\{i,r\}},\ l = \overline{1,\min\{\mu,r\}}\right\} = \sum_{t=0}^{\overline{\overline{\Gamma}}_q} \overline{\overline{\gamma}}_{t,i} x^t,$$

$$\overline{\Gamma}_{ij}(x) = \sum_{\mu=1}^{n} \sum_{l=1}^{\min\{\mu,r\}} \sum_{k=1}^{\min\{i,r\}} \left(\Gamma_{i,k,l,\mu}(x) \sum_{\lambda=1}^{n} \sum_{\kappa=1}^{m} \sum_{t=0}^{\overline{b}_q} \overline{\beta}_{t,i,j,k,l,\mu,\kappa,\lambda} x^t\right),$$

where each polynomial $\Gamma_{i,l,k,\mu}(x)$ is equal to $\overline{\overline{\Gamma}}_i(x) / \left(\sum\limits_{t=0}^{\overline{\overline{b}}_q} \overline{\overline{\beta}}_{t,i,k,l,\mu} x^t\right) = \sum\limits_{t=0}^{\overline{\overline{\Gamma}}_q - \overline{\overline{b}}_q} \gamma_{t,i,k,l,\mu} x^t$. Therefore

$$\overline{\Gamma}_{ij}(x) = \sum_{\mu=1}^{n} \sum_{l=1}^{\min\{\mu,r\}} \sum_{k=1}^{\min\{i,r\}} \sum_{\lambda=1}^{n} \sum_{\kappa=1}^{m} \sum_{t=0}^{\overline{\overline{\Gamma}}_q - \overline{\overline{b}}_q + \overline{b}_q} \left(\sum_{t_1=0}^{t} \overline{\beta}_{t_1,i,j,k,l,\mu,\kappa,\lambda} \gamma_{t-t_1,i,k,l,\mu}\right) x^t,$$

which coincides with the form (15), and the proof is complete. □

Notice that a similar theorem can be derived by observing the second statement of Theorem 3.1. Now we are able to summarize all results with the following algorithm.

Algorithm 3.1 Symbolic computation of MP-inverse using the full-rank LDL^* decomposition

Require: Polynomial matrix $A(x) \in \mathbf{C}(x)^{m\times n}_r$ with entries of the form $a_{ij}(x) = \sum\limits_{t=0}^{a_q} a_{t,i,j} x^t$.

1: Generate the full-rank LDL^* decomposition of the matrix $(A^*A)^*(A^*A)$, where $L \in \mathbf{C}(x)^{n\times r}$ and $D \in \mathbf{C}(x)^{r\times r}$ are matrices with entries of the forms (3), by applying the method provided by the equations (7)-(12) of Theorem 2.1.

2: Transform the rational matrix $M = L^*LDL^*L$ to the form: $M = \frac{1}{p(x)} M_1$, where $p(x)$ is a polynomial and M_1 is a polynomial matrix.

3: Find the inverse of the matrix M_1 using the Algorithm 3.2 from [12]. Generate the inverse matrix $N = M^{-1} = p(x) M_1^{-1}$, and reduce it to the form: $n_{ij}(x) = \left(\sum\limits_{k=0}^{\overline{n}_q} \overline{n}_{k,i,j} x^k\right) \Big/ \left(\sum\limits_{k=0}^{\overline{\overline{n}}_q} \overline{\overline{n}}_{k,i,j} x^k\right)$.

4: For each $i = \overline{1,m},\ \mu = \overline{1,n},\ k = \overline{1,\min\{i,r\}},\ l = \overline{1,\min\{\mu,r\}}$ calculate the following:

$$\overline{p}_{t,i,k,l,\mu} = \sum_{t_1=0}^{t} \sum_{t_2=0}^{t-t_1} \overline{l}_{t_1,i,k} \overline{n}_{t-t_1-t_2,k,l} \overline{l}^*_{t_2,\mu,l},\ 0 \le t \le 2\overline{l}_q + \overline{n}_q \tag{21}$$

$$\overline{\overline{\beta}}_{t,i,k,l,\mu} = \sum_{t_1=0}^{t} \sum_{t_2=0}^{t-t_1} \overline{\overline{l}}_{t_1,i,k} \overline{\overline{n}}_{t-t_1-t_2,k,l} \overline{\overline{l}}^*_{t_2,\mu,l},\ 0 \le t \le 2\overline{\overline{l}}_q + \overline{\overline{n}}_q. \tag{22}$$

5: For each $j = \overline{1,n},\ \mu = \overline{1,n},\ \kappa = \overline{1,m},\ \lambda = \overline{1,n}$ make the following calculations:

$$\alpha_{t,j,\mu,\kappa,\lambda} = \sum_{t_1=0}^{t} \sum_{t_2=0}^{t-t_1} a_{t_1,\kappa,\lambda} a^*_{t-t_1-t_2,\kappa,\mu} a^*_{t_2,j,\lambda},\ 0 \le t \le 3a_q. \tag{23}$$

6: Make the notations $\overline{b}_q = 2\overline{l}_q + \overline{n}_q + 3\overline{a}_q$, $\overline{\overline{b}}_q = 2\overline{\overline{l}}_q + \overline{\overline{n}}_q$ and for each $i = \overline{1,m},\ j = \overline{1,n},\ \mu = \overline{1,n},\ k = \overline{1,\min\{i,r\}},\ l = \overline{1,\min\{\mu,r\}},\ \kappa = \overline{1,m},\ \lambda = \overline{1,n}$ evaluate

$$\overline{\beta}_{t,i,j,k,l,\mu,\kappa,\lambda} = \sum_{t_1=0}^{t} \overline{p}_{t_1,i,k,l,\mu} \alpha_{t-t_1,j,\mu,\kappa,\lambda}, \quad 0 \le t \le \overline{b}_q. \tag{24}$$

7: For $i = \overline{1,m}$ calculate the denominator polynomials of the element $A^{\dagger}_{i,j}$ as

$$\overline{\overline{\Gamma}}_i(x) = PolynomialLCM\left\{\sum_{t=0}^{\overline{\overline{b}}_q} \overline{\overline{\beta}}_{t,i,k,l,\mu}x^t \Big| \mu = \overline{1,n},\ k = \overline{1,\min\{i,r\}},\ l = \overline{1,\min\{\mu,r\}}\right\}, \tag{25}$$

and denote it by $\overline{\overline{\Gamma}}_i(x) = \sum_{t=0}^{\overline{\overline{\Gamma}}_q} \overline{\overline{\gamma}}_{t,i}x^t$.

8: For each $i = \overline{1,m}$, $\mu = \overline{1,n}$, $k = \overline{1,\min\{i,r\}}$, $l = \overline{1,\min\{\mu,r\}}$ evaluate the following polynomial: $\overline{\overline{\Gamma}}_i(x)/\left(\sum_{t=0}^{\overline{\overline{b}}_q} \overline{\overline{\beta}}_{t,i,k,l,\mu}x^t\right)$, and denote it as $\Gamma_{i,l,k,\mu}(x) = \sum_{t=0}^{\overline{\overline{\Gamma}}_q-\overline{\overline{b}}_q} \gamma_{t,i,k,l,\mu}x^t$.

9: For $i = \overline{1,m}$, $j = \overline{1,n}$ calculate the numerator polynomials:

$$\overline{\Gamma}_{ij}(x) = \sum_{t=0}^{\overline{\overline{\Gamma}}_q-\overline{\overline{b}}_q+\overline{b}_q} \left(\sum_{\mu=1}^{n} \sum_{l=1}^{\min\{\mu,r\}} \sum_{k=1}^{\min\{i,r\}} \sum_{\lambda=1}^{n} \sum_{\kappa=1}^{m} \sum_{t_1=0}^{t} \overline{\beta}_{t_1,i,j,k,l,\mu,\kappa,\lambda}\gamma_{t-t_1,i,k,l,\mu}\right)x^t. \tag{26}$$

10: For $i = \overline{1,m}$, $j = \overline{1,n}$ set the (i,j)-th element of the generalized inverse matrix $A^{\dagger}$ to $\overline{\Gamma}_{ij}(x)/\overline{\overline{\Gamma}}_i(x)$.

LDL^* decomposition is of the same complexity as the Cholesky decomposition. Notice that LDL^* decomposition produces one more diagonal matrix, but returns square root free results, more preferable for symbolic computations. Also, the total number of non-zero entries is the same as for Cholesky decomposition. Let us observe that an arbitrary, (i,j)-th element of the matrix $(A^*A)^*(A^*A)$ can be evaluated as $\sum_{l=1}^{n}\sum_{k=1}^{m}\sum_{k'=1}^{m} a^*_{kl}a_{ki}a^*_{k'l}a_{k'j}$. These polynomials are the input of Algorithm 3.1, used for the LDL^* decomposition in Step 1. The similar idea is carried out in Step 3, when determining input of Algorithm 3.2 from [12].

4. Illustrative examples

In the next few examples we will examine our algorithm and then test different implementations and approaches in order to compare processor times for the set of test matrices.

Example 4.1. Consider the symmetric polynomial matrix of the rank 2 from [13]:

$$S_3 = \begin{bmatrix} 1+x & x & 1+x \\ x & -1+x & x \\ 1+x & x & 1+x \end{bmatrix}.$$

For $j = 1$ we set $d_{11} = 1 + x$, and therefore $l_{21}(x) = \frac{x}{1+x}$, $l_{31}(x) = \frac{1+x}{1+x} = 1$. For $j = 2$ we have that $f_{22}(x) = \frac{x^2}{1+x}$, $f_{32}(x) = x$. According to these results, it is satisfied: $d_{22}(x) = -\frac{1}{1+x}$, $l_{32}(x) = \frac{1}{d_{22}(x)}(x - f_{32}(x)) = 0$, and therefore, the following rational matrices are generated:

$$L(x) = \begin{bmatrix} 1 & 0 \\ \frac{x}{1+x} & 1 \\ 1 & 0 \end{bmatrix}, \quad D(x) = \begin{bmatrix} 1+x & 0 \\ 0 & -\frac{1}{1+x} \end{bmatrix}.$$

Example 4.2. Observe the polynomial matrix from the previous example. In order to evaluate the entries of $A^{\dagger}$, the LDL^* decomposition of the matrix $(S_3^*S_3)^*S_3^*S_3$ is determined by the direct computation of entries of matrices $L \in \mathbf{C}(x)^{3\times 2}$ and $D \in \mathbf{C}(x)^{2\times 2}$ according to Theorem 2.1. Therefore,

$$L = \begin{bmatrix} 1 & 0 \\ \frac{5x+21x^2+27x^3+27x^4}{8+32x+57x^2+54x^3+27x^4} & 1 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 8+32x+57x^2+54x^3+27x^4 & 0 \\ 0 & \frac{8}{8+32x+57x^2+54x^3+27x^4} \end{bmatrix}.$$

After performing the transformation of the matrix L^*LDL^*L and by applying the Algorithm 3.2 from [12] we get:

$$N = (L^*LDL^*L)^{-1} = \begin{bmatrix} \frac{1}{32}\left(1 - 4x + 12x^2 + 27x^4\right) & \frac{-85x-657x^2-2349x^3-5265x^4-7695x^5-8019x^6-5103x^7-2187x^8}{256+1024x+1824x^2+1728x^3+864x^4} \\ \frac{-85x-657x^2-2349x^3-5265x^4-7695x^5-8019x^6-5103x^7-2187x^8}{256+1024x+1824x^2+1728x^3+864x^4} & \frac{2048+24576x+142905x^2+532782x^3+1420335x^4+2858328x^5+4466826x^6+5484996x^7+5288166x^8+3936600x^9+2184813x^{10}+826686x^{11}+177147x^{12}}{2048+16384x+61952x^2+144384x^3+228384x^4+252288x^5+191808x^6+93312x^7+23328x^8} \end{bmatrix}$$

Computation of the coefficients from Steps 4-6 is easily processed, as well as the evaluation of the polynomial Least Common Multiplier in Step 7, but notice that the simplification is of essential importance in Step 8, upon which the coefficients $\gamma_{t,i,k,l,\mu}$, $i = \overline{1,3}$, $k = \overline{1,2}$, $l = \overline{1,3}$, $\mu = \overline{1,3}$ are computed. Finally, the generalized inverse matrix

$$S_3^\dagger = \begin{bmatrix} \frac{1-x}{4} & \frac{x}{2} & \frac{1-x}{4} \\ \frac{x}{2} & -1-x & \frac{x}{2} \\ \frac{1-x}{4} & \frac{x}{2} & \frac{1-x}{4} \end{bmatrix}$$

is obtained after performing the simplification of each entry of the fraction form $\overline{\Gamma}_{ij}(x)/\overline{\overline{\Gamma}}_i(x)$, $i = \overline{1,3}$, $j = \overline{1,3}$ by computing the greatest common divisor of each numerator and denominator pair.

Example 4.3. Observe the following 4×3 polynomial matrix A_3 from [13]:

$$A_3 = \begin{bmatrix} 3+x & 2+x & 1+x \\ 2+x & 1+x & x \\ 1+x & x & -1+x \\ x & -1+x & -2+x \end{bmatrix}$$

Since the rank of A_3 is equal to 2, our full-rank LDL^* decomposition of the matrix $(A_3^*A_3)^*A_3^*A_3$ produces matrices $L \in \mathbf{C}(x)^{3\times 2}$ and $D \in \mathbf{C}(x)^{2\times 2}$ with the following entries:

$$L = \begin{bmatrix} 1 & 0 \\ \frac{21+38x+37x^2+18x^3+6x^4}{33+60x+52x^2+24x^3+6x^4} & 1 \\ \frac{9+16x+22x^2+12x^3+6x^4}{33+60x+52x^2+24x^3+6x^4} & 2 \end{bmatrix}, \; D = \begin{bmatrix} 264 + 480x + 416x^2 + 192x^3 + 48x^4 & 0 \\ 0 & \frac{300}{33+60x+52x^2+24x^3+6x^4} \end{bmatrix}.$$

By applying Algorithm 3.1 to the matrices A_3, L and D, we get the following Moore-Penrose inverse matrix of A_3:

$$A_3^\dagger = \begin{bmatrix} -\frac{3}{20}(-1+x) & \frac{1}{60}(8-3x) & \frac{1}{60}(7+3x) & \frac{1}{20}(2+3x) \\ \frac{1}{10} & \frac{1}{30} & -\frac{1}{30} & -\frac{1}{10} \\ \frac{1}{20}(1+3x) & \frac{1}{60}(-4+3x) & \frac{1}{60}(-11-3x) & -\frac{3}{20}(2+x) \end{bmatrix}.$$

Remark. The proposed method in this work is quite fast, however it is not the fastest one. Notice that rank deficient matrices are processed faster than full-rank matrices of the same size, which is a result of the smaller dimensions of the matrices L and D used by the Algorithm 3.1. However, processor times rapidly grow with the increase of the matrix sizes and densities. Computing the inverse of a general matrix is computationally expensive (an $O(n^3)$ problem), and very sensitive to ill-conditioned matrices.

5. Conclusion

Explicit formulas for the evaluation of the coefficients in factorization matrices $L(x)$ and $D(x)$ were derived. According to this result and the method for computing generalized inverse of the given rational matrix, introduced in [6], we developed a method and an algorithm for the direct calculation of the coefficients occurring in the entries of the generalized inverse matrix $A^\dagger$, where A is a polynomial matrix. We showed that this algorithm is very efficient and suitable for the implementation in procedural languages.

A motivation for future research is to extend and generalize these results to the case of rational matrices, the 2-variable and the general n-variable case.

6. Appendix

We report the MATHEMATICA implementation of the Algorithm 3.1 as the additional information.

```
LDLGInverse[A_List] := Module[{t, i, j, k, l, mu, m = Length[A], n = Length[A[[1]]],
   r = MatrixRank[A], L, D, N, GInv, f, p, beta1, beta2, alpha, kap, lam, r1},
  {L, D} = LDLDecomposition[Conjugate[Transpose[A]].A.Conjugate[Transpose[A]].A];
  Print[m, n, r];
  L = ExpandDenominator[ExpandNumerator[L]];
  D = ExpandDenominator[ExpandNumerator[Together[D]]];
  N = Simplify[Inverse[Conjugate[Transpose[L]].L.D.Conjugate[Transpose[L]].L]];
  N = ExpandDenominator[ExpandNumerator[N]]; Print[N // MatrixForm];
  p = Table[0,
    {2 * Max[Exponent[L, x]] + Max[Exponent[N, x]] + 1}, {m + 1}, {r + 1}, {r + 1}, {n + 1}];
  alpha = Table[0, {3 * Max[Exponent[A, x]] + 35}, {n + 1}, {n + 1}, {m + 1}, {n + 1}];
  beta1 = Table[0, {m}, {n}, {r}, {r}, {n}, {m}, {n}]; beta2 = Table[0, {m}, {r}, {r}, {n}];
  GInv = Table[0, {m}, {n}];
  For[i = 1, i ≤ m, i++, For[k = 1, k ≤ Min[i, r], k++,
     For[l = 1, l ≤ Min[mu, r], l++, For[mu = 1, mu ≤ n, mu++,
        beta2[[i, k, l, mu]] = 0;
        For[t = 0, t ≤ Max[Exponent[L[[i, k]], x]] +
            Max[Exponent[N[[k, l]], x]] + Max[Exponent[L[[mu, l]], x]], t++,
          p[[t + 1, i, k, l, mu]] = ∑_{t1=0}^{t} ∑_{t2=0}^{t-t1} (Coefficient[Numerator[L[[i, k]]], x, t1] *
               Coefficient[Numerator[N[[k, l]]], x, t - t1 - t2] *
               Conjugate[Coefficient[Numerator[L[[mu, l]]], x, t2]]);
          beta2[[i, k, l, mu]] += x^t * ∑_{t1=0}^{t} ∑_{t2=0}^{t-t1} (Coefficient[Denominator[L[[i, k]]],
                 x, t1] * Coefficient[Denominator[N[[k, l]]], x, t - t1 - t2] *
                Conjugate[Coefficient[Denominator[L[[mu, l]]], x, t2]]);];];];];];
  For[j = 1, j ≤ n, j++, For[mu = 1, mu ≤ n, mu++, For[kap = 1, kap ≤ m,
       kap++, For[lam = 1, lam ≤ n, lam++,
         For[t = 0, t ≤ 3 * Max[Exponent[A, x]], t++,
           alpha[[t + 1, j, mu, kap, lam]] = ∑_{t1=0}^{t} ∑_{t2=0}^{t-t1} (Coefficient[A[[kap, lam]], x, t1] *
                 Conjugate[Coefficient[A[[kap, mu]], x, t - t1 - t2]] *
                 Conjugate[Coefficient[A[[j, lam]], x, t2]]);];];];];];
  For[i = 1, i ≤ m, i++, For[j = 1, j ≤ n, j++, For[k = 1, k ≤ Min[i, r],
       k++, For[l = 1, l ≤ Min[mu, r], l++,
         For[mu = 1, mu ≤ n, mu++, For[kap = 1, kap ≤ m, kap++,
             For[lam = 1, lam ≤ n, lam++,
               For[t = 0, t ≤ 2 * Max[Exponent[Numerator[L], x]] + Max[Exponent[
                       Numerator[N], x]] + 3 * Max[Exponent[Numerator[A], x]], t++,
                 beta1[[i, j, k, l, mu, kap, lam]] += x^t * ∑_{t1=0}^{t} (p[[t1 + 1, i, k, l,
                           mu]] * alpha[[t - t1 + 1, j, mu, kap, lam]]);];];];];];];];];
  For[i = 1, i ≤ m, i++, For[j = 1, j ≤ n, j++, For[mu = 1, mu ≤ n, mu++,
       For[l = 1, l ≤ Min[mu, r], l++,
         For[k = 1, k ≤ Min[i, r], k++,
           br = beta2[[i, k, l, mu]] /. x → 1;
           If[br ≠ 0, For[lam = 1, lam ≤ n, lam++,
              For[kap = 1, kap ≤ m, kap++,
                GInv[[i, j]] +=
                  Simplify[beta1[[i, j, k, l, mu, kap, lam]] / beta2[[i, k, l, mu]]];];];];];];];];
  Return[Simplify[GInv]];];
```

```
LDLDecomposition[A_List] := Module[{i, j, k, n = MatrixRank[A], m = Length[A], L, D},
  L = Table[0, {m}, {n}]; D = Table[0, {n}, {n}];
  For[j = 1, j ≤ n, j++, L[[j,j]] = 1; D[[j,j]] = Simplify[A[[j,j]] - Sum[(L[[j,k]])^2 D[[k,k]], {k, 1, j-1}]];
   For[i = j + 1, i ≤ m, i++, L[[i,j]] = Simplify[1/D[[j,j]] (A[[i,j]] - Sum[L[[i,k]] L[[j,k]] D[[k,k]], {k, 1, j-1}])]]];
  Return[{L, D}]]
```

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