

On the Structure of Bidegreed Graphs with Minimal Spectral Radius

Francesco Belardo^a

^aDepartment of Mathematics and Computer Science, University of Messina, 98 166 - Sant'Agata, Messina, Italy

Abstract. A graph is said to be (Δ, δ) -bidegreed if vertices all have one of two possible degrees: the maximum degree Δ or the minimum degree δ , with $\Delta \neq \delta$. We show that in the set of connected $(\Delta, 1)$ -bidegreed graphs, other than trees, with prescribed degree sequence, the graphs minimizing the adjacency matrix spectral radius cannot have vertices adjacent to $\Delta - 1$ vertices of degree 1, that is, there are not any hanging trees. Further we consider the limit point for the spectral radius of $(\Delta, 1)$ -bidegreed graphs when degree Δ vertices are inserted in each edge between any two degree Δ vertices.

1. Introduction

Let G be a simple graph (i.e. multi edges and loops are not allowed), and $V(G)$ ($E(G)$) its vertex (resp. edge) set; $|V(G)|$ ($|E(G)|$) is its order (resp. size). The degree sequence (d_1, d_2, \dots, d_n) of a graph is the sequence of degrees ordered, say, in a non-increasing way (it is a collection of graph invariants). Let $\Delta = d_1$ and $\delta = d_n$, a graph is said to be (Δ, δ) -bidegreed ((Δ, δ) is omitted if clear from the context) if and only if the vertices have degree either Δ or δ ($\Delta \neq \delta$), namely in the degree sequence there are just two different values. From the degree sequence (d_1, d_2, \dots, d_n) of a graph, it is possible to deduce the order, which is equal to n , and the size which is equal to $\frac{1}{2} \sum_{i=1}^n d_i$. In the sequel we will consider only connected simple graphs. A (connected) graph of order n and size m is said to be k -cyclic if $k = m - n + 1$; if $k = 0$ then the graph is a tree, if $k = 1$ it is said to be unicyclic, and so on. The number k is often called the *cyclomatic number*. Some graphs considered in this paper are the path P_n , the cycle C_n and the double snake W_n . Some other special graphs will be defined later. Other standard notation and basic results can be found in [7].

The literature contains a large number of papers dealing with the largest eigenvalue of some matrix associated to (simple) graphs. Most results are about the largest eigenvalue of the adjacency matrix, known as *spectral radius* or *graph index*, and the graphs, under some prescribed constraints (as the order, the size, the largest degree, and so on), maximizing or minimizing such an eigenvalue. The motivation lies in the applications, since extremal graphs, w.r.t. the spectral radius, are often related to relevant structural properties (see, for example, [7]). However, most results are given in terms of graphs maximizing the spectral radius (say *maximizers*, for short), and little is known about the graphs minimizing the spectral radius (*minimizers*), with some well-known exceptions. A motivation is that most combinatorial tools well perform when we have to increase the spectral radius, and they are not so efficient when we have to reduce the largest eigenvalue.

2010 Mathematics Subject Classification. 05C50

Keywords. bidegreed graphs, semiregular graph, degree sequence, adjacency spectrum, spectral radius.

Received: 02 August 2012; Accepted: 30 August 2013

Communicated by Dragan Stevanović

Email address: fbelardo@gmail.com (Francesco Belardo)

In the last 5 years, the attention of many researchers has been attracted by the graphs with prescribed degree sequence, for which the maximizers are identified for trees [5], unicyclic graphs [4], and bicyclic graphs [11]. Tricyclic graphs are considered in the nice paper [10]. For the minimizers the story is quite different, even in the set of trees with prescribed degree sequences minimizers seem to have a random structure [6] and the problem is hard to solve. Hence, it is natural to consider a simpler variant in which the graphs are just bidegreed. Note that the term *semiregular* has been used for the meaning of bidegreed, however we use it to denote bidegreed graphs with $\delta = \Delta - 1$. Observe that for bidegreed trees and bidegreed unicyclic graphs, the least degree δ equals 1, while $\delta \geq 1$ for k -cyclic bidegreed degree sequence with $k \geq 2$. In the case of bidegreed graphs, minimizers for trees have been independently (and with different approaches) identified in both papers [3] and [6]. However no results are known in the general case, except for some preliminary results given in [2], where minimizers are identified in the case of unicyclic graphs. Here we continue to study the connected $(\Delta, 1)$ -bidegreed graphs with prescribed number of vertices of degree Δ and given cyclomatic number, and we investigate the structure of those graphs candidate to minimize the spectral radius. Hence in the sequel, if not otherwise specified, we set $\delta = 1$. Let us denote by $\mathcal{B}^k(\Delta, n)$ the set of connected $(\Delta, 1)$ -bidegreed graphs, other than trees, with n vertices of degree Δ and cyclomatic number k (so $k > 0$); the analogous set for trees is denoted by $\mathcal{T}(\Delta, n)$. Note that fixing the number of degree Δ vertices and the cyclomatic number is equivalent to fix a (bidegreed) degree sequence.

We now introduce some additional notation. The graph matrix considered here is the well-known adjacency matrix $A = (a_{i,j})$, whose entry $a_{i,j}$ is equal to 1 when the corresponding vertices i and j are adjacent, otherwise it is 0. The *characteristic polynomial* of a graph G is $\phi(G, x) = \det(xI - A(G))$, where $A(G)$ is its adjacency matrix. The roots of $\phi(G, x)$ are the *eigenvalues* of G , and they form, together with their multiplicities, the *spectrum* of G . Note, all these eigenvalues are real since $A(G)$ is symmetric, and, since adjacency matrices on the same graph are similar, they do not depend on the particular adjacency matrix chosen.

The largest eigenvalue of $A(G)$, namely the spectral radius or index of G , will be denoted by $\rho (= \rho(G))$. If G is a connected graph on n vertices, there exists a positive eigenvector corresponding to the spectral radius, usually denoted by $x = (x_1, \dots, x_n)^T$; it is called the *principal eigenvector*, or *Perron eigenvector*, however many authors use to call it simply *Perron vector* (see, for example, Chapter 8 in [12]). The following equation,

$$\rho x_i = \sum_{i \sim j} x_j \quad (i = 1, 2, \dots, n) \tag{1}$$

is called the *eigenvalue equation* (with respect to ρ) for a vertex i of G ; x_i is usually interpreted as the weight of the vertex i (with respect to x), while \sim denotes that the corresponding vertices are adjacent.

Recall that in this paper we deal with $(\Delta, 1)$ -bidegreed graphs. Any such a graph has only two kinds of vertices: vertices of degree Δ and vertices of degree 1. In view of the latter fact, we can represent a $(\Delta, 1)$ -bidegreed graph just by looking at the vertices of degree Δ . We denote the graph induced by the vertices of degree Δ as the *skeleton* of G , and denote it by $\mathcal{S}(G)$. Clearly, if in $\mathcal{S}(G)$ we decorate the vertices of degree k with $\Delta - k$ pendant vertices, then we get again the original graph G .

We call *bouquet* the subgraph consisting of a vertex of degree Δ , adjacent to at least $\Delta - 2$ pendant vertices, together with its pendant vertices; in particular a bouquet with $\Delta - 1$ vertices of degree 1 is said to be a *pendant bouquet*; v will be called the *root* of the bouquet. So we can also say that the skeleton is the graph induced by the roots of the bouquets. Let $P(\Delta, n)$ (resp. $U(\Delta, n)$) be the $(\Delta, 1)$ -bidegreed tree (resp. unicyclic graph) such that its skeleton is the path P_n (resp. cycle C_n). In Fig. 1 we depict a bidegreed tree and its skeleton.

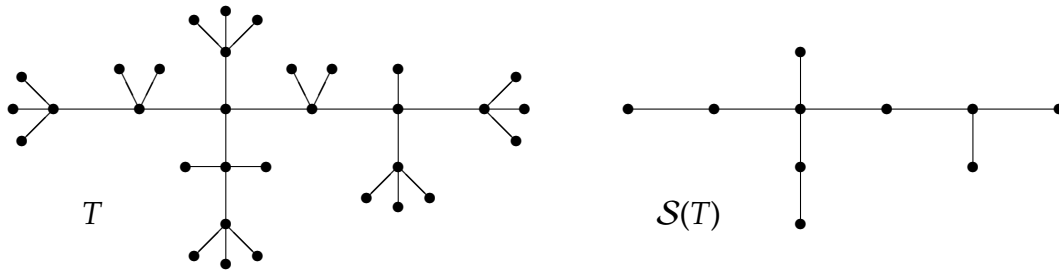


Fig. 1: A bidegreed tree and its skeleton.

In Section 2, we give a counter-example in $\mathcal{T}(\Delta, n)$ to a conjecture expressed in [2]. Still we give a new conjecture whose aim is to extend to $(\Delta, 1)$ -bidegreed graphs the well-known Hoffman-Smith graph perturbation for internal paths. Further, we will show that in $\mathcal{B}^k(\Delta, n)$, minimizers do not have any pendant bouquet, which means that in the skeleton there are not vertices of degree 1. In Section 3 we consider the limit point for the spectral radius of $(\Delta, 1)$ -bidegreed graphs when in each edge between degree Δ vertices we insert an infinite number of bouquets.

Note that some results contained in this paper have been outlined in the (non-peer reviewed) note [1]. The latter note contains the extended abstract of a contributed talk on the arguments of this paper given at the Congress “Combinatorics 2012”, held in Perugia (Italy) from 9 to 15 September 2012.

2. The skeleton of minimizers

In the literature there are several results about perturbations of graph structure and corresponding spectral radius variations, which go in the so called algebraic theory of graph perturbations (see Chapter 6 in [8]). As already observed, most of such results are about increasing the spectral radius and just a few of them are useful when looking at minimizers. One of the most important result surely is the well-known Hoffmann-Smith subdivision of an edge, by inserting a vertex of degree 2, in the *internal paths*. An internal path is a sequence of vertices v_0, v_1, \dots, v_k such that v_i is of degree 2 and both v_0 and v_k are of degree at least 3; an internal path is of Type (a) if $v_0 = v_k$, otherwise the internal path is of Type (b).

Now we state the Hoffmann-Smith result [9]:

Theorem 2.1. *Let G' be a graph obtained from a connected graph $G \neq C_n, W_n$ by inserting in an edge e a vertex of degree two. Then we have:*

- (i) *if e does not lie on an internal path then $\rho(G') > \rho(G)$;*
- (ii) *if e lies on an internal path then $\rho(G') < \rho(G)$.*

If $G = C_n, W_n$ and $G' = C_{n+1}, W_{n+1}$ then $\rho(G') = \rho(G) = 2$.

In [2] the authors considered a variant of the above result in terms of bouquets in bidegreed graphs. A *bouquet internal path* is a sequence of vertices v_0, v_1, \dots, v_k of degree Δ such that each v_1, v_2, \dots, v_{k-1} are roots of bouquets while both v_0 and v_k are not roots of bouquets. A bouquet internal path corresponds to a usual internal path in the skeleton. A bouquet internal path is of Type (a) if $v_0 = v_k$, otherwise is of Type (b). In [2] the two following results are given.

Theorem 2.2. *Let G be a connected bidegreed graph with maximum degree Δ . Let G' be a graph obtained from G by inserting a bouquet (with $\Delta - 2$ pendant vertices) in a bouquet-internal path of Type (a). Then $\rho(G) > \rho(G')$.*

Theorem 2.3. *Let U_m be the graph with minimum spectral radius in $\mathcal{B}^1(\Delta, n)$. Then $U_m = U(\Delta, n)$. Further, $\rho(U_m) = 1 + \sqrt{\Delta - 1}$.*

The authors of [2] also conjectured that Theorem 2.2 could be extended to bouquet internal paths of Type (b). The tree depicted in Fig. 2 contains a bouquet internal path of Type (b), with $v_0 = u$ and $v_k = w$, but the insertion of bouquets in that just increases the spectral radius.

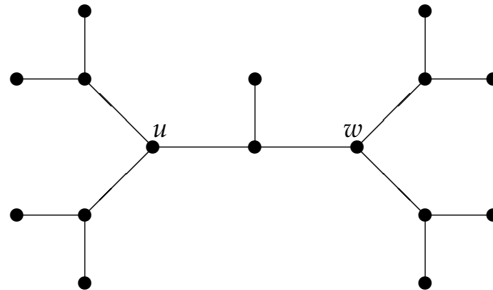


Fig. 2: The counter example for bouquet internal paths of Type (b).

In view of the above counter-example, we give the following conjecture, which covers the anomaly coming from it.

Conjecture 2.4. *Let G' be a graph obtained from a $(\Delta, 1)$ -bidegreed graph G by inserting a bouquet in an edge of a bouquet internal path. Then*

- (i) if $\rho(G) < 1 + \sqrt{\Delta - 1}$, then $\rho(G') > \rho(G)$;
- (ii) if $\rho(G) > 1 + \sqrt{\Delta - 1}$, then $\rho(G') < \rho(G)$;
- (iii) if $\rho(G) = 1 + \sqrt{\Delta - 1}$, then $\rho(G') = \rho(G)$.

We observe that if $\rho(G) < 1 + \sqrt{\Delta - 1}$, then G is a tree (see [3]).

In the sequel we will prove a weaker result w.r.t. the one described in the conjecture. The forthcoming lemmas will be useful for the latter aim.

The following lemma is well-known and it can be found in [7].

Lemma 2.5. *Let G be a connected graph and H be a vertex-deleted subgraph of G . Then $\rho(H) < \rho(G)$.*

The following lemma will play a crucial role in our proofs. A vector $x = (x_1, x_2, \dots, x_n)$ is said to be positive (i.e. $x > 0$), if $x_i \geq 0$ for $i = 1, 2, \dots, n$ and $x_j > 0$ for some $j = 1, 2, \dots, n$. Let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ be two positive vectors, then $x > y$ if and only if $x - y > 0$.

Lemma 2.6. *Let y be a positive vector and A an irreducible real symmetric matrix having μ as its largest eigenvalue. Then*

- (i) if $Ay > \rho y$ then $\mu > \rho$;
- (ii) if $Ay < \rho y$ then $\mu < \rho$.

Lemma 2.7. *Let $G \in \mathcal{B}^k(\Delta, n)$ be a graph with no pendant bouquets. If $G \neq U(\Delta, n)$, then there exists an edge e in G such that the graph G' obtained from G by inserting in e a bouquet has a smaller spectral radius. If $G = U(\Delta, n)$ and $G' = U(\Delta, n')$, then $\rho(G) = \rho(G') = 1 + \sqrt{\Delta - 1}$.*

Proof. Let v_m be the vertex of degree Δ with minimum weight in G . Without loss of generality we assume that v_m is adjacent to $(\Delta - k)$ vertices of degree 1 and k vertices of degree Δ labelled u_1, u_2, \dots, u_k , with $k = 2, 3, \dots, \Delta$. Let us denote by x the Perron vector of G , with x_m being the weight of v_m and x_1, x_2, \dots, x_k being the weights of u_1, u_2, \dots, u_k , respectively. From (1) each pendant vertex at v_m , if any, will have weight equal to x_m/ρ ,

Assume that we insert a bouquet of root v^* in the edge $v_m u_1$ (see Fig. 3), and let G' be the graph so obtained. Let G' be labelled in the same way of G , with the addition of entries corresponding to v^* and its $\Delta - 2$ pendant vertices (labelled $v_1^*, v_2^*, \dots, v_{\Delta-2}^*$). Let y be the (positive) vector defined according to the vertices of G' in the following way:

$$\begin{cases} y_v = x_v, & \text{if } v \text{ is a vertex coming from } G; \\ y_{v^*} = x_m, \\ y_{v_i^*} = \frac{x_m}{\rho}, & i = 1, 2, \dots, \Delta - 2. \end{cases}$$

Let A' be the adjacency matrix of G' , and let $y' = A'y$. According to Lemma 2.6, if we show that $y' < \rho y$, with $\rho = \rho(G)$, then we are done. Recall that G' is built from G , and y is built from the Perron vector x of G , hence $A'y|_v = \rho x_v = \rho y_v$, for each vertex v not involved in the perturbation described in this lemma, namely for each vertex $v \in G'$ coming from G such that $v \notin \{v_m, u_1\}$. So we have just to evaluate the vector $y' = A'y$ restricted to the components related to the vertices $v_m, u_1, v^*, v_1^*, v_2^*, \dots, v_{\Delta-2}^*$ of G' . Note that the vertices v_m and u_1 were adjacent in G while in G' they are not anymore adjacent, while they share v^* as common neighbor. Also we observe that, since G does not have pendant bouquets, v_m has at least two neighbors of degree Δ , namely u_1 and u_2 .

By simple computations we have:

for the vertex $u_1 \in G'$

$$y'_1 = \sum_{w \sim u_1, w \neq v^*} y_w + y_{v^*} = \sum_{w \sim u_1, w \neq v^*} x_w + x_m = \rho x_1 = \rho y_1;$$

for the vertex $v_m \in G'$

$$y'_m = \sum_{w \sim v_m, w \neq v^*} y_w + y_{v^*} = \sum_{w \sim v_m, w \neq v^*} x_w + x_m \leq \sum_{w \sim v_m, w \neq v^*} x_w + x_1 = \rho x_m = \rho y_m;$$

for the vertex $v^* \in G'$

$$\begin{aligned} y'_{v^*} &= y_1 + y_m + (\Delta - 2) \frac{y_m}{\rho} = x_1 + x_m + (\Delta - 2) \frac{x_m}{\rho} \\ &= x_1 + x_m + (k - 2) \frac{x_m}{\rho} + (\Delta - k) \frac{x_m}{\rho} \\ &\leq x_1 + x_2 + \sum_{i=3}^k x_i + (\Delta - k) \frac{x_m}{\rho} = \rho x_m = \rho y_{v^*}, \end{aligned}$$

by observing that $x_m \leq x_2$ (by minimality) and, from (1) computed at vertex u_i , we have $\rho x_i = x_m + \sum_{v \sim u_i, v \neq v_m} x_v$,

which implies $\rho x_i > x_m$ and $x_i > \frac{x_m}{\rho}$, $i = 3, 4, \dots, k$.

Finally, for each pendant vertex v_p^* at v^* ($p = 1, 2, \dots, \Delta - 2$),

$$y'_{v_p^*} = y_m = x_m = \rho y_{v_p^*}.$$

Hence we proved that $A'y \leq \rho y$ and by Lemma 2.6(ii) that $\rho(G') \leq \rho(G)$.

Assume now that $A'y = \rho y$, hence all the above inequalities are equalities and, consequently, $\rho(G') = \rho(G)$. In particular, we deduce that $x_m = x_1$, $k = 2$ and $x_2 = x_m$ as well. If so, we can compute $\rho = \rho(G') = \rho(G)$ just by the eigenvalue equation computed at v^* of G' . Hence we get $\rho x_m = 2x_m + (\Delta - 2) \frac{x_m}{\rho}$, whose largest

solution is $\rho = 1 + \sqrt{\Delta - 1}$. If G properly contains $U(\Delta, n)$ then $\rho(G) > \rho(\Delta, n) = 1 + \sqrt{\Delta - 1}$. The latter implies that $G = U(\Delta, n)$ and $G = U(\Delta, n')$.

This completes the proof. \square

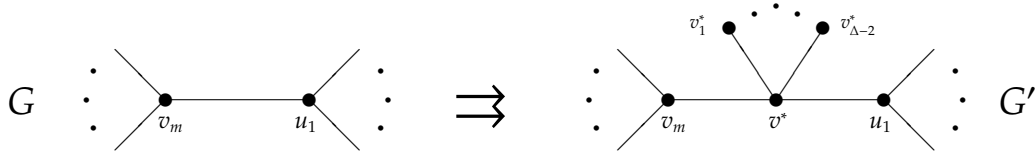


Fig. 3: The perturbation described in Lemma 2.7.

The main result of this section reads:

Theorem 2.8. *Let G_m be a bidegreed graph in $\mathcal{B}^k(\Delta, n)$ with minimal spectral radius. Then G_m does not have any pendant bouquet or, equivalently, $\mathcal{S}(G_m)$ does not have vertices of degree 1.*

Proof. We will show that for any graph $G \in \mathcal{B}^k(\Delta, n)$ with pendant bouquets, there exists a graph $\tilde{G} \in \mathcal{B}^k(\Delta, n)$ with $\rho(\tilde{G}) < \rho(G)$.

If a bidegreed graph has a pendant bouquet at v , then we can delete all vertices of degree 1 pendant at v , so that v becomes a vertex of degree 1. The graph so obtained is still bidegreed with cyclomatic number k and, according to Lemma 2.5, its spectral radius has strictly decreased. Since G has pendant bouquets, by using the latter procedure we can remove all of them, say after t steps, until we arrive at a graph $\tilde{G} \in \mathcal{B}^k(\Delta, \tilde{n})$, with $\tilde{n} < n$ such that \tilde{G} does not have any pendant bouquet. Clearly $\rho(\tilde{G}) < \rho(G)$. Now the graph \tilde{G} verifies the hypothesis of Lemma 2.7, so there exists a graph \bar{G} obtained from \tilde{G} by inserting t bouquets in some edges. The graph \bar{G} will have the same order of G and a smaller spectral radius.

This completes the proof. \square

It is worth to observe that Theorem 2.1 plays an important role in the identification of minimizers since it gives a rule for decreasing the spectral radius. We were not able to prove an analogous result to Theorem 2.1 due to the counter-example, which is given in $\mathcal{T}(\Delta, n)$, but we provide a conjecture which adapts to $(\Delta, 1)$ -bidegreed graphs. Still, we are able to exclude hanging trees in minimizers for $\mathcal{B}^k(\Delta, n)$ as a consequence of Theorem 2.8.

3. Limit points for the spectral radius

In this section we consider the limit point for the spectral radius, when we insert infinitely many bouquets in the edges coming from the skeleton of a given $(\Delta, 1)$ -bidegreed graph. Hoffmann and Smith in [9] proved that by subdividing the edges, namely a vertex of degree 2 is inserted in each edge, of a given graph infinitely many times, then the spectral radius tends to the value $\frac{\Delta}{\sqrt{\Delta-1}}$, where Δ is the largest vertex degree. Here we will consider an analogous result in which, instead of inserting vertices of degree 2, we insert bouquets between degree Δ vertices. In other words, by looking to the skeleton, we insert vertices of degree 2 in all edges of the skeleton.

In [3] the authors identified the minimizers in $\mathcal{T}(\Delta, n)$ and they proved that the minimizer is indeed $P(\Delta, n)$, the bidegreed caterpillar graph ($\mathcal{S}(P(\Delta, n)) = P_n$). The following result is Theorem 3.3 in [3].

Lemma 3.1. *Let $P(\Delta, n)$ be the minimizer in $\mathcal{T}(\Delta, n)$. Then $\rho(P(\Delta, n)) < 1 + \sqrt{\Delta - 1}$ and $\lim_{n \rightarrow \infty} \rho(P(\Delta, n)) = 1 + \sqrt{\Delta - 1}$.*

We next consider graphs whose skeleton has maximum degree at least 3. Let us consider the following lemma.

Lemma 3.2. *Let G be a $(\Delta, 1)$ -bidegreed graph with spectral radius $\rho(G) > 1 + \sqrt{\Delta - 1}$, and let $R = \rho - \frac{\Delta - 2}{\rho}$. Then $R > 2$.*

Proof. Since $\rho > 1 + \sqrt{\Delta - 1}$, then

$$R = \rho - \frac{\Delta - 2}{\rho} > 1 + \sqrt{\Delta - 1} - \frac{\Delta - 2}{1 + \sqrt{\Delta - 1}} = \frac{2 + 2\sqrt{\Delta - 1}}{1 + \sqrt{\Delta - 1}} = 2,$$

as claimed. \square

Let $T_{i,j,k}$ be the tree consisting of 3 vertex-disjoint paths P_{i+1} , P_{j+1} and P_{k+1} having one end vertex identified. Let $T(\Delta, i)$ be the $(\Delta, 1)$ -bidegreed tree whose skeleton is $T_{i,i,i}$. We now show that, for large enough i , $\rho(T(\Delta, i)) > 1 + \sqrt{\Delta - 1}$.

Lemma 3.3. *Let $T(\Delta, i)$ be the $(\Delta, 1)$ -bidegreed tree whose skeleton is $T_{i,i,i}$. If $i \geq 4$ then $\rho(T(\Delta, i)) > 1 + \sqrt{\Delta - 1}$.*

Proof. Assume first that $3 \leq \Delta \leq 5$, it is routine to check that $\rho(T(\Delta, 4)) > 1 + \sqrt{\Delta - 1}$. So in the remainder let $\Delta > 5$. Consider $T(\Delta, 3)$ and let v be the unique vertex of degree 3 in $\mathcal{S}(T(\Delta, 3))$. It is evident that all vertices of degree Δ with the sole exception of v are roots of bouquets. So consider the following equitable partition of the vertex set of $T(\Delta, 3)$. We have 8 equivalence classes, where the first one is given by v , the second one is given by the roots of bouquets at distance 1 from v , the third one is given by the roots at distance 2 from v , and the fourth one by those roots at distance 3 from v . The remaining four classes consist of the pendant vertices at each vertex of the same class. So the corresponding divisor matrix has a block form given by

$$D(T(\Delta, 3)) = \begin{pmatrix} A & B \\ I_4 & O_4 \end{pmatrix},$$

where

$$A = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$B = \text{diag}(\Delta - 3, \Delta - 2, \Delta - 2, \Delta - 1)$, I_4 is the 4×4 identity matrix and O_4 is the 4×4 matrix with zero entries.

Recall that the characteristic polynomial $\phi(D(T(\Delta, 3)), x) = \varphi(x)$ divides $\phi(T(\Delta, 3), x)$, and it contains $\rho(T(\Delta, 3))$ as well. After some computation we get

$$\begin{aligned} \varphi(x) &= x^8 + (3 - 4\Delta)x^6 + 2(\Delta - 1)(3\Delta - 4)x^4 \\ &\quad + (1 - \Delta)(4\Delta^2 - 15\Delta + 13)x^2 + (\Delta - 1)(\Delta - 3)(\Delta - 2)^2. \end{aligned}$$

It is easy to check that for any $\Delta > 5$ we have

$$\begin{aligned} \varphi(0) &= (\Delta - 1)(\Delta - 3)(\Delta - 2)^2 > 0, \\ \varphi(-1 + \sqrt{\Delta - 1}) &= -2\sqrt{\Delta - 1} - \Delta^2 + 5\Delta - 4 < 0, \\ \varphi(\sqrt{\Delta - 1}) &= (\Delta - 1)(3\Delta - 5) > 0, \\ \varphi(1 + \sqrt{\Delta - 1}) &= 2\sqrt{\Delta - 1} - \Delta^2 + 5\Delta - 4 < 0. \end{aligned}$$

Observe that $\varphi(x)$ is biquadratic, so it has just 4 non-negative roots. According to the above inequalities we deduce that the positive roots of $\varphi(x)$ lie in the real intervals $(0, 1 - \sqrt{\Delta - 1})$, $(-1 + \sqrt{\Delta - 1}, \sqrt{\Delta - 1})$, $(\sqrt{\Delta - 1}, 1 + \sqrt{\Delta - 1})$ and $(1 + \sqrt{\Delta - 1}, +\infty)$. So we can conclude that $\rho(T(\Delta, 3)) > 1 + \sqrt{\Delta - 1}$ for any $\Delta > 5$, and the same holds for $\rho(T(\Delta, i))$ with $i \geq 4$ (note $T(\Delta, 3)$ is a subgraph of $T(\Delta, i)$ for $i \geq 4$). This completes the proof. \square

We now compute the weights of the roots in paths of bouquets.

Lemma 3.4. *Let G be a $(\Delta, 1)$ -bidegreed graph with $\rho = \rho(G) > 1 + \sqrt{\Delta - 1}$. Let v_0, v_1, \dots, v_n be a sequence of vertices in G such that v_0 and v_n are vertices of degree Δ , while v_i is the root of a bouquet, for $i = 1, 2, \dots, n - 1$. If x_i is the weight of vertex v_i ($i = 0, 1, \dots, n$), then*

$$x_i = \frac{x_0 \sinh(n - i)t + x_n \sinh(it)}{\sinh nt},$$

where $t = \frac{1}{2} \ln \left(\frac{R + \sqrt{R^2 - 4}}{2} \right)$, and $R = \rho - \frac{\Delta - 2}{\rho}$.

Proof. From (1), we have, for each $i = 0, 1, \dots, n - 2$, the following relation:

$$\rho x_{i+1} = x_i + x_{i+2} + (\Delta - 2) \frac{x_{i+1}}{\rho},$$

which can be written as

$$x_{i+2} - R x_{i+1} + x_i = 0. \tag{2}$$

It is easy to see that (2) is a second order homogeneous linear recurrence equation, with characteristic equation

$$\lambda^2 - R\lambda + 1 = 0,$$

whose roots are $\lambda_{1,2} = \frac{R \pm \sqrt{R^2 - 4}}{2}$.

Since $\rho(G) > 1 + \sqrt{\Delta - 1}$, by Lemma 3.2, the roots of (2) are distinct real numbers. Note that $\lambda_2 = (\lambda_1)^{-1}$, so let $\lambda = \lambda_1 = \frac{R + \sqrt{R^2 - 4}}{2}$.

Hence the solution of (2) is given by

$$x_i = a\lambda^i + b\lambda^{-i} = a\lambda^i + b\lambda^{-i},$$

where a and b are constant that can be computed from the boundary conditions. Since $R > 2$, there exists $t > 0$ such that $R = 2 \cosh t$. Hence $t = \frac{1}{2} \ln \left(\frac{R + \sqrt{R^2 - 4}}{2} \right)$ and $\lambda = e^t$. After making the latter substitution, we obtain

$$x_i = ae^{it} + be^{-it}. \tag{3}$$

We consider the boundary conditions, that are

$$\begin{cases} x_0 = a + b \\ x_n = ae^{nt} + be^{-nt} \end{cases}$$

By solving the above system, we get $a = \frac{x_n - x_0 e^{-nt}}{e^{nt} - e^{-nt}}$ and $b = \frac{x_0 e^{nt} - x_n}{e^{nt} - e^{-nt}}$.

Finally, from (3) we get

$$\begin{aligned} x_i &= \frac{x_n - x_0 e^{-nt}}{e^{nt} - e^{-nt}} e^{it} + \frac{x_0 e^{nt} - x_n}{e^{nt} - e^{-nt}} e^{-it} \\ &= \frac{x_n e^{it} - x_0 e^{(i-n)t}}{e^{nt} - e^{-nt}} + \frac{x_0 e^{(n-i)t} - x_n e^{-it}}{e^{nt} - e^{-nt}} \\ &= \frac{x_0 (e^{(n-i)t} - e^{(i-n)t}) + x_n (e^{it} - e^{-it})}{e^{nt} - e^{-nt}} \\ &= \frac{x_0 \sinh(n - i)t + x_n \sinh(it)}{\sinh nt}, \end{aligned}$$

as claimed. This completes the proof. \square

Now we see what happens to the spectral radius when we subdivide infinitely many times the edges of the skeleton, that means, we insert infinitely many bouquets in edges between vertices of degree Δ .

Theorem 3.5. *Let $G \neq K_{1,\Delta-1}$ be a $(\Delta, 1)$ -bidegreed graph, G_n be the graph obtained from G by inserting n bouquets in each edge between two vertices of degree Δ and $\rho = \lim_{n \rightarrow \infty} \rho(G_n)$. If m is the maximum degree in $\mathcal{S}(G)$, then*

$$\rho = \frac{\sqrt{\Delta \sqrt{m-1} + m \sqrt{\Delta-1}}}{\sqrt[4]{m-1}}.$$

In addition, if $m = 2$ then $\rho = 1 + \sqrt{\Delta-1}$, and if $m = \Delta$ then $\rho = \sqrt{2\Delta}$.

Proof. Assume first that $m = 2$. If so, the graph is either the caterpillar $P(\Delta, n)$ or the unicyclic graph $U(\Delta, n)$. Since $\rho(U(\Delta, n)) = \lim_{n \rightarrow \infty} \rho(P(\Delta, n)) = 1 + \sqrt{\Delta-1}$, then we have the assertion. In the remainder of the proof let $m \geq 3$.

Consider the eigenvalue equation for the spectral radius at some vertex v of degree Δ that is not a bouquet. If so, v has $m(v) \geq 3$ neighbours v_j of degree Δ , $j = 3, 4, \dots, m(v)$. For the graph G_n , between v and v_j we have n bouquets which are a sort of path whose inner vertices are roots of bouquets and whose extremal vertices are v and v_j . For simplicity, take $\rho_n = \rho(G_n)$. For each $j = 3, 4, \dots, m(v)$, let v_j^i be the i -th root of bouquet in the j -path and let x_j^i be the corresponding weight ($i = 1, 2, \dots, n$). If we consider the eigenvalue equation for the spectral radius ρ_n at v , we have

$$\rho_n x_v = \sum_{j=1}^{m(v)} x_j^1 + (\Delta - m(v)) \frac{x_v}{\rho_n}. \tag{4}$$

Note that, for $n \geq 4$, G_n contains $T(\Delta, 4)$ which implies, in view of Lemma 3.3, that $\rho(G_n) > 1 + \sqrt{\Delta-1}$. According to Lemma 3.4,

$$x_j^1 = \frac{x_v \sinh(n-1)t + x_j \sinh(t)}{\sinh nt},$$

where t is defined according to the latter cited lemma. If $n \rightarrow \infty$, then

$$\lim_{n \rightarrow \infty} x_j^1 = e^{-t} x_v = \frac{2x_v}{R + \sqrt{R^2 - 4}}, \tag{5}$$

where $R = \rho_n - (\Delta - 2)/\rho_n$. By substituting (5) in (4), we get an equation whose largest solution is given by

$$\rho(v) = \frac{\sqrt{\Delta \sqrt{m(v)-1} + m(v) \sqrt{\Delta-1}}}{\sqrt[4]{m(v)-1}}.$$

It is not difficult to check that the above quantity is increasing in $m(v)$, so

$$\rho = \lim_{n \rightarrow \infty} \rho_n = \max_{v \in G} \rho(v).$$

If $m = \max_{v \in G} m(v)$, then we have the assertion. The additional claims can be easily verified. This completes the proof. \square

We conclude by observing that $1 + \sqrt{\Delta-1}$ is the smallest limit point for the spectral radius of $(\Delta, 1)$ -bidegreed graphs.

References

- [1] F. Belardo, Perturbations on bidegreed graphs minimizing the spectral radius, *Electron. Notes Discrete Math.* 40 (2013) 33–37.
- [2] F. Belardo, E.M. Li Marzi, Bidegreed unicyclic graphs with minimum spectral radius, *Recent Results in Designs and Graphs: a Tribute to Lucia Gionfriddo, Quaderni di Matematica, Volume 28 (2013)*, ed.s M. Buratti, C. Lindner, F. Mazzocca and N. Melone, Aracne Editrice, pp. 103–115.
- [3] F. Belardo, E.M. Li Marzi, S.K. Simić, Bidegreed trees with small index, *MATCH Comm. Math. Comput. Chem.* 61 number 2 (2009) 503–515.
- [4] F. Belardo, E.M. Li Marzi, S.K. Simić, J.F. Wang, On the spectral radius of unicyclic graphs with prescribed degree sequence, *Linear Algebra Appl.* 432 (2010) 2323–2334.
- [5] T. Biyukoglu, J. Leydold, Graphs with given degree sequence and maximal spectral radius, *Electr. J. Comb.* 15 (2008) #R119.
- [6] T. Biyukoglu, J. Leydold, Semiregular trees with minimal Laplacian spectral radius, *Linear Algebra Appl.* 432 (2010) 2335–2341.
- [7] D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs - Theory and Applications*, III revised and enlarged edition, Johan Ambrosius Bart. Verlag, Heidelberg - Leipzig, 1995.
- [8] D. Cvetković, P. Rowlinson, S. Simić, *Eigenspaces of Graphs*, Cambridge University Press, 1997.
- [9] A.J. Hoffman, J.H. Smith, On the spectral radii of topologically equivalent graphs, *Recent Advances in Graph Theory*, ed. M. Fiedler, Academia Praha, 1975, pp. 273–281.
- [10] M. Liu, The (signless Laplacian) spectral radii of connected graphs with prescribed degree sequences, *Electron. J. Combin.* 19(4) (2012) #P35.
- [11] Y. Liu, B. Liu, The spectral radius of bicyclic graphs with prescribed degree sequences, *Linear Algebra Appl.* 433 (2010) 1015–1023.
- [12] C.D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM Press, 2000.