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# Application of Fixed Point Theorems for Multivalued Maps to Anti-periodic Problems of Fractional Differential Inclusions

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**Abstract.** This paper deals with the existence and dimension of the solution set for an anti-periodic boundary value problem of fractional differential inclusions. Our results rely on Wegrzyk's fixed point theorem and a result on the topological dimension of the set of fixed points for multivalued maps.

## 1. Introduction

In this paper, we consider an anti-periodic boundary value problem of fractional differential inclusions given by

$$\begin{cases} {}^cD^q x(t) \in F(t, x(t)), \quad t \in [0, T] \ (T > 0), \ 0 < q \le 1 \\ x(0) + x(T) = 0, \end{cases} \tag{1.1}$$

where ${}^cD^q$ denote the Caputo fractional derivative of order $q$, and $F : [0, T] \times \mathbb{R}^n \to P(\mathbb{R}^n)$, where $P(\mathbb{R}^n)$ is the family of all nonempty subsets of $\mathbb{R}^n$.

The main objective of the present work is to establish the existence and dimension of the solution set for an anti-periodic boundary value problem of fractional differential inclusions by employing a method developed in recent articles ([1]-[3]).

Fractional calculus (differentiation and integration of arbitrary order) is found to be an important mathematical tool in the modeling of dynamical systems associated with phenomena such as fractal and chaos. In fact, the applications of this branch of calculus appear in various disciplines of science and engineering such as mechanics, electricity, chemistry, biology, economics, control theory, signal and image processing, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity and damping, control theory, wave propagation, percolation, identification, fitting of experimental data, etc. ([4] - [8]). For some recent works on fractional differential equations and inclusions, for instance, one can see ([9] - [17]). and the references cited therein.

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## 2. Preliminaries

Now we recall some basic definitions on multi-valued maps (see [18], [19]).

Let $\mathfrak{C}([0,T],\mathbb{R}^n)$ denote the Banach space of continuous functions from $[0,T]$ into $\mathbb{R}^n$ with the norm $\|x\|_\infty = \sup_{t\in[0,T]} \|x(t)\|$. Let $L^1([0,T],\mathbb{R}^n)$ be the Banach space of measureable functions $x : [0,T] \to \mathbb{R}^n$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^T \|x(t)\|dt$.

For a nonempty subset $C$ of a Banach space $X := (X, \|.\|)$, let $P(C) = \{Y \subseteq C : Y \neq \emptyset\}$, $P_{cl}(C) = \{Y \in P(C) : Y \text{ is closed}\}$, $P_b(C) = \{Y \in P(C) : Y \text{ is bounded}\}$, $P_{b,cl}(C) = \{Y \in P(C) : Y \text{ is bounded and closed}\}$, $P_{c,cl}(C) = \{Y \in P(C) : Y \text{is closed}$
and convex$\}$, $P_{b,c,cl}(C) = \{Y \in P(C) : Y \text{ is bounded, closed and convex}\}$, $P_{cp}(C) = \{Y \in P(C) : Y \text{ is compact}\}$, and $P_{c,cp}(C) = \{Y \in P(C) : Y \text{ is compact and convex}\}$. A multi-valued map $F : C \to P(X)$ is convex (resp. closed) valued if $F(x)$ is convex (resp. closed) for all $x \in C$. The map $F$ is bounded on bounded sets if $F(\mathbb{B}) = \cup_{x\in\mathbb{B}} F(x)$ is bounded in $X$ for all $\mathbb{B} \in P_b(C)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{\|y\| : y \in F(x)\}\} < \infty$). The map $F$ is called upper semi-continuous (u.s.c.) if $\{x \in C : Fx \subset V\}$ is open in $C$ whenever $V \subset X$ is open. $F$ is called lower semi-continuous (l.s.c.) if the set $\{y \in C : F(y) \cap V \neq \emptyset\}$ is open for any open set $V \subset X$. $F$ is called continuous if it is both l.s.c. and u.s.c. $F$ is said to be completely continuous if $F(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_b(C)$. A mapping $f : C \to X$ is called a selection of $F : C \to X$ if $f(x) \in F(x)$ for every $x \in C$. We say that the mapping $F$ has a fixed point if there is $x \in X$ such that $x \in F(x)$. The fixed points set of the multivalued operator $F$ will be denoted by $Fix(F)$. A multivalued map $F : [0,T] \to P_{cl}(\mathbb{R}^n)$ is said to be measurable if for every $y \in \mathbb{R}^n$, the function

$$t \longmapsto d(y, F(t)) = \inf\{\|y - z\| : z \in F(t)\}$$

is measurable.

**Definition 2.1.** Let $(X,d)$ be a metric space. Consider $H : P(X) \times P(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\},$$

where $d(a,B) = \inf_{b\in B} d(a,b)$. $H$ is the (generalized) Pompeiu-Hausdorff functional. It is known that $(P_{b,cl}(X), H)$ is a metric space and $(P_{cl}(X), H)$ is a generalized metric space (see [18]).

**Definition 2.2.**([20, 21]) A function $l : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a strict comparison function if it is continuous, strictly increasing and $\sum_{n=1}^{\infty} l^n(t) < \infty$, for each $t > 0$.

**Definition 2.3.** A multivalued operator $F : X \to P_{cl}(X)$ is called

(**a**) $\gamma-$Lipschitz if there exists $\gamma > 0$ such that

$$H(F(x), F(y)) \leq \gamma d(x,y) \text{ for each } x, y \in X;$$

(**b**) a contraction if and only if it is $\gamma-$Lipschitz with $\gamma < 1$;

(**c**) a generalized contraction if and only if there is a strict comparison function $l : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$H(F(x), F(y)) \leq l(d(x,y)) \text{ for each } x, y \in X.$$

It is known that $F : X \to P_{cp}(X)$ is continuous on $X$ if and only if $F$ is continuous on $X$ with respect to Hausdorff metric. Also, if $F : X \to P_{b,cl}(X)$ is $k$-Lipschitz, then $F$ is continuous with respect to Hausdorff metric.

**Definition 2.4.** [22] Let $X$ be a normed space and $C$ a nonempty subset of $X$. The closure, the convex hull and the closed convex hull of $C$ in $X$ are denoted by $\overline{C}$, $co\, C$ and $\overline{co}C$. Let $\Psi$ be a collection of subsets of $\overline{co}C$ with the property that for any $M \in \Psi$, the sets $\overline{M}$, $coM$, $M \cup \{u\}$ $(u \in C)$ and every subset of $M$ belong to $\Psi$. Let $A$ be a partially ordered set with the partial ordering $\leq$, and $\varphi : A \to A$ a function. A function $\gamma : \Psi \to A$ is said to be a $\varphi$-measure of noncompactness on $C$ if the following conditions are satisfied for any $M \in \Psi$:
(i) $\gamma(\overline{M}) = \gamma(M)$;
(ii) if $u \in C$, then $\gamma(M \cup \{u\}) = \gamma(M)$;
(iii) if $N \subset M$, then $\gamma(N) \leq \gamma(M)$;
(iv) $\gamma(coM) \leq \varphi(\gamma(M))$.
$\gamma$ is called a measure of noncompactness if instead of (iv), we have the following
(iv)* $\gamma(coM) \leq \gamma(M)$.

Let $\gamma$ be a $\varphi$-measure of noncompactness on $C$. A map $F : C \to P_{c,cl}(X)$ is said to be $(\gamma, \varphi)$-condensing if for every $S \subset C$, the inequality $\gamma(S) \leq \varphi(\gamma(F(S)))$ implies that $F(S)$ is relatively compact. In particular, if $\varphi$ is the identity map, then $F$ is called $\gamma$-condensing.

**Lemma 2.5.** (Wegrzyk's fixed point theorem [20, 21]). Let $(X, d)$ be a complete metric space. If $F : X \to P_{cl}(X)$ is a generalized contraction, then $Fix(F) \neq \emptyset$.

**Lemma 2.6.** ([1]). Let $C$ be a nonempty closed convex subset of a Banach space $X$. Suppose that $\gamma$ is a $\varphi$-measure of noncompactness on $C$ and $F : C \to P_{b,c,cl}(C)$ is a continuous $(\gamma, \varphi)$-condensing map. If $\dim F(x) \geq n$ for each $x \in C$, then $\dim Fix(F) \geq n$.

Let us recall some definitions on fractional calculus [4–6].

**Definition 2.7.** The Riemann-Liouville fractional integral of order $q$ for a continuous function $w$ is defined as

$$I^q w(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{w(s)}{(t-s)^{1-q}} ds, \quad q > 0,$$

provided the right hand side is pointwise defined on $(0, \infty)$.

**Definition 2.8.** For at least $(n-1)$–times continuously differentiable function $w : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order $q$ is defined as

$$^cD^q w(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} w^{(n)}(s) ds, \quad n-1 < q < n, n = [q]+1, \ q > 0,$$

where $[q]$ denotes the integer part of the real number $q$ and $\Gamma$ denotes the gamma function.

To define the solution of (1.1), we consider the following lemma.

**Lemma 2.9.** For a given $\sigma \in \mathfrak{C}([0, T], \mathbb{R}^n)$, the unique solution of the problem

$$\begin{cases} {}^cD^q x(t) = \sigma(t), \ t \in [0, T] \ (T > 0), \ 0 < q \leq 1 \\ x(0) + x(T) = 0, \end{cases} \tag{2.2}$$

is given by

$$x(t) = x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s) ds. \tag{2.3}$$

**Definition 2.10.** A function $x \in \mathfrak{C}([0,T],\mathbb{R}^n)$ is a solution of the problem (1.1) if there exists a function $f \in L^1([0,T],\mathbb{R}^n)$ such that $f(t) \in F(t,x(t))$ a.e. on $[0,T]$ and

$$x(t) = -\frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)ds + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds.$$

Let $S_T([0,\alpha])$ denote the set of all solutions of (1.1) on the interval $[0,\alpha]$, where $0 < \alpha \le T$.

## 3. Fractional Differential Inclusion with anti-periodic condition

**Lemma 3.1.** Assume that

**(K$_1$)** $F : [0,T] \times \mathbb{R}^n \to P_{c,cp}(\mathbb{R}^n)$ is such that $F(.,x) : [0,T] \to P_{c,cp}(\mathbb{R}^n)$ is measurable for each $x \in \mathbb{R}^n$;

**(K$_2$)** $H(F(t,x),F(t,\bar{x})) \le \kappa_1(t)l(\|x-\bar{x}\|)$ for almost all $t \in [0,1]$ and $x,\bar{x} \in \mathbb{R}^n$ with $\kappa_1 \in L^1([0,1],\mathbb{R}_+)$ and $d(0,F(t,0)) \le \kappa_1(t)$ for almost all $t \in [0,1]$, where $l : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing.

Then the problem (1.1) has at least one solution on $[0,T]$ if $\gamma l : \mathbb{R}_+ \to \mathbb{R}_+$ is a strict comparison function, where $\gamma = \dfrac{3T^{q-1}}{2\Gamma(q)}\|\kappa_1\|_{L^1}$.

**Proof.** For each $y \in \mathfrak{C}([0,T],\mathbb{R}^n)$, define the set of selections of $F$ by

$$S_{F,y} := \{v \in L^1([0,T],\mathbb{R}^n) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in [0,T]\}.$$

Suppose that $\gamma l : \mathbb{R}_+ \to \mathbb{R}_+$ is a strict comparison function. Observe that by the assumptions $(K_1)$ and $(K_2)$, $F(.,x(.))$ is measurable and has a measureable selection $v(.)$ (see Theorem III.6 [23]). Also $\kappa_1 \in L^1([0,1],\mathbb{R}_+)$ and

$$\begin{aligned}\|v(t)\| &\le d(0,F(t,0)) + H(F(t,0),F(t,x(t))) \le \kappa_1(t) + \kappa_1(t)l(\|x(t)\|)\\ &\le (1 + l(\|x\|_\infty))\kappa_1(t).\end{aligned}$$

Thus the set $S_{F,x}$ is nonempty for each $x \in \mathfrak{C}([0,T],\mathbb{R}^n)$. Define an operator $\Omega$ as

$$\Omega(x) = \left\{h \in C([0,T],\mathbb{R}^n) : h(t) = -\frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)ds + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds, f \in S_{F,x}\right\}.$$

Now we show that the operator $\Omega$ satisfies the assumptions of Lemma 2.3. To show that $\Omega(x) \in P_{cl}(\mathfrak{C}([0,T],\mathbb{R}^n))$ for each $x \in \mathfrak{C}([0,1],\mathbb{R}^n)$, let $\{u_n\}_{n\ge 0} \in \Omega(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $\mathfrak{C}([0,T],\mathbb{R}^n)$. Then $u \in \mathfrak{C}([0,T],\mathbb{R}^n)$ and there exists $v_n \in S_{F,x}$ such that, for each $t \in [0,T]$,

$$u_n(t) = -\frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v_n(s)ds + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_n(s)ds.$$

As $F$ has compact values, we pass to a subsequence to obtain that $v_n$ converges to $v$ in $L^1([0,T],\mathbb{R}^n)$. Thus, $v \in S_{F,x}$ and for each $t \in [0,T]$,

$$u_n(t) \to u(t) = -\frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v(s)ds + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v(s)ds.$$

Hence $u \in \Omega(x)$.
Next we show that

$$H(\Omega(x),\Omega(\bar{x})) \le \gamma l(\|x-\bar{x}\|_\infty) \text{ for each } x,\bar{x} \in \mathfrak{C}([0,T],\mathbb{R}^n).$$

Let $x,\bar{x} \in \mathfrak{C}([0,T],\mathbb{R}^n)$ and $h_1 \in \Omega(x)$. Then there exists $v_1(t) \in S_{F,x}$ such that, for each $t \in [0,T]$,

$$h_1(t) = -\frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v_1(s)ds + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_1(s)ds..$$

By $(K_2)$, we have

$$H(F(t,x),F(t,\bar{x})) \leq \kappa_1(t)l(\|x(t)-\bar{x}(t)\|).$$

So, there exists $w \in F(t,\bar{x}(t))$ such that

$$\|v_1(t)-w\| \leq \kappa_1(t)l(\|x(t)-\bar{x}(t)\|), \quad t \in [0,T].$$

Define $V:[0,T] \to P(\mathbb{R}^n)$ by

$$V(t)=\{w \in \mathbb{R}^n : \|v_1(t)-w\| \leq \kappa_1(t)l(\|x(t)-\bar{x}(t)\|)\}.$$

Since the nonempty closed valued operator $V(t) \cap F(t,\bar{x}(t))$ is measurable (Proposition III.4 [23]), there exists a function $v_2(t)$ which is a measurable selection for $V(t) \cap F(t,\bar{x}(t))$. So $v_2(t) \in F(t,\bar{x}(t))$ and for each $t \in [0,T]$, we have $\|v_1(t)-v_2(t)\| \leq \kappa_1(t)l(\|x(t)-\bar{x}(t)\|)$.
For each $t \in [0,T]$, let us define

$$h_2(t) = -\frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}v_2(s)ds + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}v_2(s)ds..$$

Thus

$$\|h_1(t)-h_2(t)\| \leq \frac{1}{2}\int_0^T \frac{|T-s|^{q-1}}{\Gamma(q)}\|v_1(s)-v_2(s)\|ds + \int_0^t \frac{|t-s|^{q-1}}{\Gamma(q)}\|v_1(s)-v_2(s)\|ds.$$

Hence

$$\|h_1-h_2\|_\infty \leq \frac{3T^{q-1}}{2\Gamma(q)}\|\kappa_1\|_{L^1}l(\|x-\overline{x}\|_\infty).$$

Analogously, interchanging the roles of $x$ and $\overline{x}$, we obtain

$$H(\Omega(x),\Omega(\bar{x})) \leq \gamma l(\|x-\bar{x}\|_\infty)\text{for each } x,\bar{x} \in C([0,T],\mathbb{R}^n).$$

Since $\Omega$ is a contraction, it follows by Lemma 2.5 that $\Omega$ has a fixed point $x$ which is a solution of (1.1). This completes the proof.

**Lemma 3.2.** Let $F:[0,T]\times\mathbb{R}^n \to P_{c,cp}(\mathbb{R}^n)$ satisfy $(K_1)$ and $(K_2)$ and suppose that $\Omega : \mathfrak{C}([0,T],\mathbb{R}^n) \to P(\mathfrak{C}([0,T],\mathbb{R}^n))$ is defined by

$$\Omega(x) = \left\{h \in \mathfrak{C}([0,T],\mathbb{R}^n) : h(t) = -\frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}f(s)ds + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s)ds, f \in S_{F,x}\right\}.$$

Then $\Omega(x) \in P_{c,cp}(\mathfrak{C}([0,T],\mathbb{R}^n))$ for each $x \in \mathfrak{C}(([0,T],\mathbb{R}^n))$.

**Proof.** First we show that $\Omega(x)$ is convex for each $x \in \mathfrak{C}([0,T],\mathbb{R}^n)$. For that, let $h_1,h_2 \in \Omega(x)$. Then there exist $f_1,f_2 \in S_{F,x}$ such that for each $t \in [0,T]$, we have

$$h_i(t) = -\frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}f_i(s)ds + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f_i(s)ds, \qquad i=1,2.$$

Let $0 \leq \lambda \leq 1$. Then, for each $t \in [0,T]$, we have

$$[\lambda h_1+(1-\lambda)h_2](t) = -\frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}[\lambda f_1(s)+(1-\lambda)f_2(s)]ds + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}[\lambda f_1(s)+(1-\lambda)f_2(s)]ds.$$

Since $S_{F,x}$ is convex ($F$ has convex values), therefore it follows that $\lambda h_1+(1-\lambda)h_2 \in \Omega(x)$.
Next, we show that $\Omega$ maps bounded sets into bounded sets in $\mathfrak{C}([0,T],\mathbb{R}^n)$. For a positive number $r$, let

$B_r = \{x \in \mathfrak{C}([0,T],\mathbb{R}^n) : \|x\|_\infty \leq r\}$ be a bounded set in $\mathfrak{C}([0,T],\mathbb{R}^n)$. Then, for each $h \in \Omega(x), x \in B_r$, there exists $f \in S_{F,x}$ such that

$$h(t) = -\frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)ds + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds$$

and in view of $(H_1)$, we have

$$\begin{aligned}\|h(t)\| &\leq \frac{1}{2}\int_0^T \frac{|T-s|^{q-1}}{\Gamma(q)}\|f(s)\|ds + \int_0^t \frac{|t-s|^{q-1}}{\Gamma(q)}\|f(s)\|ds \\ &\leq \frac{3T^{q-1}}{2\Gamma(q)}\int_0^T \kappa_1(s)ds.\end{aligned}$$

Thus,

$$\|h\|_\infty \leq \frac{3T^{q-1}}{2\Gamma(q)}\|\kappa_1\|_{L^1}.$$

Now we show that $\Omega$ maps bounded sets into equicontinuous sets in $\mathfrak{C}([0,T],\mathbb{R}^n)$. Let $t', t'' \in [0,T]$ with $t' < t''$ and $x \in B_r$, where $B_r$ is a bounded set in $\mathfrak{C}([0,T],\mathbb{R}^n)$. For each $h \in \Omega(x)$, we obtain

$$\begin{aligned}\|h(t'') - h(t')\| &= \Big\|\int_0^{t''} \frac{(t''-s)^{q-1}}{\Gamma(q)} f(s)ds - \int_0^{t'} \frac{(t'-s)^{q-1}}{\Gamma(q)} f(s)ds\Big\| \\ &\leq \Big\|\int_0^{t'} \frac{[(t''-s)^{q-1} - (t'-s)^{q-1}]}{\Gamma(q)} f(s)ds\Big\| + \Big\|\int_{t'}^{t''} \frac{(t''-s)^{q-1}}{\Gamma(q)} f(s)ds\Big\|.\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_{r'}$ as $t'' - t' \to 0$.. By the Arzela-Ascoli Theorem, $\Omega : \mathfrak{C}([0,T],\mathbb{R}^n) \to P(\mathfrak{C}([0,T],\mathbb{R}^n))$ is completely continuous. As in Lemma 3.1, $\Omega$ is closed-valued. Consequently, $\Omega(x) \in P_{c,cp}(\mathfrak{C}([0,T],\mathbb{R}^n))$ for each $x \in \mathfrak{C}(([0,T],\mathbb{R}^n))$.

For $0 < \alpha \leq T$, let us consider the operator

$$\Omega(x) = \left\{h \in \mathfrak{C}([0,\alpha],\mathbb{R}^n) : h(t) = -\frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)ds + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds, f \in S_{F,x}\right\}.$$

It is well-known that $Fix(\Omega) = S_{x_0}([0,\alpha])$ and, in view of Lemma 3.1, it is nonempty for each $0 < \alpha \leq T$.

The following lemma due to Dzedzej and Gelman [19] is useful in the sequel.

**Lemma 3.3.** Let $F : [0,\alpha] \to P_{c,cp}(\mathbb{R}^n)$ be a measurable map such that the Lebesgue measure $\mu$ of the set $\{t : \dim F(t) < 1\}$ is zero. Then there are arbitrarily many linearly independent measurable selections $x_1(.), x_2(.), ..., x_m(.)$ of $F$.

**Theorem 3.4.** Let $F : [0,\alpha] \times \mathbb{R}^n \to P_{c,cp}(\mathbb{R}^n)$ satisfy $(K_1)$ and $(K_2)$ and suppose that the Lebesgue measure $\mu$ of the set$\{t : \dim F(t,x) < 1$ for some $x \in \mathbb{R}^n\}$ is zero. Then for each $\alpha$, $0 < \alpha < T$, the set $S_T([0,\alpha])$ of solutions of (1.1) has an infinite dimension.

**Proof.** Define the operator $\Omega$ by

$$\Omega(x) = \left\{h \in \mathfrak{C}([0,\alpha],\mathbb{R}^n) : h(t) = -\frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)ds + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds, f \in S_{F,x}\right\}.$$

Then by Lemma 3.2, $\Omega(x) \in P_{c,cp}(\mathfrak{C}([0,\alpha],\mathbb{R}^n))$ for each $x \in \mathfrak{C}([0,\alpha],\mathbb{R}^n)$ and it is a generalized contraction by Lemma 3.1. We shall show that $\dim\Omega(x) \geq m$ for any $x \in \mathfrak{C}([0,\alpha],\mathbb{R}^n)$ and arbitrary $m \in \mathbb{N}$. Consider

$G(t) = F(t, x(t))$. By Lemma 3.3, there exist linearly independent measurable selections $x_1(.), x_2(.), ..., x_m(.)$ of $G$. Set $y_i(t) = -\frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} x_i(s)ds + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} x_i(s)ds \in \Omega(x)$. Assume that $\sum_{i=1}^m a_i y_i(t) = 0$ a.e. in $[0, \alpha]$. Taking Caputo derivatives a.e. in $[0, \alpha]$, we have $\sum_{i=1}^m a_i x_i(t) = 0$ a.e. in $[0, \alpha]$ and hence $a_i = 0$ for all $i$. As a result, $y_i(.)$ are linearly independent. Thus $\Omega(x)$ contains an $m$-dimensional simplex. So $\dim\Omega(x) \geq m$. By Lemma 2.6, $Fix(\Omega) = S_T([0, \alpha])$ is infinite dimensional.

A metric space $X$ is an AR-space if, whenever it is nonempty closed subset of another metric space $Y$, then there exists a continuous retraction $r : Y \to X$, $r(x) = x$ for $x \in X$. In particular, it is contractible (and hence connected).

**Theorem 3.5.** [24] Let $C$ be a nonempty closed convex subset of a Banach space $X$ and $F : C \to P_{c,cp}(C)$ a contraction. Then $Fix(F)$ is a nonempty AR-space.

**Remark 3.6.** Under the assumption of Theorem 3.5, $Fix(F)$ is also compact. It follows by the arguments used in the proof of Theorem 3.9 of reference [1] as $\Omega$ is condensing with respect to Hausdorff measure of noncompactness.

The following result follows from Theorem 3.4 and Theorem 3.5.

**Corollary 3.7.** Let $F : [0, \alpha] \times \mathbb{R}^n \to P_{c,cp}(\mathbb{R}^n)$ satisfy $(K_1)$ and $(K_2)$ and suppose that the Lebesgue measure $\mu$ of the set$\{t : \dim F(t, x) < 1$ for some $x \in \mathbb{R}^n\}$ is zero. Then for each $\alpha$, $0 < \alpha < T$, the set $S_T([0, \alpha])$ of solutions of (1.1) is a compact and infinite dimensional AR-space.

## References


[1] R.P. Agarwal, B. Ahmad, A. Alsaedi, N. Shahzad, Dimension of the solution set for fractional differential inclusions, J. Nonlinear Convex Anal. 14 (2013), 319-329.

[2] R.P. Agarwal, B. Ahmad, A. Alsaedi, N. Shahzad, On the dimension of the solution set for semilinear fractional differential inclusions, Abstr. Appl. Anal. Article 305924 (2012).

[3] R.P. Agarwal, B. Ahmad, A. Alsaedi, N. Shahzad, Existence and dimension of the set of mild solutions to semilinear fractional differential inclusions, Adv. Difference Equ. (2012) 2012:74.

[4] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives, Theory and Applications, Gordon and Breach, Yverdon, 1993.

[5] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[6] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo,Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[7] J. Sabatier, O.P. Agrawal, J.A.T. Machado (Eds.), Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering, Springer, Dordrecht, 2007.

[8] V. Lakshmikantham, S. Leela, J. Vasundhara Devi, Theory of Fractional Dynamic Systems, Cambridge Academic Publishers, Cambridge, 2009.

[9] G. M. N'Guerekata, A Cauchy problem for some fractional abstract differential equation with non local conditions, Nonlinear Anal.70 (2009), 1873-1876.

[10] B. Ahmad, J.J. Nieto, Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions, Comput. Math. Appl. 58 (2009), 1838-1843.

[11] B. Ahmad and N. Shahzad, A nonlocal boundary value problem for fractional differential inclusions of arbitrary order involving convex and non-convex valued maps, Veitnam J. Math. 38 (2010), 435-451.

[12] Y. Wang, L. Liu, Y. Wu, Positive solutions for a nonlocal fractional differential equation,Nonlinear Anal. 74 (2011), 3599-3605.

[13] N.J. Ford, M.L. Morgado, Fractional boundary value problems: Analysis and numerical methods, Fract. Calc. Appl. Anal. 14 (2011), 554-567.

[14] B. Ahmad, S.K. Ntouyas, Nonlinear fractional differential equations and inclusions of arbitrary order and multi-strip boundary conditions, Electron. J. Differential Equations (2012),No. 98, 22 pp.

[15] Z.B. Bai, W. Sun, Existence and multiplicity of positive solutions for singular fractional boundary value problems, Comput. Math. Appl. 63 (2012), 1369–1381.

[16] R. Sakthivel, N.I. Mahmudov, J.J. Nieto, Controllability for a class of fractional-order neutral evolution control systems, Appl. Math. Comput.218 (2012), 10334–10340.

[17] R.P. Agarwal, D. O'Regan, S. Stanek, Positive solutions for mixed problems of singular fractional differential equations, Math. Nachr. 285 (2012), no. 1, 27–41.

[18] S. Hu and N. Papageorgiou, Handbook of Multivalued Analysis, Theory I , Kluwer, Dordrecht, 1997.
[19] J. Dugundji and A. Granas, Fixed Point Theory, Springer-Verlag, New York, 2005.
[20] T. A. Lazar, A. Petrusel and N. Shahzad, Fixed points for non-self operators and domain invariance theorems, Nonlinear Anal 70(2009) 117–125.
[21] R. Wegrzyk, Fixed point theorems for multifunctions and their applications to functional equations, Dissertationes Math. (Rozprawy Mat.)201(1982) 28.
[22] I.S. Kim, Fixed point theorems of quasicompact multivalued mappings in general topological vector spaces.Bull. Korean Math. Soc38(2001),113-120.
[23] C. Castaing and M. Valadier, Convex Analysis and Measurable Multifunctions, Lecture Notes in Mathematics 580, Springer-Verlag, Berlin-Heidelberg-New York, 1977.
[24] B. Ricceri, Une proprit topologique de l'ensemble des points fixes d'une contraction multivoque valeurs convexes. (French) [A topological property of the set of fixed points of a multivalued contraction with convex values] Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.(8) 81 (1987), no. 3, 283-286.