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Approximation of Analytic Functions by Sequences of Linear Operators

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Abstract. In the present paper, we investigate approximation of analytic functions and their derivatives in a bounded simply connected domain by the sequences of linear operators without the properties of $k-$positivity.

1. Introduction

The problem of approximation of analytic functions by the sequences of linear operators was studied in several papers using the additional properties of $k-$positivity of linear operators. The concept of $k-$positivity was first introduced in [4] to obtain Korovkin-type approximation theorems in the space of complex-valued analytic functions. Using this definition of $k-$positivity in [4], some results on approximation of analytic functions in the unit disk by means of $k-$positive linear operators were obtained. Then, various approximation problems of analytic functions by $k-$positive linear operators have been studied intensively by several authors (see [1], [2], [5]-[10]). We may remark here that recently, general theorems were proved for linear operators acting on the space of analytic functions in a simply connected bounded domain in [6]. The recent works concerning $k-$positive linear operators motivated us to study the sequences of linear operators acting in the space of analytic functions without properties of $k-$positivity and investigate the problem of approximation of analytic functions and their derivatives by these type operators. Therefore, the obtained results are more general than above-mentioned results for k-positive cases.

For that purpose firstly we recall the necessary background material used throughout the paper.

Let D be a bounded simply connected domain in the complex plane and $A(D)$ denote the space of all analytic functions in D. Let $\phi(z)$ be any function mapping D conformally and one to one on the unit disk. It is known that the system of functions $\phi^k(z)$, $k = 0, 1, 2, \ldots$ is a basis in the space $A(D)$ (see [3]). Then, for every $f \in A(D)$, the Taylor expansion of f is given by

$$f(z) = \sum_{k=0}^{\infty} f_k \phi^k(z), \tag{1}$$

where f_k is the Taylor coefficients of f and satisfies

$$\limsup_{k\to\infty} \left| f_k \right|^{\frac{1}{k}} = 1. \tag{2}$$

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Note that Taylor coefficients of f are calculated by

$$f_k = \frac{1}{2\pi i}\int_C \frac{f(z)\,\phi'(z)}{\left(\phi(z)\right)^{k+1}}dz,$$

where C is any contour lying in the interior of D. The series (1) under the condition (2) is uniformly convergent if $\left|\phi(z)\right| \le r < 1$. Denoting

$$\left\|f\right\|_{A(D),r} = \max_{\left|\phi(z)\right|\le r<1}\left|f(z)\right|, \tag{3}$$

we transfer $A(D)$ in a Fréchet space with the family of norms $\|.\|_{A(D),r}$ depending on r.

Let T_n be a sequence of linear operators acting from $A(D)$ into $A(D)$. Then by (1), for any function $f \in A(D)$ we can write Taylor expansion of $T_n f(z)$ as follows:

$$T_n f(z) = \sum_{k=0}^{\infty}\phi^k(z)\sum_{m=0}^{\infty} f_m T_{k,m}^{(n)} \tag{4}$$

where $T_{k,m}^{(n)}$ is the Taylor coefficient of $T_n\phi^k(z)$ such that

$$\limsup_{k\to\infty}\left|\sum_{m=0}^{\infty} f_m T_{k,m}^{(n)}\right|^{\frac{1}{k}} = 1.$$

2. Approximation by Linear Operators

In this section, firstly we give a lemma that will be needed in the proof of our main theorems.

Lemma 2.1. *([3]) In order that the sequence $f_n(z)$ tends towards zero in $A(D)$ it is necessary and sufficient that*

$$f_n(z) = \sum_{k=0}^{\infty} f_{n,k}\phi^k(z)\ , \qquad \limsup_{k\to\infty}\left|f_{n,k}\right|^{\frac{1}{k}} = 1$$

for any n and

$$\left|f_{n,k}\right| \le \varepsilon_n (1+\delta_n)^k, \tag{5}$$

where

$$\lim_{n\to\infty}\varepsilon_n = \lim_{n\to\infty}\delta_n = 0. \tag{6}$$

Note that the example given in the work [6] shows that even the convergence of operators (4) on the counting system of basis functions $\phi^k(z)$, $k = 0, 1, 2, ...$ is not sufficient for approval of convergence of any analytic function. In this paper, we present some sufficient conditions for approximation of analytic functions belonging to the following subspace of $A(D)$.

Let $g_k \ge 1$ be an increasing sequence of real numbers, $\limsup_{k\to\infty} g_k^{\frac{1}{k}} = 1$ and let $A_g(D)$ be the subspace of functions in $A(D)$ with Taylor coefficients f_k satisfying the inequality $\left|f_k\right| \le M_f g_k$ $(k = 0, 1, 2...)$, where M_f is a constant depending only on f. Also, assume that g_k satisfies the condition

$$\limsup_{k\to\infty}\left(\sqrt{g_k} - \sqrt{g_{k-1}}\right)^{\frac{1}{k}} = 1. \tag{7}$$

In this sense we present the following results.

Theorem 2.2. *Let (T_n) be a sequence of linear operators $T_n : A(D) \to A(D)$. If there exist sequences ε_n and δ_n satisfying* (6) *such that the following inequalities*

$$\left| \sum_{m=0}^{\infty} T_{k,m}^{(n)} - 1 \right| < \varepsilon_n (1 + \delta_n)^k \tag{8}$$

$$\left| \sum_{m=0}^{\infty} \left| T_{k,m}^{(n)} \right| - 1 \right| < \varepsilon_n (1 + \delta_n)^k \tag{9}$$

$$\left| \sum_{m=0}^{\infty} \sqrt{g_m} \left| T_{k,m}^{(n)} \right| - \sqrt{g_k} \right| < \varepsilon_n (1 + \delta_n)^k \tag{10}$$

$$\left| \sum_{m=0}^{\infty} g_m \left| T_{k,m}^{(n)} \right| - g_k \right| < \varepsilon_n (1 + \delta_n)^k \tag{11}$$

hold, then for any function $f \in A_g(D)$, we have

$$\lim_{n\to\infty} \left\| T_n f(z) - f(z) \right\|_{A(D),r} = 0.$$

Remark 2.3. *We note that the conditions* (8) *and* (9) *do not contain each other and both of these conditions are essential for the proof of main results.*

Proof. Let $f \in A_g(D)$. By (1) and (4), we get

$$T_n f(z) - f(z) = \sum_{k=0}^{\infty} \phi^k(z) \sum_{m=0}^{\infty} (f_m - f_k) T_{k,m}^{(n)} + \sum_{k=0}^{\infty} \phi^k(z) f_k \left(\sum_{m=0}^{\infty} T_{k,m}^{(n)} - 1 \right).$$

Therefore, for any $r < 1$ we have the inequality

$$\left\| T_n f - f \right\|_{A(D),r} \leq \sum_{k=0}^{\infty} r^k \sum_{m=0}^{\infty} \left| f_m - f_k \right| \left| T_{k,m}^{(n)} \right| + \sum_{k=0}^{\infty} r^k \left| f_k \right| \left| \sum_{m=0}^{\infty} T_{k,m}^{(n)} - 1 \right| = S_n' + S_n''. \tag{12}$$

Using (8), we immediately obtain

$$S_n'' \leq M_f \varepsilon_n \sum_{k=0}^{\infty} r^k g_k (1 + \delta_n)^k.$$

From the conditions on g_k, the series converges for any $0 < r < 1$ since δ_n is infinitely small sequence. Hence the right-hand side of the last inequality tends to zero as $n \to \infty$ together with ε_n.

Now we estimate S_n'. By simple calculations, we can write

$$\left| f_m - f_k \right| \leq 8 M_f \frac{g_k^3}{\Delta_k^2(g)} \left(\sqrt{g_m} - \sqrt{g_k} \right)^2 \tag{13}$$

for all $m, k \in \mathbb{N}_0$ where

$$\Delta_k(g) = \min \left\{ \sqrt{g_k} - \sqrt{g_{k-1}}; \sqrt{g_{k+1}} - \sqrt{g_k} \right\}. \tag{14}$$

Also, from the inequalities (9), (10) and (11), we easily obtain

$$\begin{aligned}\sum_{m=0}^{\infty}\left(\sqrt{g_m}-\sqrt{g_k}\right)^2\left|T_{k,m}^{(n)}\right| &< \varepsilon_n\left(1+\delta_n\right)^k\left(1+\sqrt{g_k}\right)^2\\ &< 4\varepsilon_n\left(1+\delta_n\right)^k g_k.\end{aligned}$$

Writing respectively (13) and the last inequality in the term S_n', it follows

$$\begin{aligned}S_n' &= \sum_{k=0}^{\infty} r^k \sum_{m=0}^{\infty}\left|f_m-f_k\right|\left|T_{k,m}^{(n)}\right|\\ &\leq 32M_f\varepsilon_n\sum_{k=0}^{\infty} r^k\left(1+\delta_n\right)^k\frac{g_k^4}{\Delta_k^2\left(g\right)}.\end{aligned}$$

By the conditions on g_k and (7), we get that the series on the right-hand side of the last inequality converges for any $0 < r < 1$, which implies $\lim_{n\to\infty} S_n' = 0$. This completes the proof. □

Now, let us suppose that the sequence g_k has the form

$$g_k = 1 + h_k,$$

where h_k is an increasing sequence. In this case the condition (7) has the form

$$\limsup_{k\to\infty}\left(\sqrt{h_k}-\sqrt{h_{k-1}}\right)^{\frac{1}{k}} = 1. \tag{15}$$

The inequalities (10) and (11) in Theorem 2.2 take the forms

$$\left|\sum_{m=0}^{\infty}\sqrt{1+h_m}\left|T_{k,m}^{(n)}\right|-\sqrt{1+h_k}\right| < \varepsilon_n\left(1+\delta_n\right)^k$$

$$\left|\sum_{m=0}^{\infty}\left(1+h_m\right)\left|T_{k,m}^{(n)}\right|-\left(1+h_k\right)\right| < \varepsilon_n\left(1+\delta_n\right)^k.$$

We shall show that in this particular case the conditions given above may be chosen in a simple form as

$$\left|\sum_{m=0}^{\infty}\sqrt{h_m}\left|T_{k,m}^{(n)}\right|-\sqrt{h_k}\right| < \varepsilon_n\left(1+\delta_n\right)^k$$

$$\left|\sum_{m=0}^{\infty}h_m\left|T_{k,m}^{(n)}\right|-h_k\right| < \varepsilon_n\left(1+\delta_n\right)^k.$$

Then, we have the next theorem for this particular case.

Theorem 2.4. *Let (T_n) be a sequence of linear operators $T_n : A(D) \to A(D)$. If there exist sequences ε_n and δ_n satisfying* (6) *such that the inequalities* (8), (9),

$$\left|\sum_{m=0}^{\infty}\sqrt{h_m}\left|T_{k,m}^{(n)}\right|-\sqrt{h_k}\right| < \varepsilon_n\left(1+\delta_n\right)^k, \tag{16}$$

and

$$\left|\sum_{m=0}^{\infty}h_m\left|T_{k,m}^{(n)}\right|-h_k\right| < \varepsilon_n\left(1+\delta_n\right)^k \tag{17}$$

hold, then for any function $f \in A_g(D)$, *with* $g_k = 1 + h_k$ *we get*

$$\lim_{n\to\infty} \left\| T_n f(z) - f(z) \right\|_{A(D),r} = 0.$$

Proof. For $f \in A_g(D)$, we evidently have (12) so that we need estimates for S_n' and S_n''. For this purpose, using (8) we easily obtain that $\lim_{n\to\infty} S_n'' = 0$. In order to estimate S_n', we will use the same methods as in the proof of Theorem 2.2. Taking into account hypothesis on f, by making rearrangements we arrive to form

$$\left| f_m - f_k \right| \le 4M_f \frac{g_k^2}{\Delta_k^2(h)} \left(\sqrt{h_m} - \sqrt{h_k} \right)^2 \left[1 + \Delta_k^2(h) \right]$$

where $\Delta_k(h)$ is given as at the above (14). From $1 + \Delta_k^2(h) \le g_k$, it follows

$$\left| f_m - f_k \right| \le 4M_f \frac{g_k^3}{\Delta_k^2(h)} \left(\sqrt{h_m} - \sqrt{h_k} \right)^2. \tag{18}$$

On the other hand, by using (9), (16) and (17) we easily get

$$\begin{aligned} \sum_{m=0}^{\infty} \left(\sqrt{h_m} - \sqrt{h_k} \right)^2 \left| T_{k,m}^{(n)} \right| &< \varepsilon_n (1 + \delta_n)^k \left(1 + \sqrt{h_k} \right)^2 \\ &< 4\varepsilon_n (1 + \delta_n)^k g_k. \end{aligned}$$

Hence, taking into account the last inequality and (18) we immediately obtain

$$S_n' \le 16 M_f \varepsilon_n \sum_{k=0}^{\infty} r^k (1 + \delta_n)^k \frac{g_k^4}{\Delta_k^2(h)}.$$

Since by the conditions on g_k and (15) the last series is convergent, this implies $\lim_{n\to\infty} S_n' = 0$ which proves the theorem. □

Now, we will present the following general result on approximation in $A(D)$.

Theorem 2.5. *Let* b_k *be an increasing sequence of positive numbers such that* $\limsup_{k\to\infty} b_k^{\frac{1}{k}} = 1$ *and* g_k *is defined as above. If the sequence of linear operators* $T_n : A(D) \to A(D)$ *satisfies the conditions* (8),

$$\left| \sum_{m=0}^{\infty} \frac{g_m}{1 + b_m} \left| T_{k,m}^{(n)} \right| - \frac{g_k}{1 + b_k} \right| < \varepsilon_n (1 + \delta_n)^k, \tag{19}$$

$$\left| \sum_{m=0}^{\infty} \frac{\sqrt{b_m}}{1 + b_m} g_m \left| T_{k,m}^{(n)} \right| - \frac{\sqrt{b_k}}{1 + b_k} g_k \right| < \varepsilon_n (1 + \delta_n)^k \tag{20}$$

and

$$\left| \sum_{m=0}^{\infty} \frac{b_m}{1 + b_m} g_m \left| T_{k,m}^{(n)} \right| - \frac{b_k}{1 + b_k} g_k \right| < \varepsilon_n (1 + \delta_n)^k, \tag{21}$$

then for any function $f \in A_g(D)$ *we get*

$$\lim_{n\to\infty} \left\| T_n f(z) - f(z) \right\|_{A(D),r} = 0,$$

where ε_n *and* δ_n *are the same as* (6).

Proof. Assume that the inequalities (8), (19), (20) and (21) are satisfied. Let $f \in A_g(D)$. We may write the inequality (12) as

$$\left\|T_n f - f\right\|_{A(D),r} \le S_n' + S_n''.$$

Firstly we estimate the term S_n'. Taking into account $g_k \ge 1$ for $k = 0, 1, 2...$, we get

$$\left|f_m - f_k\right| \le 2M_f g_k g_m.$$

By similar reasonings with those in the proof of Theorem 2.2, we obtain

$$\left|f_m - f_k\right| \le 8M_f g_k \frac{1+b_k}{\Delta_k^2(b)} \frac{\left(\sqrt{b_m} - \sqrt{b_k}\right)^2}{1+b_m} g_m \tag{22}$$

where $\Delta_k(b)$ is given as (14). Also, using the inequalities (19), (20) and (21), it follows by simple computation the following relation

$$\sum_{m=0}^{\infty} \frac{\left(\sqrt{b_m} - \sqrt{b_k}\right)^2}{1+b_m} g_m \left|T_{k,m}^{(n)}\right| < \varepsilon_n (1+\delta_n)^k \left(1 + \sqrt{b_k}\right)^2.$$

Now using the last inequality and (22), we have

$$S_n' \le 8M_f \varepsilon_n \sum_{k=0}^{\infty} r^k (1+\delta_n)^k g_k \frac{1+b_k}{\Delta_k^2(b)} \left(1 + \sqrt{b_k}\right)^2.$$

From the conditions on g_k and b_k, we get $\lim_{n\to\infty} S_n' = 0$. Finally, applying (8) in the term S_n'', it is clear that $\lim_{n\to\infty} S_n'' = 0$. Therefore the theorem is proved. □

Here, let us note that for example in Theorem 2.5, choosing $g_k = 1 + b_k$, we get a theorem on convergence in the subspace of $A(D)$ of functions with Taylor coefficients satisfying $\left|f_k\right| \le M_f(1+b_k)$.

3. Approximation by Derivatives of Linear Operators

In this section, we will obtain similar results with those stated in the first section for simultaneous approximation by linear operators.

We first need the following auxiliary results.

Lemma 3.1. *The sequence $f_n^{(p)}(z)$ tends to zero as $n \to \infty$ in $A(D)$ if and only if*

$$\left|f_{n,k+p}\right| \le \frac{k!}{(k+p)!} \varepsilon_n (1+\delta_n)^{k+p} \tag{23}$$

for all $n \in \mathbb{N}$, $k, p \in \mathbb{N}_0$ where ε_n and δ_n are as in (6).

Proof. Let $f_n^{(p)}(z)$ tends to zero as $n \to \infty$ in $A(D)$. Considering Taylor expansion of f, we immediately get

$$f_n^{(p)}(z) = \sum_{k=0}^{\infty} (k+1)(k+2)...(k+p) f_{n,k+p} \phi^k(z). \tag{24}$$

Also, using Lemma 2.1, there exist sequences ε_n and δ_n satisfying (6) such that the inequality

$$(k+1)(k+2)...(k+p)\left|f_{n,k+p}\right| \le \varepsilon_n (1+\delta_n)^k$$

holds. This implies

$$\left|f_{n,k+p}\right| \leq \frac{k!}{(k+p)!}\varepsilon_n (1+\delta_n)^{k+p}.$$

Now, assume that (23) holds. Then by (3), we obtain that

$$\begin{aligned}\left\|f_n^{(p)}(z)\right\|_{A(D),r} &\leq \sum_{k=0}^{\infty}(k+1)\ldots(k+p)\left|f_{n,k+p}\right| r^k \\ &\leq \varepsilon_n (1+\delta_n)^p \sum_{k=0}^{\infty}(1+\delta_n)^k r^k \\ &= \varepsilon_n \frac{(1+\delta_n)^p}{1-(1+\delta_n) r}.\end{aligned}$$

Hence the proof is completed. □

Lemma 3.2. *For any $p = 0, 1, 2...$*

$$\lim_{n\to\infty}\left\|f_n^{(p)}\right\|_{A(D),r} = 0 \tag{25}$$

if and only if

$$\lim_{n\to\infty}\left\|f_n\right\|_{A(D),r} = 0. \tag{26}$$

Proof. Assume that (26) holds. Then, in view of Lemma 2.1, there exist sequences ε_n and δ_n such that the conditions (5) and (6) hold. Thus, using (24), for any $p \in \mathbb{N}_0$ we get

$$\left\|f_n^{(p)}\right\|_{A(D),r} \leq \varepsilon_n (1+\delta_n)^p \sum_{k=0}^{\infty}(k+1)\ldots(k+p)(1+\delta_n)^k r^k. \tag{27}$$

On the other hand, since for $\left|\phi(z)\right| = r < 1$

$$\frac{1}{1-\phi(z)} = \sum_{k=0}^{\infty}\phi^k(z),$$

by a simple calculation it follows that

$$\frac{p!}{\left(1-\phi(z)\right)^{p+1}} = \sum_{k=0}^{\infty}(k+1)\ldots(k+p)\phi^k(z)$$

for any $p \in \mathbb{N}_0$ with $\phi'(z) \neq 0$.

Applying the last equality in (27), we obtain

$$\left\|f_n^{(p)}\right\|_{A(D),r} \leq \varepsilon_n (1+\delta_n)^p \frac{p!}{(1-(1+\delta_n) r)^{p+1}},$$

which implies (25). □

We note that the Lemmas 3.1 and 3.2 were proved in the paper [5] for so-called statistical convergence of analytic functions in the unit disk by the sequences of $k-$positive linear operators.

Let T_n be a sequence of linear operators mapping from $A(D)$ into $A(D)$. By (4) and (24), it is easily verified that for each $p \in \mathbb{N}_0$,

$$\frac{d^p}{dz^p} T_n f(z) := T_n^{(p)} f(z) = \sum_{k=0}^{\infty} \phi^k(z)(k+1)\ldots(k+p) \sum_{m=0}^{\infty} T_{k+p,m}^{(n)} f_m.$$

Using Lemma 3.2, we can obtain some results in the case of simultaneous approximation. These results are correspondingly the corollaries of Theorems 2.2, 2.4 and 2.5.

Proposition 3.3. *Let* (T_n) *be a sequence of linear operators from* $A(D)$ *to itself. If there exist sequences* ε_n *and* δ_n *satisfying* (6) *such that the inequalities* (8), (9), (10) *and* (11) *hold, then for any function* $f \in A_g(D)$, *we have*

$$\lim_{n\to\infty} \left\| T_n^{(p)} f(z) - f^{(p)}(z) \right\|_{A(D),r} = 0.$$

Proposition 3.4. *Suppose that the hypothesis of Theorem 2.4 or Theorem 2.5 holds. Then for any function* $f \in A_g(D)$, *we have*

$$\lim_{n\to\infty} \left\| T_n^{(p)} f(z) - f^{(p)}(z) \right\|_{A(D),r} = 0.$$

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