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A p-adic Montel Theorem and Locally Polynomial Functions

J. M. Almira[a], Kh. F. Abu-Helaiel[a]

[a]*Universidad de Jaén*

Abstract. We prove a version of Montel's Theorem for the case of continuous functions defined over the field $\mathbb{Q}_p$ of p-adic numbers. In particular, we prove that, if $\Delta_{h_0}^{m+1} f(x) = 0$ for all $x \in \mathbb{Q}_p$, and h_0 satisfies $|h_0|_p = p^{-N_0}$, then, for all $x_0 \in \mathbb{Q}_p$, the restriction of f over the set $x_0 + p^{N_0}\mathbb{Z}_p$ coincides with a polynomial $p_{x_0}(x) = a_0(x_0) + a_1(x_0)x + \cdots + a_m(x_0)x^m$. Motivated by this result, we compute the general solution of the functional equation with restrictions given by

$$\Delta_h^{m+1} f(x) = 0 \quad (x \in X \text{ and } h \in B_X(r) = \{x \in X : \|x\| \leq r\}),$$

whenever $f : X \to Y$, X is an ultrametric normed space over a non-Archimedean valued field $(\mathbb{K}, |\cdot|)$ of characteristic zero, and Y is a $\mathbb{Q}$-vector space. By obvious reasons, we call these functions *uniformly locally polynomial*.

1. Motivation

Given a commutative group $(G, +)$, a nonempty set Y, and a function $f : G \to Y$, we consider the set of periods of f, $\mathfrak{P}_0(f) = \{g \in G : f(w + g) = f(w) \text{ for all } w \in G\}$. Obviously, $\mathfrak{P}_0(f)$ is always a subgroup of G and, in some special cases, these groups are well known and, indeed, have a nice structure. For example, a famous result proved by Jacobi in 1834 claims that if $f : \mathbb{C} \to \widehat{\mathbb{C}}$ is a non constant meromorphic function defined on the complex numbers, then $\mathfrak{P}_0(f)$ is a discrete subgroup of $(\mathbb{C}, +)$. This reduces the possibilities to the following three cases: $\mathfrak{P}_0(f) = \{0\}$, or $\mathfrak{P}_0(f) = \{nw_1 : n \in \mathbb{Z}\}$ for a certain complex number $w_1 \neq 0$, or $\mathfrak{P}_0(f) = \{n_1w_1 + n_2w_2 : (n_1, n_2) \in \mathbb{Z}^2\}$ for certain complex numbers w_1, w_2 satisfying $w_1w_2 \neq 0$ and $w_1/w_2 \notin \mathbb{R}$. In particular, these functions cannot have three independent periods and there exist functions $f : \mathbb{C} \to \widehat{\mathbb{C}}$ with two independent periods w_1, w_2 as soon as $w_1/w_2 \notin \mathbb{R}$. These functions are called doubly periodic (or elliptic) and have an important role in complex function theory [8]. Analogously, if the function $f : \mathbb{R} \to \mathbb{R}$ is continuous and non constant, it does not admit two $\mathbb{Q}$-linearly independent periods.

These results can be formulated in terms of functional equations since h is a period of $f : G \to Y$ if and only if f solves the functional equation $\Delta_h f(x) = 0$ $(x \in G)$. Thus, Jacobi's theorem can be formulated as a result which characterizes the constant functions as those meromorphic functions $f : \mathbb{C} \to \widehat{\mathbb{C}}$ which solve a system of functional equations of the form

$$\Delta_{h_1} f(z) = \Delta_{h_2} f(z) = \Delta_{h_3} f(z) = 0 \quad (z \in \mathbb{C}) \tag{1}$$

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Email addresses: jmalmira@ujaen.es (J. M. Almira), kabu@ujaen.es (Kh. F. Abu-Helaiel)

for three independent periods $\{h_1, h_2, h_3\}$ (i.e., $h_1\mathbb{Z} + h_2\mathbb{Z} + h_3\mathbb{Z}$ is a dense subset of $\mathbb{C}$). For the real case, the result states that, if $\dim_{\mathbb{Q}} \mathbf{span}_{\mathbb{Q}}\{h_1, h_2\} = 2$, the continuous function $f : \mathbb{R} \to \mathbb{R}$ is a constant function if and only if it solves the system of functional equations

$$\Delta_{h_1} f(x) = \Delta_{h_2} f(x) = 0 \ (x \in \mathbb{R}). \tag{2}$$

In 1937 Montel [13] proved an interesting nontrivial generalization of Jacobi's theorem. Concretely, he substituted in the equations (1), (2) above the first difference operator Δ_h by the higher differences operator Δ_h^{m+1} (which is inductively defined by $\Delta_h^{n+1} f(x) = \Delta_h(\Delta_h^n f)(x)$, $n = 1, 2, \cdots$) and proved that these equations are appropriate for the characterization of ordinary polynomials. Concretely, he proved the following result:

Theorem 1.1 (Montel). *Assume that $f : \mathbb{C} \to \mathbb{C}$ is an analytic function which solves a system of functional equations of the form*

$$\Delta_{h_1}^{m+1} f(z) = \Delta_{h_2}^{m+1} f(z) = \Delta_{h_3}^{m+1} f(z) = 0 \ \ (z \in \mathbb{C}) \tag{3}$$

for three independent periods $\{h_1, h_2, h_3\}$. Then $f(z) = a_0 + a_1 z + \cdots + a_m z^m$ is an ordinary polynomial with complex coefficients and degree $\leq m$. Furthermore, if $\{h_1, h_2\} \subset \mathbb{R}$ satisfy $\dim_{\mathbb{Q}} \mathbf{span}_{\mathbb{Q}}\{h_1, h_2\} = 2$, the continuous function $f : \mathbb{R} \to \mathbb{R}$ is an ordinary polynomial with real coefficients and degree $\leq m$ (i.e., $f(x) = a_0 + a_1 x + \cdots + a_m x^m$) if and only if it solves the system of functional equations

$$\Delta_{h_1}^{m+1} f(x) = \Delta_{h_2}^{m+1} f(x) = 0 \ (x \in \mathbb{R}). \tag{4}$$

The functional equation $\Delta_h^{m+1} f(x) = 0$ had already been introduced in the literature by M. Fréchet in 1909 as a particular case of the functional equation

$$\Delta_{h_1 h_2 \cdots h_{m+1}} f(x) = 0 \ \ (x, h_1, h_2, \ldots, h_{m+1} \in \mathbb{R}), \tag{5}$$

where $f : \mathbb{R} \to \mathbb{R}$ and $\Delta_{h_1 h_2 \cdots h_s} f(x) = \Delta_{h_1} \left(\Delta_{h_2 \cdots h_s} f\right)(x)$, $s = 2, 3, \cdots$. In particular, after Fréchet's seminal paper [3], the solutions of (5) are named "polynomials" by the Functional Equations community, since it is known that, under very mild regularity conditions on f, if $f : \mathbb{R} \to \mathbb{R}$ satisfies (5), then $f(x) = a_0 + a_1 x + \cdots a_{m-1} x^{m-1}$ for all $x \in \mathbb{R}$ and certain constants $a_i \in \mathbb{R}$. For example, in order to have this property, it is enough for f being locally bounded [3], [1], but there are stronger results [4], [11], [12]. The equation (5) can be studied for functions $f : X \to Y$ whenever X, Y are two $\mathbb{Q}$-vector spaces and the variables $x, h_1, \cdots, h_{m+1}$ are assumed to be elements of X:

$$\Delta_{h_1 h_2 \cdots h_{m+1}} f(x) = 0 \ \ (x, h_1, h_2, \ldots, h_{m+1} \in X). \tag{6}$$

In this context, the general solutions of (6) are characterized as functions of the form $f(x) = A_0 + A_1(x) + \cdots + A_m(x)$, where A_0 is a constant and $A_k(x) = A^k(x, x, \cdots, x)$ for a certain k-additive symmetric function $A^k : X^k \to Y$ (we say that A_k is the diagonalization of A^k). In particular, if $x \in X$ and $r \in \mathbb{Q}$, then $f(rx) = A_0 + rA_1(x) + \cdots + r^m A_m(x)$. Furthermore, it is known that $f : X \to Y$ satisfies (6) if and only if it satisfies

$$\Delta_h^{m+1} f(x) := \sum_{k=0}^{m+1} \binom{m+1}{k} (-1)^{m+1-k} f(x + kh) = 0 \ \ (x, h \in X). \tag{7}$$

A proof of this fact follows directly from Djoković's Theorem [2] (see also [7, Theorem 7.5, page 160], [10, Theorem 15.1.2., page 418]), which states that the operators $\Delta_{h_1 h_2 \cdots h_s}$ satisfy the equation

$$\Delta_{h_1 \cdots h_s} f(x) = \sum_{\epsilon_1, \ldots, \epsilon_s = 0}^{1} (-1)^{\epsilon_1 + \cdots + \epsilon_s} \Delta^s_{\alpha_{(\epsilon_1, \ldots, \epsilon_s)}(h_1, \cdots, h_s)} f(x + \beta_{(\epsilon_1, \ldots, \epsilon_s)}(h_1, \cdots, h_s)), \tag{8}$$

where

$$\alpha_{(\epsilon_1,\dots,\epsilon_s)}(h_1,\cdots,h_s)=(-1)\sum_{r=1}^{s}\frac{\epsilon_r h_r}{r}$$

and

$$\beta_{(\epsilon_1,\dots,\epsilon_s)}(h_1,\cdots,h_s)=\sum_{r=1}^{s}\epsilon_r h_r.$$

In section 2 of this paper we prove a version of both Jacobi's and Montel's Theorems for the case of continuous functions defined over the field $\mathbb{Q}_p$ of p-adic numbers. In particular, we prove that, if

$$\Delta_{h_0}^{m+1}f(x)=0 \quad \text{for all } x\in\mathbb{Q}_p,$$

and $|h_0|_p=p^{-N_0}$ then, for all $x_0\in\mathbb{Q}_p$, the restriction of f over the set $x_0+p^{N_0}\mathbb{Z}_p$ coincides with a polynomial $p_{x_0}(x)=a_0(x_0)+a_1(x_0)x+\cdots+a_m(x_0)x^m$. Motivated by this result, we compute, in the last section of this paper, the general solution of the functional equation with restrictions given by

$$\Delta_h^{m+1}f(x)=0 \ \ (x\in X \text{ and } h\in B_X(r)=\{x\in X:\|x\|\leq r\}), \tag{9}$$

whenever $f:X\to Y$, X is an ultrametric normed space over a non-Archimedean valued field $(\mathbb{K},|\cdot|)$ of characteristic zero (so that, it contains a copy of $\mathbb{Q}$), and Y is a $\mathbb{Q}$-vector space. By obvious reasons, we call these functions *uniformly locally polynomial*.

The definition and basic properties of $\mathbb{Q}_p$ and ultrametric normed spaces over non-Archimedean valued fields can be found, for example, in [6], [14] and [15]. In any case, we would like to stand up the fact that, if X is an ultrametric normed space over a non-Archimedean valued field $(\mathbb{K},|\cdot|)$ and $x,y\in X$ satisfy $\|x\|>\|y\|$, then $\|x+y\|=\|x\|$ (see, e.g., [14, page 22]).

2. *p*-adic Montel's Theorem

Theorem 2.1. *Let Y be a topological space with infinitely many points, and let $N\in\mathbb{Z}$. Then there are continuous functions $f:\mathbb{Q}_p\to Y$ such that*

$$\Delta_h f(x)=0\Leftrightarrow h\in p^N\mathbb{Z}_p.$$

These functions are obviously non-constant.

Proof. We know that $p^N\mathbb{Z}_p$ is an additive subgroup of $\mathbb{Q}_p$. Moreover, the quotient group $\mathbb{Q}_p/p^N\mathbb{Z}_p$ is isomorphic to the well known Prüfer group $C_{p^{(\infty)}}=\bigcup_{k=0}^{\infty}C_{p^k}$ (here, C_{p^k} denotes the cyclic group of order p^k). In particular, there exists an infinite countable set $S_N\subset\mathbb{Q}_p$ such that $\left\{s+p^N\mathbb{Z}_p\right\}_{s\in S_N}$ defines a partition of $\mathbb{Q}_p$ in clopen sets. If $\lambda:S_N\to Y$ is any inyective map, the function $f:\mathbb{Q}_p\to Y$ defined by $f(x)=\lambda(s)$ if and only if $x\in s+p^N\mathbb{Z}_p$, $s\in S_N$, satisfies our requirements. □

Lemma 2.2. *Let (Y,d) be a metric space. If $f:\mathbb{Q}_p\to Y$ is continuous and $h\in\mathfrak{P}_0(f)$, $|h|_p=p^{-N}$, then $p^N\mathbb{Z}_p\subseteq\mathfrak{P}_0(f)$. In particular, $\mathfrak{P}_0(f)$ is a clopen additive subgroup of $\mathbb{Q}_p$.*

Proof. The continuity of f implies that $\mathfrak{P}_0(f)$ is closed. Let us include, for the sake of completeness, the proof of this fact. Let $\{h_k\}\subset\mathfrak{P}_0(f)$, $\lim_{k\to\infty}h_k=h$. Then

$$\begin{aligned} 0 &\leq d(f(x+h),f(x))\leq d(f(x+h),f(x+h_k))+d(f(x+h_k),f(x))\\ &= d(f(x+h),f(x+h_k))\to 0 \ \ (\text{ for } k\to\infty), \end{aligned}$$

Hence $f(x+h)=f(x)$ for all $x\in\mathbb{Q}_p$. Thus $h\in\mathfrak{P}_0(f)$.

Take $h\in\mathfrak{P}_0(f)$, $|h|_p=p^{-N}$. Then $\overline{\{kh\}_{k=1}^{\infty}}^{\mathbb{Q}_p}=p^N\mathbb{Z}_p\subset\mathfrak{P}_0(f)$. This ends the proof. □

Corollary 2.3 (p-adic version of Jacobi's Theorem). *Let (Y,d) be a metric space. If $f:\mathbb{Q}_p\to Y$ is continuous and non-constant, then $\mathfrak{P}_0(f)=\{0\}$ or $\mathfrak{P}_0(f)=p^N\mathbb{Z}_p$ for a certain $N\in\mathbb{Z}$. In particular, the continuous function $f:\mathbb{Q}_p\to Y$ is a constant if and only if it contains an unbounded sequence of periods.*

Proof. It is well known (and easy to prove) that every proper nontrivial closed additive subgroup of $\mathbb{Q}_p$ is of the form $p^N\mathbb{Z}_p$ for a certain $N\in\mathbb{Z}$ (see, e.g., [16, p. 283, Proposition 52.3]). □

Theorem 2.4 (p-adic version of Montel's Theorem). *Let $(\mathbb{K},|\cdot|_{\mathbb{K}})$ be a valued field such that $\mathbb{Q}_p\subseteq\mathbb{K}$ and the inclusion $\mathbb{Q}_p\hookrightarrow\mathbb{K}$ is continuous. Let us assume that $f:\mathbb{Q}_p\to\mathbb{K}$ is continuous, and define*

$$\mathfrak{P}_m(f)=\{h\in\mathbb{Q}_p:\Delta_h^{m+1}f=0\}.$$

Then either $\mathfrak{P}_m(f)=\{0\}$, $\mathfrak{P}_m(f)=\mathbb{Q}_p$, or $\mathfrak{P}_m(f)=p^N\mathbb{Z}_p$ for a certain $N\in\mathbb{Z}$. Furthermore, all these cases are effectively attained by some appropriate instances of the function f. Finally, for all $a\in\mathbb{Q}_p$ there exists constants $a_0,a_1,\cdots,a_m\in\mathbb{K}$ such that $f(x)=a_0+\cdots+a_mx^m$ for all $x\in a+\mathfrak{P}_m(f)$. In particular, f is a polynomial of degree $\leq m$ if and only if $\mathfrak{P}_m(f)$ contains an unbounded sequence.

Proof. Assume $\mathfrak{P}_m(f)\neq\{0\}$. Let $h_0\in\mathfrak{P}_m(f)$, $h_0\neq 0$. Then $\Delta_{h_0}^{m+1}f(x)=0$ for all $x\in\mathbb{Q}_p$. Let $x_0\in\mathbb{Q}_p$ and let $p_0(t)\in\mathbb{K}[t]$ be the polynomial of degree $\leq m$ such that $f(x_0+kh_0)=p_0(x_0+kh_0)$ for all $k\in\{0,1,\cdots,m\}$ (this polynomial exists and it is unique, thanks to Lagrange's interpolation formula). Then

$$\begin{aligned} 0 &= \Delta_{h_0}^{m+1}f(x_0)=\sum_{k=0}^{m}\binom{m+1}{k}(-1)^{m+1-k}f(x_0+kh_0)+f(x_0+(m+1)h_0)\\ &= \sum_{k=0}^{m}\binom{m+1}{k}(-1)^{m+1-k}p_0(x_0+kh_0)+f(x_0+(m+1)h_0)\\ &= -p_0(x_0+(m+1)h_0)+f(x_0+(m+1)h_0), \end{aligned}$$

since $0=\Delta_{h_0}^{m+1}p(x_0)=\sum_{k=0}^{m+1}\binom{m+1}{k}(-1)^{m+1-k}p_0(x_0+kh_0)$. This means that $f(x_0+(m+1)h_0)=p_0(x_0+(m+1)h_0)$. In particular, $p_0=q$, where q denotes the polynomial of degree $\leq m$ which interpolates f at the nodes $\{x_0+kh_0\}_{k=1}^{m+1}$. This argument can be repeated to prove that p_0 interpolates f at all the nodes $x_0+h_0\mathbb{N}$. On the other hand, if $|h_0|_p=p^{-N}$, then $h_0\mathbb{N}$ is a dense subset of $p^N\mathbb{Z}_p$. It follows that $f_{|x_0+p^N\mathbb{Z}_p}=(p_0)_{|x_0+p^N\mathbb{Z}_p}$, since f is continuous. Thus, we have proved that the restrictions of f over the sets of the form $x_0+p^N\mathbb{Z}_p$ are polynomials of degree $\leq m$. On the other hand, we have already shown the existence an infinite countable set $S_N\subset\mathbb{Q}_p$ such that $\{s+p^N\mathbb{Z}_p\}_{s\in S_N}$ is a partition of $\mathbb{Q}_p$ in clopen sets. Hence there exists a family of polynomials $\{p_s(t)\}_{s\in S_N}\subset\mathbb{K}[t]$ such that $\deg p_s\leq m$ for all $s\in S_N$ and $f(x)=p_s(x)$ if and only if $x\in s+p^N\mathbb{Z}_p$, $s\in S_N$. Let $h\in p^N\mathbb{Z}_p$. We want to show that $h\in\mathfrak{P}_m(f)$. Now, given $x\in\mathbb{Q}_p$, there exists $s\in S_N$ such that $x+p^N\mathbb{Z}_p=s+p^N\mathbb{Z}_p$. In particular, $f_{|\{x,x+h,x+2h,\cdots,x+mh,x+(m+1)h\}}=(p_s)_{|\{x,x+h,x+2h,\cdots,x+mh,x+(m+1)h\}}$, so that $\Delta_h^{m+1}f(x)=\Delta_h^{m+1}p_s(x)=0$. This proves that $p^N\mathbb{Z}_p\subseteq\mathfrak{P}_m(f)$.

We may summarize the the arguments above by claiming that if $h_0\in\mathfrak{P}_m(f)$ and $|h_0|_p=p^{-N}$, then $p^N\mathbb{Z}_p\subseteq\mathfrak{P}_m(f)$ and

$$f(x)=p_s(x)\Leftrightarrow x\in s+p^N\mathbb{Z}_p \text{ and } s\in S_N, \tag{10}$$

where $\{p_s(t)\}_{s\in S_N}$ is a family of polynomials $p_s\in\mathbb{K}[t]$ verifying $\deg p_s\leq m$ for all $s\in S_N$, and $\{s+p^N\mathbb{Z}_p\}_{s\in S_N}$ is a partition of $\mathbb{Q}_p$. Furthermore, for any function f satisfying (10), we have that $p^N\mathbb{Z}_p\subseteq\mathfrak{P}_m(f)$.

Thus, there are just two possibilities we may consider:

Case 1: $\inf\{N\in\mathbb{Z}:p^N\mathbb{Z}_p\subseteq\mathfrak{P}_m(f)=\}-\infty$.

In this case $\mathfrak{P}_m(f)=\mathbb{Q}_p$ and f is a polynomial of degree $\leq m$.

Case 2: $\inf\{N\in\mathbb{Z}:p^N\mathbb{Z}_p\subseteq\mathfrak{P}_m(f)\}=N_0$.

In this case $\mathfrak{P}_m(f)=p^{N_0}\mathbb{Z}_p$ and f satisfies (10) with $N=N_0\in\mathbb{Z}$.

This ends the proof. □

Definition 2.5. *Given $f : \mathbb{Q}_p \to K$ a continuous function, we say that f is locally an ordinary polynomial if for each $x_0 \in \mathbb{Q}_p$ there exist a positive radius $r > 0$ and constants $a_0, a_1, \cdots, a_m \in \mathbb{K}$ such that $f(x) = a_0 + \cdots + a_m x^m$ for all $x \in x_0 + B_{\mathbb{Q}_p}(r)$. We say that f is uniformly locally an ordinary polynomial if, furthermore, the radius $r > 0$ can be chosen the same for all $x_0 \in \mathbb{Q}_p$.*

Corollary 2.6. *If $f : \mathbb{Q}_p \to K$ is continuous, then f is uniformly locally an ordinary polynomial if and only if $\mathfrak{P}_m(f) \neq \{0\}$. There are locally ordinary polynomial functions $f : \mathbb{Q}_p \to K$ such that $\mathfrak{P}_m(f) = \{0\}$.*

Proof. The first claim is an easy consequence of Theorem 2.4. To prove the existence of locally ordinary polynomials which are not uniformly locally ordinary polynomials it will be enough to construct an example. With this objective in mind, we define $f : \mathbb{Q}_p \to \mathbb{Q}_p$ as follows:

$$f(x) = \begin{cases} p^n x^m, & \text{if} \quad n \in \mathbb{N} \text{ and } x \in p^{-n} + p^n\mathbb{Z}_p \\ 0, & \text{if} \quad x \notin \bigcup_{n=0}^{\infty}(p^{-n} + p^n\mathbb{Z}_p) \end{cases}.$$

Obviously, $f(x)$ is locally an ordinary polynomial of degree $\leq m$. On the other hand, if $N \geq 1$ is a natural number, $h \in \mathbb{Q}_p$, $|h| = p^{-N}$, then $p^{-N(m+1)} + kh \notin \bigcup_{n=0}^{\infty}(p^{-n} + p^n\mathbb{Z}_p)$, $k = 1, 2, \cdots, m+1$. Hence

$$\begin{aligned} \Delta_h^{m+1} f(p^{-N(m+1)}) &= \sum_{k=0}^{m+1} \binom{m+1}{k} (-1)^{m+1-k} f(p^{-N(m+1)} + kh) \\ &= (-1)^{m+1} f(p^{-N(m+1)}) = p^{N(m+1)} p^{-Nm(m+1)} \neq 0 \end{aligned}$$

and $\mathfrak{P}_m(f) = \{0\}$. □

3. Characterization of uniformly locally polynomial functions

The results of the section above and, in particular, the p-adic Montel's Theorem and Corollary 2.6, motivate us to study, for functions $f : X \to Y$ (where X is an ultrametric normed space over a non-Archimedean valued field $(\mathbb{K}, |\cdot|)$ of characteristic zero, and Y is a $\mathbb{Q}$-vector space), the functional equation with restrictions

$$\Delta_h^{m+1} f(x) = 0 \quad (x \in X, h \in B_X(r) = \{x : \|x\| \leq r\}). \tag{11}$$

Definition 3.1. *We say that $f : X \to Y$ is an uniformly locally polynomial function if it solves the functional equation* (11) *for a certain $r > 0$.*

The best motivation for the concept above should be found in the statement of the following theorem, which is the main result of this section:

Theorem 3.2 (Characterization of uniformly locally polynomial functions). *Assume that $f : X \to Y$ satisfies* (11) *and let*

$$\phi(r, m) = r \left(\prod_{k=2}^{m+1} \max\{|1/t| : t = 1, 2, \cdots, k\} \right)^{-1}. \tag{12}$$

Then for all $x_0 \in X$ there exists a constant A_{0,x_0} and k-additive symmetric maps

$$A^{k,x_0} : B_X(\phi(r,m)) \times \cdots^{(k\ times)} \times B_X(\phi(r,m)) \to Y$$

for $k = 1, 2, \cdots, m$, such that

$$f(x_0 + z) = A_{0,x_0} + \sum_{k=1}^{m} A_{k,x_0}(z) \quad \text{for all} \;\; z \in B_X(\phi(r,m));$$

where $A_{k,x_0}(z) = A^{k,x_0}(z, z, \cdots, z)$ is the diagonalization of $A^{k,x_0}(z_1, \cdots, z_k)$, $k = 1, \cdots, m$.

Lemma 3.3. *Assume that $f : X \to Y$ satisfies* (11) *and let $\phi(r, m)$ be defined by equation* (12). *Then there exist a constant A_0 and k-additive symmetric maps*

$$A^k : B_X(\phi(r, m)) \times \cdots^{(k \text{ times})} \times B_X(\phi(r, m)) \to Y$$

for $k = 1, 2, \cdots, m$, such that

$$f(z) = A_0 + \sum_{k=1}^{m} A_k(z) \quad \text{for all } z \in B_X(\phi(r, m));$$

where $A_k(z) = A^k(z, z, \cdots, z)$ is the diagonalization of $A^k(z_1, \cdots, z_k)$, $k = 1, \cdots, m$.

Proof. Assume that $f : X \to Y$ satisfies (11), and consider the function $A^m(x_1, \cdots, x_m) = \frac{1}{m!}\Delta_{x_1 x_2 \cdots x_m} f(0)$. Then A^m is symmetric since the operators Δ_{x_i}, Δ_{x_j} commute. Furthermore, the identity $\Delta_{x+y} = \Delta_x + \Delta_y + \Delta_{xy}$ implies that

$$\begin{aligned} &A^m(x_1, \cdots, x_{k-1}, x + y, x_{k+1}, \cdots, x_m) - A^m(x_1, \cdots, x_{k-1}, x, x_{k+1}, \cdots, x_m) - A^m(x_1, \cdots, x_{k-1}, y, x_{k+1}, \cdots, x_m) \\ &= \frac{1}{m!}\left(\Delta_{x_1 \cdots x_{k-1} x_{k+1} \cdots x_m}(\Delta_{x+y} - \Delta_x - \Delta_y) f(0)\right) \\ &= \frac{1}{m!}\left(\Delta_{x_1 \cdots x_{k-1} x_{k+1} \cdots x_m xy} f(0)\right). \end{aligned}$$

If we apply Djoković's theorem to the operator $\Delta_{z_1 z_2 \cdots z_{m+1}}$, we conclude that, if

$$z_1, \cdots, z_{m+1} \in B_X\left(r / \max_{1 \le t \le m+1} |1/t|\right),$$

then

$$\|\alpha_{(\epsilon_1, \cdots, \epsilon_{m+1})}(z_1, \cdots, z_{m+1})\| = \left\|(-1)\sum_{t=1}^{m+1} \frac{\epsilon_t}{t} z_t\right\| \le \max_{1 \le t \le m+1} |1/t| \max_{1 \le t \le m+1} \|z_t\| \le r,$$

so that $\Delta_{z_1 z_2 \cdots z_{m+1}} f(x) = 0$ for all $x \in X$. Hence the application A^m is m-additive on $B_X\left(r / \max_{1 \le t \le m+1} |1/t|\right)$ and, consequently, on all its additive subgroups. In particular, it is m-additive on the balls $B_X(\rho)$ for all $\rho \le r / \max_{1 \le t \le m+1} |1/t|$.

Let us define the function

$$f_1(x) = \begin{cases} f(x) - A_m(x), & \text{if } \ x \in B_X\left(r / \max_{1 \le t \le m+1} |1/t|\right) \\ 0, & \text{if } \ x \notin B_X\left(r / \max_{1 \le t \le m+1} |1/t|\right) \end{cases},$$

where $A_m(x) = A^m(x, x, \cdots, x)$ is the diagonalization of A^m, and let us compute $\Delta_h^m f_1(x)$ for $x \in X$ and $h \in B_X\left(r / \max_{1 \le t \le m+1} |1/t|\right)$. We divide this computation into two steps:

Step 1: Assume $x \in B_X\left(r / \max_{1 \le t \le m+1} |1/t|\right)$.

In this case, $x + kh \in B_X\left(r / \max_{1 \le t \le m+1} |1/t|\right)$ for all $h \in B_X\left(r / \max_{1 \le t \le m+1} |1/t|\right)$ and all $k \in \mathbb{N}$, so that $\Delta_h^m f_1(x) = \Delta_h^m f(x) - \Delta_h^m A_m(x)$. We compute separately each summand of the second member of this identity. Obviously,

$$0 = \Delta_{hh \cdots hx} f(0) = \Delta_h^m \Delta_x f(0) = \Delta_h^m f(x) - \Delta_h^m f(0),$$

since $x, h \in B_X\left(r/\max_{1\leq t\leq m+1}|1/t|\right)$. This means that $\Delta_h^m f(x) = \Delta_h^m f(0) = m!A^m(h,\cdots,h)$. On the other hand, a direct computation shows that $\Delta_h^m A_m(x) = m!A^m(h,\cdots,h)$, which proves that

$$\Delta_h^m f_1(x) = 0 \text{ for all } x, h \in B_X\left(r/\max_{1\leq t\leq m+1}|1/t|\right).$$

Step 2: Assume $x \notin B_X\left(r/\max_{1\leq t\leq m+1}|1/t|\right)$.

Obviously, $\|x\| > \|h\| \geq \|kh\|$ for all $h \in B_X\left(r/\max_{1\leq t\leq m+1}|1/t|\right)$ and $k \in \mathbb{N}$, so that $\|x+kh\| = \|x\|$ and $\{x+kh\}_{k=0}^m \subset X \setminus B_X\left(r/\max_{1\leq t\leq m+1}|1/t|\right)$. Hence $\Delta_h^m f_1(x) = 0$ also in this case.

Thus, we have proved that

$$f(x) = f_1(x) + A_m(x) \quad \text{for all } x \in B_X\left(r/\max_{1\leq t\leq m+1}|1/t|\right),$$

and

$$\Delta_h^m f_1(x) = 0 \quad \text{for all } x \in X \text{ and } h \in B_X\left(r/\max_{1\leq t\leq m+1}|1/t|\right).$$

A repetition of the same arguments will show that $f_1(x)$ admits a decomposition $f_1(x) = f_2(x) + A_{m-1}(x)$ on the ball $B_X\left(\frac{r}{\max_{1\leq t\leq m+1}|1/t|\max_{1\leq t\leq m}|1/t|}\right)$, with A_{m-1} the diagonalization of an $(m-1)$-additive symmetric function

$$A^{m-1} : B_X\left(\frac{r}{\max_{1\leq t\leq m+1}|1/t|\max_{1\leq t\leq m}|1/t|}\right)^{m-1} \to Y,$$

and f_2 satisfying

$$\Delta_h^m f_2(x) = 0 \quad \text{for all } x \in X \text{ and } h \in B_X\left(\frac{r}{\max_{1\leq t\leq m+1}|1/t|\max_{1\leq t\leq m}|1/t|}\right).$$

The iteration of this process leads to a decomposition

$$f(x) = f_m(x) + A_1(x) + A_2(x) + \cdots + A_m(x), \quad \text{for all } x \in B_X(\phi(r,m)),$$

with $A_k(z) = A^k(z,z,\cdots,z)$ being the diagonalization of the k-additive symmetric map $A^k : B_X(\phi(r,m)) \times \cdots^{(k \text{ times})} \times B_X(\phi(r,m)) \to Y$, $k = 1,2,\cdots,m$; and $\Delta_h^1 f_m(x) = 0$ for all $x \in X$ and $h \in B_X(\phi(r,m))$. In particular, this last formula implies that, for $x \in B_X(\phi(r,m))$, $f_m(x) = f_m(0) = A_0$ is a constant. □

Proof of Theorem 3.2: Let us define, for $x_0 \in X$, the function $g(x) = f(x_0 + x)$. Then $g = \tau_{x_0}(f)$, where $\tau_{x_0}(f)(x) = f(x_0 + x)$ is a translation operator. Obviously, the operators τ_{x_0} and Δ_h commute, so that

$$\Delta_h^{m+1} g(x) = \Delta_h^{m+1}\tau_{x_0} f(x) = \tau_{x_0}\Delta_h^{m+1} f(x) = 0 \;\; (x \in X, h \in B_X(r) = \{x : \|x\| \leq r\}).$$

Hence, we can use Lemma 3.3 with g to conclude that there exist a constant A_{0,x_0} and k-additive maps $A^{k,x_0} : B_X(\phi(r,m)) \times \cdots^{(k \text{ times})} \times B_X(\phi(r,m)) \to Y$, $k = 1,2,\cdots,m$, such that

$$f(x_0 + z) = A_{0,x_0} + \sum_{k=1}^m A_{k,x_0}(z) \quad \text{for all } \; z \in B_X(r);$$

where $A_{k,x_0}(z) = A^{k,x_0}(z,z,\cdots,z)$ is the diagonalization of $A^{k,x_0}(z_1,\cdots,z_k)$, $k = 0,1,\cdots,m$. □

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