

Univalence Criteria for a General Integral Operator

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Abstract. In this paper the author introduces a general integral operator and determines conditions for the univalence of this integral operator. Also, the significant relationships and relevance with other results are also given.

1. Introduction and Preliminaries

Let \mathcal{A} be the class of functions $f(z)$ which are analytic in the open unit disk

$$\mathcal{U} = \{z : |z| < 1\} \text{ and } f(0) = f'(0) - 1 = 0.$$

We denote by \mathcal{S} the subclass of \mathcal{A} consisting of functions $f(z) \in \mathcal{A}$ which are univalent in \mathcal{U} .

Miller and Mocanu [11] have considered the integral operator M_α given by

$$M_\alpha(z) = \left\{ \frac{1}{\alpha} \int_0^z (f(u))^{\frac{1}{\alpha}} u^{-1} du \right\}^\alpha, \quad z \in \mathcal{U} \quad (1)$$

for functions $f(z)$ belonging to the class \mathcal{A} and for some complex numbers α , $\alpha \neq 0$. It is well known that $M_\alpha(z) \in \mathcal{S}$ for $f(z) \in \mathcal{S}^*$ and $\alpha > 0$, where \mathcal{S}^* denotes the subclass of \mathcal{S} consisting of all starlike functions $f(z)$ in \mathcal{U} .

In this present investigation, we introduce a general integral operator as follows:

$$\mathcal{J}_{\gamma_1, \gamma_2, \dots, \gamma_n, \beta}(z) = \left\{ \left(\sum_{i=1}^n \frac{1}{\gamma_i} - n + 1 \right) \int_0^z u^{-n\beta} \prod_{i=1}^n (f_i(u))^{\frac{1}{\gamma_i} + \beta - 1} du \right\}^{\frac{1}{\sum_{i=1}^n \frac{1}{\gamma_i} - n + 1}} \quad (2)$$

where $z \in \mathcal{U}$, $f_i \in \mathcal{A}$ and $\gamma_i, \beta \in \mathbb{C}$, ($\gamma_i \neq 0$), $i = \overline{1, n}$ which is a generalization of integral operator M_α .

From (2), for $n = 1$, $\beta = 1$, $f_1 = f$, $\gamma_1 = \alpha$, we obtain the integral operator M_α .

If $\gamma_i = \gamma$ for each $i = \overline{1, n}$, from (2) we obtain the integral operator

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$$\mathcal{J}_{n,\gamma,\beta}(z) = \left\{ \left(\frac{n(1-\gamma) + \gamma}{\gamma} \right) \int_0^z u^{-n\beta} \prod_{i=1}^n (f_i(u))^{\frac{1}{\gamma} + \beta - 1} du \right\}^{\frac{\gamma}{n(1-\gamma) + \gamma}}. \quad (3)$$

If $\beta = 0$, from (2) we obtain the integral operator define by

$$\mathcal{J}_{\gamma_1, \gamma_2, \dots, \gamma_n, 0}(z) = \left\{ \left(\sum_{i=1}^n \frac{1}{\gamma_i} - n + 1 \right) \int_0^z \prod_{i=1}^n (f_i(u))^{\frac{1}{\gamma_i} - 1} du \right\}^{\frac{1}{\sum_{i=1}^n \frac{1}{\gamma_i} - n + 1}}. \quad (4)$$

If $\beta = 1$, from (2) we obtain the integral operator

$$\mathcal{J}_{\gamma_1, \gamma_2, \dots, \gamma_n, 1}(z) = \left\{ \left(\sum_{i=1}^n \frac{1}{\gamma_i} - n + 1 \right) \int_0^z u^{-n} \prod_{i=1}^n (f_i(u))^{\frac{1}{\gamma_i}} du \right\}^{\frac{1}{\sum_{i=1}^n \frac{1}{\gamma_i} - n + 1}}. \quad (5)$$

If $n = 1$ and $f_1 = f$, from (2) we obtain the integral operator

$$\mathcal{J}_{\gamma, \beta}(z) = \left\{ \frac{1}{\gamma} \int_0^z u^{-\beta} (f(u))^{\frac{1}{\gamma} + \beta - 1} du \right\}^{\gamma}, \quad z \in \mathcal{U} \quad (6)$$

which was introduced and studied by Pescar and Breaz [17].

If $n = 1$, $f_1 = f$, $\frac{1}{\gamma_1} = 1$ and $\beta \in \mathbb{C} - \{0, 1\}$, from (2) we obtain the integral operator

$$\mathcal{J}_{\beta}(z) = \int_0^z \left(\frac{f(u)}{u} \right)^{\beta} du, \quad z \in \mathcal{U} \quad (7)$$

which was introduced Kim-Merkes [9].

From (2), for $n = 1$, $f_1 = f$, $\frac{1}{\gamma_1} = 1$ and $\beta = 1$ we obtain Alexander integral operator defined by

$$\mathcal{J}(z) = \int_0^z \frac{f(u)}{u} du. \quad (8)$$

Recently the problem of univalence of some generalized integral operators have discussed by many authors such as: (see [1]-[8], [10], [15]-[17], [19] and [20]). In our paper, we consider the general integral operator of the type (2) and obtain some sufficient conditions for this integral operator to be univalent in \mathcal{U} . In particular, we obtain simple sufficient conditions for some integral operators which involve special cases n, β and γ_i ($\gamma_i \neq 0$), $i = \overline{1, n}$ of the integral operator (2).

2. Preliminary Results

To discuss our problems for univalence of the integral operator $\mathcal{J}_{\gamma_1, \gamma_2, \dots, \gamma_n, \beta}$, we need the following lemmas.

Lemma 2.1. (Pascu [14]). Let $h(z) \in \mathcal{A}$ and $\gamma \in \mathbb{C}$. If $\Re(\gamma) > 0$ and

$$\frac{1 - |z|^{2\Re(\gamma)}}{\Re(\gamma)} \left| \frac{zh''(z)}{h'(z)} \right| \leq 1, \quad z \in \mathcal{U}$$

then the function $H_\gamma(z)$ given by

$$\mathcal{H}_\gamma(z) = \left(\gamma \int_0^z t^{\gamma-1} h'(t) dt \right)^{1/\gamma}, \quad z \in \mathcal{U} \tag{9}$$

is in the class \mathcal{S} .

Lemma 2.2. (Ozaki and Nunokawa [13]). Let $f \in \mathcal{A}$ satisfy the following condition

$$\left| \frac{z^2 f'(z)}{f^2(z)} \right| \leq 1, \quad z \in \mathcal{U}$$

then the function f is in the class \mathcal{S} .

Let $\mathcal{S}(p)$ denote the class of functions $f \in \mathcal{A}$ which satisfies the conditions

$$\begin{cases} f(z) \neq 0, & \text{for } 0 < |z| < 1 \\ \left| \left(\frac{z}{f(z)} \right)'' \right| \leq p, & \text{for } z \in \mathcal{U} \end{cases}$$

for some real p with $0 < p \leq 2$. The class $\mathcal{S}(p)$ is defined by Yang and Liu [21]. Sign [18] has shown that if $f \in \mathcal{S}(p)$, then f satisfies

$$\left| \frac{z^2 f'(z)}{f^2(z)} \right| \leq p |z|^2, \quad z \in \mathcal{U}. \tag{10}$$

Lemma 2.3. (General Schwarz Lemma [16]). Let $f(z)$ be the function regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If $f(z)$ has at $z = 0$ one zero with multiplicity greater than m , then

$$|f(z)| \leq \frac{M}{R^m} |z|^m, \quad z \in \mathcal{U}_R \tag{11}$$

the equality (in the equality (11) for $z \neq 0$) can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

Lemma 2.4. (Caratheodory [1], [12]). Let f be analytic function in \mathcal{U} , with $f(0) = 0$. If f satisfies

$$\Re f(z) \leq M \tag{12}$$

for some $M > 0$, then

$$(1 - |z|) |f(z)| \leq 2M |z|, \quad z \in \mathcal{U}. \tag{13}$$

3. Main Results

Theorem 3.1. Let γ_i be complex numbers $\Re(\gamma_i) \neq 0, \delta = \sum_{i=1}^n \Re\left(\frac{1}{\gamma_i}\right) - n + 1 > 0, f_i \in \mathcal{A}, f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + \dots, i = \overline{1, n}$. If

$$\Re \left\{ e^{i\theta} \left(\frac{zf'_i(z)}{f_i(z)} - 1 \right) \right\} \leq \frac{|\gamma_i| \left(\Re\left(\frac{1}{\gamma_i}\right) - 1 + \frac{1}{n} \right)}{4(1 + |\gamma_i| |\beta - 1|)}, \quad (i = \overline{1, n}; 0 < \delta < 1) \tag{14}$$

or

$$\Re \left\{ e^{i\theta} \left(\frac{zf'_i(z)}{f_i(z)} - 1 \right) \right\} \leq \frac{|\gamma_i|}{4n(1 + |\gamma_i| |\beta - 1|)}, \quad (i = \overline{1, n}; \delta \geq 1), \tag{15}$$

for all $z \in \mathcal{U}, \theta \in [0, 2\pi]$ and $\beta \in \mathbb{C}$, then the integral operator $\mathcal{J}_{\gamma_1, \gamma_2, \dots, \gamma_n, \beta}$ defined by (2) is in the class \mathcal{S} .

Proof. The integral operator $\mathcal{J}_{\gamma_1, \gamma_2, \dots, \gamma_n, \beta}$ has the form

$$\mathcal{J}_{\gamma_1, \gamma_2, \dots, \gamma_n, \beta}(z) = \left\{ \left(\sum_{i=1}^n \frac{1}{\gamma_i} - n + 1 \right) \int_0^z u^{\sum_{i=1}^n \frac{1}{\gamma_i} - n} \prod_{i=1}^n \left(\frac{f_i(u)}{u} \right)^{\frac{1}{\gamma_i} + \beta - 1} du \right\}^{\frac{1}{\sum_{i=1}^n \frac{1}{\gamma_i} - n + 1}}. \tag{16}$$

We consider the function

$$g(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(u)}{u} \right)^{\frac{1}{\gamma_i} + \beta - 1} du \tag{17}$$

analytic in \mathcal{U} .

We have

$$\frac{zg''(z)}{g'(z)} = \sum_{i=1}^n \left(\frac{1}{\gamma_i} + \beta - 1 \right) \left(\frac{zf'_i(z)}{f_i(z)} - 1 \right). \tag{18}$$

Let us consider the function

$$\varphi_i(z) = e^{i\theta} \left(\frac{zf'_i(z)}{f_i(z)} - 1 \right), \quad z \in \mathcal{U}, i = \overline{1, n}, \theta \in [0, 2\pi] \tag{19}$$

and we observe that $\varphi_i(0) = 0$ for all $i = \overline{1, n}$.

By (14) and Lemma 2.4 for $\delta \in (0, 1)$ we obtain

$$|\varphi_i(z)| \leq \frac{|\gamma_i| \left(\Re\left(\frac{1}{\gamma_i}\right) - 1 + \frac{1}{n} \right)}{2(1 - |z|)(1 + |\gamma_i| |\beta - 1|)}, \quad z \in \mathcal{U}, \beta \in \mathbb{C}, i = \overline{1, n}. \tag{20}$$

From (15) and Lemma 2.4, for $\delta \in [1, \infty)$ we have

$$|\varphi_i(z)| \leq \frac{|\gamma_i|}{2(1 - |z|)(1 + |\gamma_i| |\beta - 1|)}, \quad z \in \mathcal{U}, \beta \in \mathbb{C}, i = \overline{1, n}. \tag{21}$$

From (18) and (20) we get

$$\frac{1 - |z|^{2\delta}}{\delta} \left| \frac{zg''(z)}{g'(z)} \right| \leq \frac{1 - |z|^{2\delta}}{2(1 - |z|)} |z|, \quad \delta \in (0, 1). \tag{22}$$

Because $1 - |z|^{2\delta} \leq 1 - |z|^2$ for $\delta \in (0, 1)$, $z \in \mathcal{U}$ from (22) we have

$$\frac{1 - |z|^{2\delta}}{\delta} \left| \frac{zg''(z)}{g'(z)} \right| \leq 1 \tag{23}$$

for all $z \in \mathcal{U}$, $\delta \in (0, 1)$.

For $\delta \in [1, \infty)$, we have $\frac{1 - |z|^{2\delta}}{\delta} \leq 1 - |z|^2$, $z \in \mathcal{U}$ and from (18) and (21), we obtain

$$\frac{1 - |z|^{2\delta}}{\delta} \left| \frac{zg''(z)}{g'(z)} \right| \leq 1 \tag{24}$$

for all $z \in \mathcal{U}$, $\delta \in [1, \infty)$.

From (17) we have $g'(z) = \prod_{i=1}^n \left(\frac{f_i(z)}{z} \right)^{\frac{1}{\gamma_i} + \beta - 1}$ and using (23) and (24), by Lemma 2.1 it results that $\mathcal{J}_{\gamma_1, \gamma_2, \dots, \gamma_n, \beta}$ given by (2) is in the class \mathcal{S} . \square

Letting $\gamma_i = \gamma$ for $i = \overline{1, n}$ in Theorem 3.1, we have

Corollary 3.2. Let γ be complex number $\Re(\gamma) \neq 0$, $\delta = n\Re(\frac{1-\gamma}{\gamma}) + 1 > 0$, $f_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + \dots$, $i = \overline{1, n}$. If

$$\Re \left\{ e^{i\theta} \left(\frac{zf'_i(z)}{f_i(z)} - 1 \right) \right\} \leq \frac{|\gamma| \left(\Re(\frac{1}{\gamma}) - 1 + \frac{1}{n} \right)}{4(1 + |\gamma| |\beta - 1|)}, \quad (i = \overline{1, n}; 0 < \delta < 1)$$

or

$$\Re \left\{ e^{i\theta} \left(\frac{zf'_i(z)}{f_i(z)} - 1 \right) \right\} \leq \frac{|\gamma|}{4n(1 + |\gamma| |\beta - 1|)}, \quad (i = \overline{1, n}; \delta \geq 1),$$

for all $z \in \mathcal{U}$, $\theta \in [0, 2\pi]$ and $\beta \in \mathbb{C}$, then the integral operator $\mathcal{J}_{n, \gamma, \beta}$ defined by (3) is in the class \mathcal{S} .

Letting $\beta = 0$ in Theorem 3.1, we have

Corollary 3.3. Let γ_i be complex numbers $\Re(\gamma_i) \neq 0$, $\delta = \sum_{i=1}^n \Re(\frac{1}{\gamma_i}) - n + 1 > 0$, $f_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + \dots$, $i = \overline{1, n}$. If

$$\Re \left\{ e^{i\theta} \left(\frac{zf'_i(z)}{f_i(z)} - 1 \right) \right\} \leq \frac{|\gamma_i| \left(\Re(\frac{1}{\gamma_i}) - 1 + \frac{1}{n} \right)}{4(1 + |\gamma_i|)}, \quad (i = \overline{1, n}; 0 < \delta < 1)$$

or

$$\Re \left\{ e^{i\theta} \left(\frac{zf'_i(z)}{f_i(z)} - 1 \right) \right\} \leq \frac{|\gamma_i|}{4n(1 + |\gamma_i|)}, \quad (i = \overline{1, n}; \delta \geq 1),$$

for all $z \in \mathcal{U}$, $\theta \in [0, 2\pi]$, then the integral operator $\mathcal{J}_{\gamma_1, \gamma_2, \dots, \gamma_n, 0}$ defined by (4) is in the class \mathcal{S} .

Letting $\beta = 1$ in Theorem 3.1, we have

Corollary 3.4. Let γ_i be complex numbers $\Re(\gamma_i) \neq 0, \delta = \sum_{i=1}^n \Re\left(\frac{1}{\gamma_i}\right) - n + 1 > 0, f_i \in \mathcal{A}, f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + \dots, i = \overline{1, n}$. If

$$\Re \left\{ e^{i\theta} \left(\frac{zf'_i(z)}{f_i(z)} - 1 \right) \right\} \leq \frac{|\gamma_i| \left(\Re\left(\frac{1}{\gamma_i}\right) - 1 + \frac{1}{n} \right)}{4}, \quad (i = \overline{1, n}; 0 < \delta < 1)$$

or

$$\Re \left\{ e^{i\theta} \left(\frac{zf'_i(z)}{f_i(z)} - 1 \right) \right\} \leq \frac{|\gamma_i|}{4n}, \quad (i = \overline{1, n}; \delta \geq 1),$$

for all $z \in \mathcal{U}, \theta \in [0, 2\pi]$, then the integral operator $\mathcal{J}_{\gamma_1, \gamma_2, \dots, \gamma_n, 1}$ defined by (5) is in the class \mathcal{S} .

Theorem 3.5. Let γ_i be complex numbers $\Re(\gamma_i) \neq 0, \delta = \sum_{i=1}^n \Re\left(\frac{1}{\gamma_i}\right) - n + 1 > 0, f_i \in \mathcal{A}, f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + \dots, i = \overline{1, n}$. If

$$\left| \frac{zf'_i(z)}{f_i(z)} - 1 \right| \leq \frac{(2\delta + 1)^{\frac{2\delta+1}{2\delta}}}{2n} \frac{|\gamma_i|}{(1 + |\gamma_i||\beta - 1|)}, \quad (i = \overline{1, n}) \tag{25}$$

for all $z \in \mathcal{U}, \theta \in [0, 2\pi]$ and $\beta \in \mathbb{C}$, then the integral operator $\mathcal{J}_{\gamma_1, \gamma_2, \dots, \gamma_n, \beta}$ defined by (2) is in the class \mathcal{S} .

Proof. We observe that $\mathcal{J}_{\gamma_1, \gamma_2, \dots, \gamma_n, \beta}$ has the form (16). Consider the function $g(z)$ defined by (17). We define the function $\psi(z) = \frac{zg''(z)}{g'(z)}$ for all $z \in \mathcal{U}$. From (18) and (25) we have

$$|\psi(z)| \leq \frac{(2\delta + 1)^{\frac{2\delta+1}{2\delta}}}{2}$$

for all $z \in \mathcal{U}$.

The function ψ satisfies the condition $\psi(0) = 0$ and applying Lemma 2.3 we obtain

$$|\psi(z)| \leq \frac{(2\delta + 1)^{\frac{2\delta+1}{2\delta}}}{2} |z|, \quad z \in \mathcal{U}. \tag{26}$$

From (26) we get

$$\frac{1 - |z|^{2\delta}}{\delta} \left| \frac{zg''(z)}{g'(z)} \right| \leq \frac{(2\delta + 1)^{\frac{2\delta+1}{2\delta}}}{2} \frac{1 - |z|^{2\delta}}{\delta} |z|. \tag{27}$$

Because

$$\max_{|z| \leq 1} \left\{ \frac{1 - |z|^{2\delta}}{\delta} |z| \right\} = \frac{2}{(2\delta + 1)^{\frac{2\delta+1}{2\delta}}}$$

from (27) we obtain

$$\frac{1 - |z|^{2\delta}}{\delta} \left| \frac{zg''(z)}{g'(z)} \right| \leq 1. \tag{28}$$

From (28) and because $g'(z) = \prod_{i=1}^n \left(\frac{f_i(z)}{z} \right)^{\frac{1}{\gamma_i} + \beta - 1}$, by Lemma 2.1 we obtain that the integral operator $\mathcal{J}_{\gamma_1, \gamma_2, \dots, \gamma_n, \beta}$ is in the class \mathcal{S} . \square

Letting $\gamma_i = \gamma$ for $i = \overline{1, n}$ in Theorem 3.5, we have

Corollary 3.6. Let γ be complex number $\Re(\gamma) \neq 0$, $\delta = n\Re(\frac{1-\gamma}{\gamma})+1 > 0$, $f_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + \dots$, $i = \overline{1, n}$. If

$$\left| \frac{zf'_i(z)}{f_i(z)} - 1 \right| \leq \frac{(2\delta + 1)^{\frac{2\delta+1}{2\delta}}}{2n} \frac{|\gamma|}{(1 + |\gamma| |\beta - 1|)}, \quad (i = \overline{1, n})$$

for all $z \in \mathcal{U}$, $\theta \in [0, 2\pi]$ and $\beta \in \mathbb{C}$, then the integral operator $\mathcal{J}_{n, \gamma, \beta}$ defined by (3) is in the class \mathcal{S} .

Letting $\beta = 0$ in Theorem 3.5, we have

Corollary 3.7. Let γ_i be complex numbers $\Re(\gamma_i) \neq 0$, $\delta = \sum_{i=1}^n \Re\left(\frac{1}{\gamma_i}\right) - n + 1 > 0$, $f_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + \dots$, $i = \overline{1, n}$. If

$$\left| \frac{zf'_i(z)}{f_i(z)} - 1 \right| \leq \frac{(2\delta + 1)^{\frac{2\delta+1}{2\delta}}}{2n} \frac{|\gamma|}{(1 + |\gamma|)}, \quad (i = \overline{1, n})$$

for all $z \in \mathcal{U}$, $\theta \in [0, 2\pi]$ and $\beta \in \mathbb{C}$, then the integral operator $\mathcal{J}_{\gamma_1, \gamma_2, \dots, \gamma_n}$ defined by (4) is in the class \mathcal{S} .

Letting $\beta = 1$ in Theorem 3.5, we have

Corollary 3.8. Let γ_i be complex numbers $\Re(\gamma_i) \neq 0$, $\delta = \sum_{i=1}^n \Re\left(\frac{1}{\gamma_i}\right) - n + 1 > 0$, $f_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + \dots$, $i = \overline{1, n}$. If

$$\left| \frac{zf'_i(z)}{f_i(z)} - 1 \right| \leq \frac{|\gamma_i| (2\delta + 1)^{\frac{2\delta+1}{2\delta}}}{2n}, \quad (i = \overline{1, n})$$

for all $z \in \mathcal{U}$, $\theta \in [0, 2\pi]$, then the integral operator $\mathcal{J}_{\gamma_1, \gamma_2, \dots, \gamma_n, 1}$ defined by (5) is in the class \mathcal{S} .

Remark 3.9. For $n = 1$ in Theorem 3.1 and Theorem 3.5, we obtain Theorem 3.2 and Theorem 3.5 in [17] respectively.

Theorem 3.10. Let the functions $f_i \in \mathcal{S}(p_i)$, $(i = \overline{1, n})$ which satisfy the inequality (10) with $0 < p_i \leq 2$ and $M_i > 0$ for $i = \overline{1, n}$. Furthermore γ_i be complex numbers with $\lambda = \sum_{i=1}^n \frac{1}{\gamma_i} - n + 1$ and $\Re(\lambda) > 0$. If

$$|f_i(z)| \leq M_i \quad (z \in \mathcal{U}, i = \overline{1, n}) \tag{29}$$

and

$$\sum_{i=1}^n \left| \frac{1}{\gamma_i} + \beta - 1 \right| (M_i(p_i + 1) + 1) \leq \Re(\lambda) \tag{30}$$

then the integral operator $\mathcal{J}_{\gamma_1, \gamma_2, \dots, \gamma_n, \beta}$ defined by (2) is in the class \mathcal{S} .

Proof. We observe that $\mathcal{J}_{\gamma_1, \gamma_2, \dots, \gamma_n, \beta}$ has the form (16). Consider the function $g(z)$ defined in the formula (17). Equation (18), yields

$$\begin{aligned} \left| \frac{zg''(z)}{g'(z)} \right| &\leq \sum_{i=1}^n \left| \frac{1}{\gamma_i} + \beta - 1 \right| \left(\left| \frac{zf'_i(z)}{f_i(z)} \right| + 1 \right) \\ &\leq \sum_{i=1}^n \left| \frac{1}{\gamma_i} + \beta - 1 \right| \left(\left| \frac{z^2 f'_i(z)}{(f_i(z))^2} \right| \left| \frac{f_i(z)}{z} \right| + 1 \right). \end{aligned}$$

The hypothesis will then yield $|f_i(z)| \leq M_i$ ($z \in \mathcal{U}$, $i = \overline{1, n}$). By General Schwarz Lemma, we can obtain that

$$|f_i(z)| \leq M_i |z| \tag{31}$$

for all $z \in \mathcal{U}$, $i = \overline{1, n}$. Therefore by using the inequality (31), we obtain

$$\begin{aligned} \left| \frac{zg''(z)}{g'(z)} \right| &\leq \sum_{i=1}^n \left| \frac{1}{\gamma_i} + \beta - 1 \right| \left(\left| \frac{z^2 f'_i(z)}{(f_i(z))^2} \right| M_{i+1} \right) \\ &\leq \sum_{i=1}^n \left| \frac{1}{\gamma_i} + \beta - 1 \right| \left(\left| \frac{z^2 f'_i(z)}{(f_i(z))^2} - 1 \right| M_i + M_{i+1} \right). \end{aligned} \tag{32}$$

Since $f \in \mathcal{S}(p_i)$, in view of (30), using (10), (32) may be written as

$$\begin{aligned} \left| \frac{zg''(z)}{g'(z)} \right| &\leq \sum_{i=1}^n \left| \frac{1}{\gamma_i} + \beta - 1 \right| (M_i p_i |z|^2 + M_{i+1}) \\ &\leq \sum_{i=1}^n \left| \frac{1}{\gamma_i} + \beta - 1 \right| (M_i(p_i + 1) + 1) \leq \Re(\lambda). \end{aligned} \tag{33}$$

On multiplying (33) by $\frac{(1 - |z|^{2\Re(\lambda)})}{\Re(\lambda)}$, the following inequality is obtained

$$\frac{1 - |z|^{2\Re(\lambda)}}{\Re(\lambda)} \left| \frac{zg''(z)}{g'(z)} \right| \leq 1 - |z|^{2\Re(\lambda)} \leq 1 \quad (z \in \mathcal{U}).$$

So, according to the Lemma 2.1, the integral operator $\mathcal{J}_{\gamma_1, \gamma_2, \dots, \gamma_n, \beta}$ belongs to the class \mathcal{S} . \square

Letting $\gamma_i = \gamma$ for $i = \overline{1, n}$ in Theorem 3.10, we have

Corollary 3.11. *Let the functions $f_i \in \mathcal{S}(p_i)$, ($i = \overline{1, n}$) which satisfy the inequality (10) with $0 < p_i \leq 2$ and $M_i > 0$ for $i = \overline{1, n}$. Furthermore γ be complex number with $\arg \gamma = \theta$ and $\Re\left(\frac{n(1-\gamma)+\gamma}{\gamma}\right) > 0$. If*

$$|f_i(z)| \leq M_i \quad (z \in \mathcal{U}, i = \overline{1, n})$$

and

$$\left| 1 + \gamma(\beta - 1) \sum_{i=1}^n (M_i(p_i + 1) + 1) \right| \leq n \cos \theta + (1 - n) |\gamma|$$

then the integral operator $\mathcal{J}_{n, \gamma, \beta}$ defined by (3) is in the class \mathcal{S} .

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