

On Locally 1-connectedness of Quotient Spaces and Its Applications to Fundamental Groups

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Abstract. Let X be a locally 1-connected metric space and A_1, A_2, \dots, A_n be connected, locally path connected and compact pairwise disjoint subspaces of X . In this paper, we show that the quotient space $X/(A_1, A_2, \dots, A_n)$ obtained from X by collapsing each of the sets A_i 's to a point, is also locally 1-connected. Moreover, we prove that the induced continuous homomorphism of quasitopological fundamental groups is surjective. Finally, we give some applications to find out some properties of the fundamental group of the quotient space $X/(A_1, A_2, \dots, A_n)$.

1. Introduction and motivation

Let \sim be an equivalent relation on a topological space X . Then one can consider the quotient topological space X/\sim and the quotient map $p : X \rightarrow X/\sim$. In general, quotient spaces are not well behaved and it seems interesting to determine which topological properties of the space X may be transferred to the quotient space X/\sim . For example, a quotient space of a simply connected or contractible space need not share those properties. Also, a quotient space of a locally compact space need not be locally compact.

Let A_1, A_2, \dots, A_n be a finite collection of pairwise disjoint subspaces of a topological space X . Then by $X/(A_1, A_2, \dots, A_n)$ we mean the quotient space obtained from X by identifying each of the sets A_i 's to a point. The main result of this paper is as follows:

Theorem A. *If X is a locally 1-connected metric space and A_1, A_2, \dots, A_n are connected, locally path connected and compact pairwise disjoint subspaces of X , then the quotient space $X/(A_1, A_2, \dots, A_n)$ is locally 1-connected.*

The quasitopological fundamental group $\pi_1^{qtop}(X, x)$ of a based space (X, x) is the familiar fundamental group $\pi_1(X, x)$ endowed with the quotient topology with respect to the canonical function $\Omega(X, x) \rightarrow \pi_1(X, x)$ identifying path components on the loop space $\Omega(X, x)$ with the compact-open topology (see [3]). It is known that this construction gives rise a homotopy invariant functor $\pi_1^{qtop} : hTop_* \rightarrow qTopGrp$ from the homotopy category of based spaces to the category of quasitopological group and continuous homomorphism [4]. By applying the above functor on the quotient map $p : X \rightarrow X/\sim$, we have a continuous homomorphism $p_* : \pi_1^{qtop}(X, x) \rightarrow \pi_1^{qtop}(X/\sim, *)$. Recently, the authors [10] proved that if X is a first countable, connected,

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locally path connected space and A_1, A_2, \dots, A_n are disjoint path connected, closed subspaces of X , then the image of p_* is dense in $\pi_1^{qtop}(X/(A_1, A_2, \dots, A_n))$. By using the results of [10] and Theorem A we have the primary application of the main result of this paper as follows:

Theorem B. *If X is a locally 1-connected metric space and A_1, A_2, \dots, A_n are disjoint connected, locally path connected and compact subspaces of X , then for each $a \in \bigcup_{i=1}^n A_i$ the continuous homomorphism*

$$p_* : \pi_1^{qtop}(X, a) \longrightarrow \pi_1^{qtop}(X/(A_1, A_2, \dots, A_n), *)$$

is an epimorphism.

Also, by some examples, we show that p_* is not necessarily onto if the hypotheses of the above theorem do not hold.

Finally, we give some applications of the above results to explore the properties of the fundamental group of the quotient space $X/(A_1, A_2, \dots, A_n)$. In particular, we prove that if X is a separable, connected, locally 1-connected metric space and A_1, \dots, A_n are connected, locally path connected and compact subsets of X , then $\pi_1(X/(A_1, A_2, \dots, A_n), *)$ is countable. Moreover, $\pi_1(X/(A_1, A_2, \dots, A_n), *)$ is finitely presented. Note that with the recent assumptions on X and the A_i 's, $X/(A_1, A_2, \dots, A_n)$ is simply connected when X is simply connected.

2. Notations and preliminaries

For a pointed topological space (X, x) , by a path we mean a continuous map $\alpha : [0, 1] \longrightarrow X$. The points $\alpha(0)$ and $\alpha(1)$ are called the initial point and the terminal point of α , respectively. A loop α is a path with $\alpha(0) = \alpha(1)$. For a path $\alpha : [0, 1] \longrightarrow X$, $\bar{\alpha}$ denotes a path such that $\bar{\alpha}(t) = \alpha(1 - t)$, for all $t \in [0, 1]$. Denote $[0, 1]$ by I , two paths $\alpha, \beta : I \longrightarrow X$ with the same initial and terminal points are called homotopic relative to end points if there exists a continuous map $F : I \times I \longrightarrow X$ such that

$$F(t, s) = \begin{cases} \alpha(t) & s = 0 \\ \beta(t) & s = 1 \\ \alpha(0) = \beta(0) & t = 0 \\ \alpha(1) = \beta(1) & t = 1. \end{cases}$$

The homotopy class containing a path α is denoted by $[\alpha]$. Since most of the homotopies that appear in this paper have this property and end points are the same, we drop the term "relative homotopy" for simplicity. For paths $\alpha, \beta : I \longrightarrow X$ with $\alpha(1) = \beta(0)$, $\alpha * \beta$ denotes the concatenation of α and β , that is, a path from I to X such that $(\alpha * \beta)(t) = \alpha(2t)$, for $0 \leq t \leq 1/2$ and $(\alpha * \beta)(t) = \beta(2t - 1)$, for $1/2 \leq t \leq 1$.

For a space (X, x) , let $\Omega(X, x)$ be the space of based maps from I to X with the compact-open topology. A subbase for this topology consists of neighborhoods of the form $\langle K, U \rangle = \{\gamma \in \Omega(X, x) \mid \gamma(K) \subseteq U\}$, where $K \subseteq I$ is compact and U is open in X . We will consistently denote the constant path at x by e_x . The quasitopological fundamental group of a pointed space (X, x) may be described as the usual fundamental group $\pi_1(X, x)$ with the quotient topology with respect to the canonical map $\Omega(X, x) \longrightarrow \pi_1(X, x)$ identifying homotopy classes of loops, denoted by $\pi_1^{qtop}(X, x)$. A basic account of quasitopological fundamental groups may be found in [3], [4] and [5]. For undefined notation, see [9].

Definition 2.1. ([1]) A quasitopological group G is a group with a topology such that inversion $G \longrightarrow G$, $g \longrightarrow g^{-1}$, is continuous and multiplication $G \times G \longrightarrow G$ is continuous in each variable. A morphism of quasitopological groups is a continuous homomorphism.

Theorem 2.2. ([4]) π_1^{qtop} is a functor from the homotopy category of based topological spaces to the category of quasitopological groups.

A space X is *semi-locally simply connected* if for each point $x \in X$, there is an open neighborhood U of x such that the inclusion $i : U \longrightarrow X$ induces the zero map $i_* : \pi_1(U, x) \longrightarrow \pi_1(X, x)$ or equivalently a loop in U can be contracted inside X .

Theorem 2.3. ([4]) *Let X be a connected and locally path connected topological space. The quasitopological fundamental group $\pi_1^{qtop}(X, x)$ is discrete for some $x \in X$ if and only if X is semi-locally simply connected.*

Theorem 2.4. ([10]) *Let A_1, A_2, \dots, A_n be disjoint path connected, closed subsets of a first countable, connected, locally path connected space X such that $X/(A_1, A_2, \dots, A_n)$ is semi-locally simply connected, then for each $a \in \bigcup_{i=1}^n A_i$, $p_* : \pi_1(X, a) \rightarrow \pi_1(X/(A_1, A_2, \dots, A_n, *))$ is an epimorphism.*

In this note all homotopies are relative. Also, when $\alpha : [a, b] \rightarrow X$ is a path, for brevity by $\hat{\alpha}$ we mean $\alpha \circ \varphi$ where $\varphi : I \rightarrow [a, b]$ is a suitable linear homeomorphism.

3. Constructing homotopies

This section is dedicated to some technical lemmas about homotopy properties of loops.

Assume that α is a path in X and t_0, t_1, \dots, t_n are real numbers such that $0 = t_0 < t_1 < \dots < t_n = 1$. Let $\alpha_i = \alpha|_{[t_i, t_{i+1}]}$, for $0 \leq i \leq n - 1$. Then we know that ([9, Theorem 51.3])

$$\alpha \simeq \hat{\alpha}_0 * \hat{\alpha}_1 * \dots * \hat{\alpha}_n.$$

Corollary 3.1. *Let α be a path in X and I_1, I_2, \dots, I_n be closed disjoint subintervals of I . If $\alpha_j = \alpha|_{I_j}$ and $\beta_j : I_j \rightarrow X$ are homotopic relative to end points for $j = 1, \dots, n$, then $\alpha \simeq \beta$, where*

$$\beta(t) = \begin{cases} \beta_j(t) & t \in I_j \\ \alpha(t) & \text{otherwise.} \end{cases}$$

If A is a subset of a topological space X , then we denote the complement of A in X by $A^c = X - A$.

Lemma 3.2. *Let α be a loop in X based at x_0 . Let $\{t_i\} \subseteq (0, 1)$ be an increasing sequence such that $t_i \rightarrow 1$ and $\{V_i\}$ form a nested neighborhood basis at x_0 such that $\alpha(t_i) \in V_i$ for every integer i . If $\beta_i : [t_i, 1] \rightarrow V_i$ is a path from $\alpha(t_i)$ to x_0 such that $\hat{\beta}_i * \hat{\alpha}|_{[t_i, t_{i+1}]} * \hat{\beta}_{i+1}$ is a null loop in V_i , for every integer i , then $\alpha \simeq \alpha_i$, where*

$$\alpha_i(t) = \begin{cases} \alpha(t) & t \leq t_i \\ \beta_i(t) & t \geq t_i. \end{cases}$$

Proof. It is obvious that $\alpha_i \simeq \alpha_j$, for every integers i, j and hence it suffices to prove $\alpha \simeq \alpha_1$. Let $\alpha_1 \simeq \alpha_1$ by $F(t, s) = \alpha_1(t)$. Since $([t_1, 1] \times \{0\}) \cup (\{1\} \times I)$ is compact and V_1 is an open neighborhood of $F([t_1, 1] \times \{0\} \cup \{1\} \times I)$, there exist $\varepsilon_1 \in (0, 1)$ such that $F(B_1) \subseteq V_1$, where $B_1 = \{(x, y) \in I \times I \mid (1 - t_1)y \leq \varepsilon_1(x - t_1)\}$. Let $\eta_1 : I \rightarrow V_1$ be the path defined by $\eta_1(t) = F(t(t_1, 0) + (1 - t)(1, \varepsilon_1))$. By definition of η_1 and since $F(t, 0) = \beta_1(t)$ for $t \geq t_1$, $\eta_1 * \hat{\beta}_1 \simeq e_{x_0}$ by a homotopy in V_1 . Therefore

$$\eta_1 * \hat{\alpha}|_{[t_1, t_2]} * \hat{\beta}_2 \simeq \eta_1 * \hat{\beta}_1 * \hat{\beta}_1^{-1} * \hat{\alpha}|_{[t_1, t_2]} * \hat{\beta}_2 \simeq e_{x_0}$$

by a homotopy in V_1 and hence there exists a continuous map $H_1 : B_1 \rightarrow V_1$ such that

- (i) $H_1(t(t_1, 0) + (1 - t)(1, \varepsilon_1)) = \eta_1(t)$ for $t \in I$,
- (ii) $H_1(t, 0) = \alpha(t)$ for $t \in [t_1, t_2]$,
- (iii) $H_1(t, 0) = \beta_2(t)$ for $t \geq t_2$,
- (iv) $H_1(1, t) = x_0$.

Since $([t_2, 1] \times \{0\}) \cup (\{1\} \times I)$ is compact and V_2 is an open neighborhood of $H_1([t_2, 1] \times \{0\} \cup \{1\} \times I)$, there exist $\varepsilon_2 < \varepsilon_1 \in (0, 1)$ such that $H_1(B_2) \subseteq V_2$, where $B_2 = \{(x, y) \in I \times I \mid (1 - t_2)y \leq \varepsilon_2(x - t_2)\}$. Let $\eta_2 : I \rightarrow V_2$ be the path defined by $\eta_2(t) = H_1(t(t_2, 0) + (1 - t)(1, \varepsilon_2))$. Then $\eta_1 * \hat{\alpha}|_{[t_1, t_2]} * \eta_2^{-1}$ is null in V_1 by the homotopy $K_1 = H_1|_{B_1 - B_2}$ (note that $B_1 - B_2$ is homeomorphic to $I \times I$). Also, $\eta_2 * \hat{\beta}_2$ is null in V_2 because $H_1(B_2) \subseteq V_2$ and hence $\eta_2 * \hat{\alpha}|_{[t_2, t_3]} * \hat{\beta}_3 \simeq e_{x_0}$ by homotopy $H_2 : B_2 \rightarrow V_2$. By the similar way, for every $n \in \mathbb{N}$ we have the path $\eta_n : I \rightarrow V_n$ by $\eta_n(t) = H_{n-1}(t(t_n, 0) + (1 - t)(1, \varepsilon_n))$ for which $\eta_{n-1} * \alpha|_{[t_{n-1}, t_n]} * \eta_n^{-1}$ is null in V_n by a homotopy $K_n = H_n|_{B_n - B_{n+1}}$, where $B_n = \{(x, y) \in I \times I \mid (1 - t_n)y \leq \varepsilon_n(x - t_n)\}$ and $\varepsilon_n \rightarrow 0$ (see Figure 1).

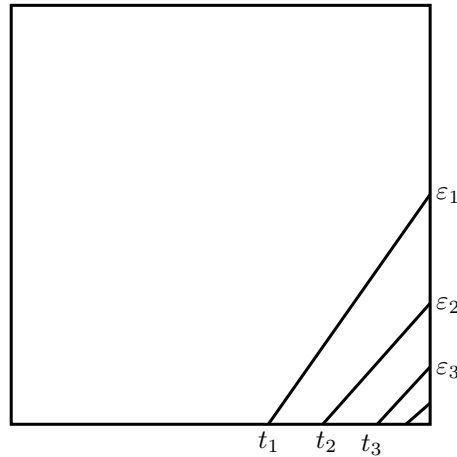


Figure 1:

Define $G : I \times I \rightarrow X$ by

$$G(t, s) = \begin{cases} K_n(t, s) & (t, s) \in B_n - B_{n+1} \\ F(t, s) & o.w. \end{cases}$$

Then $G(t, 0) = \alpha(t)$ and $G(t, 1) = \alpha_1(t)$. Let $(r_n, s_n) \rightarrow (1, 0)$ and U be an open neighborhood of x_0 . There exist $n_0 \in \mathbb{N}$ such that $V_{n_0} \subseteq U$. By the construction of B_n 's, there exist $n_1 > n_0$ such that $(r_n, s_n) \in B_{n_1}$ for each $n > n_1$ which implies that $G(r_n, s_n) \in V_{n_1} \subseteq U$ for each $n > n_1$. Hence G is continuous (preserving the other convergent sequences by G is obvious).

□

In the following examples we show that the above lemma does not hold if $\bar{\beta}_i * \hat{\alpha}|_{[t_i, t_{i+1}]} * \beta_{i+1}$ is only null in X .

Example 3.3. Let X be the Griffiths' space: the one-point union $C(HE) \vee C(HE)$ of two copies of the cone over the Hawaiian earring as described in [8]. Let θ_i 's and λ_i 's be the consecutive loops of the right and left Hawaiian earring, respectively. Then all loops θ_i and λ_i are null by homotopies in X . Let $I_j = [1 - \frac{1}{2^{j-1}}, 1 - \frac{1}{2^j}]$, for every $j \in \mathbb{N}$ and define $\alpha : I \rightarrow X$ by

$$\alpha(t) = \begin{cases} \theta_j \circ \varphi_{2^{j-1}}(t) & t \in I_{2^{j-1}}, \\ \lambda_j \circ \varphi_{2^j}(t) & t \in I_{2^j}, \end{cases}$$

where φ_j 's are suitable linear homeomorphisms from I_j to I . Note that α is not null in X [13, Example 2.5.18]. Let $t_i = 1 - \frac{1}{2^{i-1}}$, $\beta_{2i-1} = \theta_i \circ \varphi_{2^{i-1}}$ and $\beta_{2i} = \lambda_i \circ \varphi_{2^i}$. Then $\bar{\beta}_i * \hat{\alpha}|_{[t_i, t_{i+1}]} * \beta_{i+1}$ is homotopic to a null loop θ_j or λ_j (depended on i is even or odd). Also, every loop α_i defined as like as in Lemma 3.3 is null and hence α is not homotopic to α_i , for every $i \in \mathbb{N}$.

Lemma 3.4. Let α be a loop in X based at x_0 , $\{x_0\}$ be closed and $\{V_i\}$ form a nested neighborhood basis at x_0 . If $\alpha^{-1}(\{x_0\}^c) = \bigsqcup_{n=1}^{\infty} (a_n, b_n)$ and $\hat{\alpha}|_{[a_n, b_n]} \simeq e_{x_0}$ by a homotopy in V_n ($\alpha([a_n, b_n]) \subseteq V_n$), then α is nullhomotopic.

Proof. Let $I_n = [a_n, b_n]$ and $\alpha_n = \alpha|_{I_n}$. Assume $\hat{\alpha}_n \simeq e_{x_0}$ by a homotopy $F_n : I_n \times I \rightarrow V_n$. Define $F : I \times I \rightarrow V_1$ by

$$F(t, s) = \begin{cases} F_n(t, s) & t \in I_n, \text{ for } n \in \mathbb{N} \\ x_0 & \text{otherwise} \end{cases}$$

Since $F(t, 0) = \alpha(t)$ and $F(t, 1) = x_0$, it remains to show that F is continuous. Let $(t_n, s_n) \rightarrow (t, s)$. If $F(t, s) \neq x_0$, then $t_n \in (a_k, b_k)$, for a $k \in \mathbb{N}$ and every $n > k$. Continuity of F_k implies that $F(t_n, s_n) = F_k(t_n, s_n) \rightarrow F_k(t, s) = F(t, s)$. If $F(t, s) = x_0$, then the construction of F , F_n 's and nested property of V_n 's imply that $F(t_n, s_n) \in U$, for an open neighborhood U of x_0 and sufficiently large n . \square

By the next example we show that Lemma 3.4 does not hold if $\hat{\alpha}|_{[a_n, b_n]}$ is a null loop in X instead of V_n .

Example 3.5. Let X , α , θ_i and λ_i be defined as in Example 3.3 and x_0 be the common point of two cones. We have all the hypothesis of the lemma except θ_i and λ_i are null in X and α is not null.

4. The main result

This section dedicated to prove Theorem A by introducing notions π_1 -contained for subspaces of a given space X and nested basis for a subspace $A \subseteq X$.

Definition 4.1. Let X be a topological space, Z, K be path connected subset of $Y \subseteq X$. We say that Z is π_1 -contained in K at Y if $i_*\pi_1(Z, x) \leq j_*\pi_1(K, x)$ for every $x \in Z \cap K$, where $i : Z \rightarrow Y$, $j : K \rightarrow Y$ are inclusion maps. Equivalently, every loop in Z is homotopic to a loop in K by a relative homotopy in Y .

For a topological space X and open subsets $G' \subseteq G \subseteq X$, we say G' is simply connected in G if every loop in G' is nullhomotopic in G . If A is a subset of a topological space X , then we denote the boundary of A in X by ∂A . Also, in a metric space (X, d) , by $S_r(a)$ we mean the open ball $\{x \in X | d(x, a) < r\}$.

Lemma 4.2. Let X be a metric space and A be a locally path connected, closed subset of X . Let $U \subseteq X$ be an open neighborhood of $a \in \partial A$ which is simply connected in an open subset W of X . Then there exists an open neighborhood V of the point a such that $V \subseteq U$ and $A \cup V$ is π_1 -contained in A at $A \cup W$.

Proof. Since X is metric, there exists $U' \subseteq U$ such that $\overline{U'} \subseteq U$. Since A is locally path connected, there exists a neighborhood $V' \subseteq U'$ of a such that $V' \cap A$ is path connected. There exists $r > 0$ such that the closed ball $B_r(a)$ is contained in V' . Let V be the open ball $S_r(a)$, then we show that $A \cup V$ is π_1 -contained in A at $A \cup W$. Let α be a loop in $A \cup V$ with base point in A . If (r, s) is a component of $\alpha^{-1}(V \cap A^c) = \alpha^{-1}(A^c)$, then $\alpha(r), \alpha(s) \in \overline{V} \cap A \subseteq V' \cap A$. Since $V' \cap A$ is path connected, there exists a path in $V' \cap A$ from $\alpha(r)$ to $\alpha(s)$.

Case 1:

Let $\alpha^{-1}(A^c) = \bigcup_{n=1}^N (a_n, b_n)$. If for every $1 \leq n \leq N$, $\beta_n : [a_n, b_n] \rightarrow V' \cap A$ is a path from $\alpha(a_n)$ to $\alpha(b_n)$, then $\alpha|_{[a_n, b_n]} \simeq \beta_n$ by a homotopy in W since $\hat{\alpha}|_{[a_n, b_n]} * \hat{\beta}_n$ is a loop in U . Using Corollary 3.1, α is homotopic to a loop in A by a homotopy in $A \cup W$.

Case 2:

Let $\alpha^{-1}(A^c) = \bigcup_{n=1}^\infty (a_n, b_n)$. Since α is uniformly continuous, there exists $\delta > 0$ such that if $|t - s| < \delta$, then $d(\alpha(t), \alpha(s)) < r/2$. Also, there exists $n_0 \in \mathbb{N}$ such that for every $n > n_0$, $b_n - a_n < \delta$. Similar to Case 1, there are paths $\beta_i : [a_i, b_i] \rightarrow V' \cap A$ such that $\beta_i \simeq \alpha|_{[a_i, b_i]}$, for $1 \leq i \leq n_0$ by homotopies in W . Therefore α is homotopic to a loop $\alpha' : I \rightarrow A \cup V$ by a homotopy in $A \cup W$ and $\alpha^{-1}(A^c) = \bigcup_{n=n_0+1}^\infty (a_n, b_n)$.

Let $F = \{a_n | n > n_0\}$ and $E = \{b_n | n > n_0\}$. If $t_1 = \inf\{\alpha^{-1}(\overline{A^c}) \cap F\}$ and $s_1 = \sup\{[t_1, t_1 + \delta] \cap E\}$, then by definition of E and F , t_1, s_1 exist and $s_1 - t_1 < \delta$ which implies that $\alpha'|_{[t_1, s_1]} \subseteq S_r(a) \subseteq V'$. Since $V' \cap A$ is path connected, there exists a path $\lambda_1 : I \rightarrow V' \cap A$ with $\lambda_1(0) = \alpha'(t_1)$ and $\lambda_1(1) = \alpha'(s_1)$ such that $\hat{\alpha}'|_{[t_1, s_1]} \simeq \lambda_1$ by a homotopy in W because $\hat{\alpha}'|_{[t_1, s_1]} * \overline{\lambda_1}$ is a loop in $V' \subseteq U$. Now, let $t_n = \inf\{[s_{n-1}, 1] \cap \alpha^{-1}(\overline{A^c}) \cap F\}$ and $s_n = \sup\{[t_n, t_n + \delta] \cap E\}$. Similarly, $\alpha'([t_n, s_n]) \subseteq S_r(a) \subseteq V'$ and there exists a path $\lambda_n : I \rightarrow V' \cap A$ with $\lambda_n(0) = \alpha'(t_n)$ and $\lambda_n(1) = \alpha'(s_n)$ such that $\hat{\alpha}'|_{[t_n, s_n]} \simeq \lambda_n$ by a homotopy in W because $\hat{\alpha}'|_{[t_n, s_n]} * \overline{\lambda_n}$ is a loop in $V' \subseteq U$. Since $I \setminus \bigcup_{i \leq n_0} (a_i, b_i)$ is compact and $\delta > 0$, there exists $n_1 > 0$ such that $[s_{n_1-1}, 1] \cap \alpha^{-1}(\overline{A^c}) \cap F = \emptyset$. Hence, we can replace $\hat{\alpha}'|_{[t_i, s_i]}$, by the path λ_i , for $1 \leq i \leq n_1$, with homotopies in W . Therefore α is homotopic to a loop in A by a homotopy in $A \cup W$, as desired. \square

Definition 4.3. For a topological space X , a family $\mathcal{U} = \{U_n\}_{n \in \mathbb{N}}$ of open neighborhoods of a subset A is called a **nested basis of A** if $U_{n+1} \subseteq U_n$ for all $n \in \mathbb{N}$ and for every open set G containing A , there is $n_0 \in \mathbb{N}$ such that $U_{n_0} \subseteq G$. Let a nested basis $\{V_n\}_{n \in \mathbb{N}}$ of A be a refinement of a nested basis $\{U_n\}_{n \in \mathbb{N}}$ of A i.e $V_n \subseteq U_n$ for each n . Then we say that $\{V_n\}_{n \in \mathbb{N}}$ is π_1 -refinement in A at $\{U_n\}_{n \in \mathbb{N}}$ if V_n is π_1 -contained in A at U_n for each n .

There is two natural questions that whether a subset of a topological space X has a nested basis and whether every nested basis has a π_1 -refinement? Obviously, compact subsets of metric spaces have nested basis and for the second question we have the following theorem. We recall that a topological space X is said to be locally 1-connected if it is locally path connected and locally simply connected.

Theorem 4.4. *Let X be a locally 1-connected metric space and A be a locally path connected compact subset of X . Then every nested basis $\{U_n\}_{n \in \mathbb{N}}$ of A has a π_1 -refinement in A at $\{U_n\}_{n \in \mathbb{N}}$.*

Proof. Let $U_n = S_{\frac{1}{n}}(A)$ be the open ball with radius $1/n$ of A , then $\{U_n\}_{n \in \mathbb{N}}$ is a nested basis for A . Since X is locally 1-connected, for every $n \in \mathbb{N}$ and $a \in \partial A$ there is an open neighborhood $G_n(a) \subseteq U_n$ at a such that $G_n(a)$ is simply connected in U_n . By Lemma 4.2, there exists $G'_n(a) \subseteq G_n(a)$ such that $A \cup G'_n(a)$ is π_1 -contained in A at U_n . By local 0-connectivity of X we can assume every $G'_n(a)$ is path connected. Since A is compact, for every $n \in \mathbb{N}$ there exists $m_n \in \mathbb{N}$ and $a_n^1, a_n^2, \dots, a_n^{m_n}$ such that $A \subseteq \text{int}(A) \cup (\cup_{i=1}^{m_n} G'_n(a_n^i))$.

We claim for every $n \in \mathbb{N}$ there exists integer $k_n \geq n$ such that for every $m > k_n$ and every couple $G_m^i(a_m^i), G_m^j(a_m^j)$ with nonempty intersection, $A \cup G_m^i(a_m^i) \cup G_m^j(a_m^j)$ is π_1 -contained in A at U_n . By contradiction assume there exists no such k_{n_0} for an $n_0 \in \mathbb{N}$, then there are a sequence $\{s_n\}_{n \in \mathbb{N}}$, a couple $G_{s_n}^i(a_{s_n}^i), G_{s_n}^j(a_{s_n}^j)$ and loops $\alpha_n : I \rightarrow A \cup G_{s_n}^i(a_{s_n}^i) \cup G_{s_n}^j(a_{s_n}^j)$ such that there is no loop in A homotopic to α_n by a homotopy in U_{n_0} . Let $x_n \in \text{Im} \alpha_n \cap \partial A$. Since ∂A is compact, there is a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ converging to $x \in \partial A$. Let G be a simply connected neighborhood of x in U_{n_0} . By Lemma 4.2 there exists an open neighborhood $x \in G' \subseteq G$ such that $A \cup G'$ is π_1 -contained in A at U_{n_0} . There exists $k_0 \in \mathbb{N}$ such that for each $k > k_0$, $G_{s_{n_k}}^i(a_{s_{n_k}}^i) \cup G_{s_{n_k}}^j(a_{s_{n_k}}^j) \subseteq G' \cup A$ which implies $\text{Im} \alpha_{n_k} \subseteq A \cup G'$, for $k > k_0$. Therefore α_{n_k} is homotopic to a loop in A by a homotopy in U_{n_0} which is a contradiction.

Let $V_n := A \cup (\cup_{i=1}^{m_{k_n}} G_{k_n}^i(a_{k_n}^i))$. We show that V_n is π_1 -contained in A at U_n . Consider $\alpha : I \rightarrow V_n$ as a loop at $a \in A$. Since $\text{Im} \alpha \subseteq \text{int}(A) \cup (\cup_{i=1}^{m_{k_n}} G_{k_n}^i(a_{k_n}^i))$, there is the Lebesgue number δ for this open cover. Let $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \delta$, then there is $\{G_1, G_2, \dots, G_{n_0}\} \subseteq \{\text{int}(A), G_{k_n}^1(a_{k_n}^1), G_{k_n}^2(a_{k_n}^2), \dots, G_{k_n}^{m_{k_n}}(a_{k_n}^{m_{k_n}})\}$ such that $\alpha([\frac{i-1}{n_0}, \frac{i}{n_0}]) \subseteq G_i$. Let $\alpha_i = \alpha|_{[\frac{i-1}{n_0}, \frac{i}{n_0}]} \circ \varphi_i$, where $\varphi_i : [0, 1] \rightarrow [\frac{i-1}{n_0}, \frac{i}{n_0}]$ is linear homeomorphism for $i = 1, \dots, n_0$. Since $a = \alpha(0) \in G_1$ and $\alpha(\frac{1}{n_0}) \in G_1 \cap G_2$, if $\beta_1 : I \rightarrow A \cup G_2$ is a path from $\alpha(\frac{1}{n_0})$ to a (G_i 's and A are path connected), then $\alpha_1 * \beta_1$ is a loop in $A \cup G_1 \cup G_2$ which implies that $\alpha_1 * \beta_1$ is homotopic to a loop $\gamma_1 : I \rightarrow A$ by a homotopy in U_n . Also, there exists $\beta_2 : I \rightarrow A \cup G_3$ such that $\beta_1 * \alpha_2 * \beta_2$ is homotopic to a loop in A by a homotopy in U_n and similarly there are β_i 's and γ_i 's for $i = 1, 2, \dots, n_0$ such that

$$\alpha \simeq (\alpha_1 * \beta_1) * (\overline{\beta_1} * \alpha_2 * \beta_2) * \dots * (\overline{\beta_{n_0-2}} * \alpha_{n_0-1} * \beta_{n_0-1}) * (\overline{\beta_{n_0-1}} * \alpha_{n_0}) \simeq \gamma_1 * \gamma_2 * \dots * \gamma_{n_0}$$

which implies that α is homotopic to a loop in A by a homotopy in U_n , as desired. Note that if $\{V_n\}_{n \in \mathbb{N}}$ is not decreasing, then we can put V'_n be the path component of $V_1 \cap V_2 \cap \dots \cap V_n$ that contains A .

Let $\{U'_n\}_{n \in \mathbb{N}}$ be an arbitrary nested basis of A . Since $\{U_n\}_{n \in \mathbb{N}}$ is a nested basis of A , there exists a subsequence $\{k_n\}_{n \in \mathbb{N}}$ of \mathbb{N} such that $U_{k_n} \subseteq U'_n$ for every $n \in \mathbb{N}$. Let $\{V_{k_n}\}$ be the π_1 -refinement of $\{U_{k_n}\}$. Then $\{V'_n\}_{n \in \mathbb{N}}$ is a π_1 -refinement of $\{U'_n\}_{n \in \mathbb{N}}$, where $V'_n = V_{k_n}$ for each n . \square

Remark 4.5. If X is a metric space with a metric d , then for the quotient space X/A , the map $\bar{d} : X/A \times X/A \rightarrow [0, +\infty)$ defined by

$$\bar{d}(x, y) = \begin{cases} \min\{d(x, y), d(x, A) + d(y, A)\} & x, y \neq * \\ d(A, x) & x \neq y = * \\ 0 & x = y = *, \end{cases}$$

where $*$ = $p(A)$ and $p : X \rightarrow X/A$ is the natural quotient map, is a metric and the topology induced by this metric on X/A is equivalent to the quotient topology on X/A since A is compact (see [2, Section 3.1]).

For a topological space X , a loop α in X based at x is called *semi-simple* if $\alpha^{-1}(\{x\}^c) = (0, 1)$ and is called *geometrically simple* if $\alpha^{-1}(\{x\}^c) = (a, b)$ for $a, b \in [0, 1]$. Also, every geometrically simple loop is homotopic to a semi-simple loop [10].

Theorem 4.6. *Let X be a locally 1-connected metric space and A be a connected, locally path connected and compact subset of X . Then X/A is locally 1-connected space.*

Proof. Since X is locally path connected and A is connected, X/A is locally path connected. Clearly $X - A$ is an open subset of X and hence it is locally 1-connected. Since $q := p|_{X-A} : X - A \rightarrow (X/A) - \{*\}$ is homeomorphism, $(X/A) - \{*\}$ is locally 1-connected.

Let U be an open neighborhood of $*$. By the proof of Theorem 4.4, the nested basis $\{U_n = S_{\frac{1}{n}}(A)\}_{n \in \mathbb{N}}$ for A has a refinement $\{V_n\}_{n \in \mathbb{N}}$ as its π_1 -refinement in A at $\{U_n\}_{n \in \mathbb{N}}$ such that V_n 's are path connected. We can assume that $U_1 \subseteq p^{-1}(U)$ and show that $V := p(V_1)$ is a desired open neighborhood. Let $\alpha : I \rightarrow V$ be a loop at $*$, then we must show that $(i_V)_*(\alpha) = 0$, where $i : V \hookrightarrow U$ is the inclusion map.

Case 1:

Suppose that α is a semi-simple loop and $a_0 = \lim_{t \rightarrow 0} p^{-1}(\alpha(t))$ and $a_1 = \lim_{t \rightarrow 1} p^{-1}(\alpha(t))$ exist. Obviously $a_0, a_1 \in \partial A$ and since A is closed path connected, there exists a loop $\tilde{\alpha}$ in V_1 with base point $a \in A$ such that $p \circ \tilde{\alpha} \simeq \alpha$ by a homotopy in V . Since V_1 is π_1 -contained in A at U_1 , there is a loop β in A such that $\beta \simeq \tilde{\alpha}$ by a homotopy in U_1 . Therefore $\alpha \simeq p \circ \tilde{\alpha} \simeq p \circ \beta \simeq e_*$ by homotopies in $p(U_1) \subseteq U$, as desired.

Case 2:

Suppose that α is a semi-simple loop and at least one of the above limits does not exist. Let $\tilde{\alpha} = q^{-1}(\alpha|_{(0,1)})$. Without loss of generality, assume $a_0 = \lim_{t \rightarrow 0} \tilde{\alpha}(t)$ exists and $a_1 \in A$ is a limit point of $\tilde{\alpha}((1/2, 1))$. The point a_1 is a limit point and α is continuous, hence there is an increasing sequence $\{t_n\}_{n \in \mathbb{N}}$ in $(1/2, 1)$ such that $\tilde{\alpha}([t_n, 1]) \subseteq V_n$, for every $n \in \mathbb{N}$ and $t_n \rightarrow 1$. Since the V_n 's are path connected, there are paths $\gamma_n : [t_n, 1] \rightarrow V_n$ from $\tilde{\alpha}(t_n)$ to a_1 . If $\beta_n := p \circ \gamma_n$, then $Im \beta_n \subseteq p(V_n)$ and $\tilde{\beta}_n * \hat{\alpha}|_{[t_n, t_{n+1}]} * \beta_{n+1}$ is null in $p(U_n)$ since $\tilde{\gamma}_n * \hat{\alpha}|_{[t_n, t_{n+1}]} * \gamma_{n+1}$ is a loop in V_n . Let

$$\alpha_n(t) = \begin{cases} \alpha(t) & t \leq t_n \\ \beta_n(t) & t \geq t_n. \end{cases}$$

By Lemma 3.2, $\alpha \simeq \alpha_n$ for each n and by Case 1, α_n 's are nullhomotopic in $p(U_1)$ which implies that α is nullhomotopic.

Case 3:

If α is not a semi-simple loop in V , then $\alpha^{-1}(X/A - \{*\}) = \bigcup_{i \in \mathbb{N}} (a_i, b_i)$. Obviously $\alpha|_{[a_i, b_i]}$ is null in a $p(U_{n_i})$, where $n_i = \max\{n \in \mathbb{N} | \alpha([a_i, b_i]) \subseteq p(V_n)\}$. By Lemma 3.4, α is null.

Similarly, the result holds if α is a geometrically simple loop. Also, in Case 3, if $\alpha^{-1}(X/A - \{*\}) = \bigcup_{i=1}^n (a_i, b_i)$, by Corollary 3.1 the result holds. \square

Proof of Theorem A. By Remark 4.5 and Theorem 4.6, X/A_1 is a locally 1-connected metric space, hence we can use Theorem 4.6 again and by induction the result holds.

In the following examples, we show that locally path connectedness and locally simply connectedness of X are necessary conditions in Theorem A.

Example 4.7. Let $Y = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = (1/2 + 1/n)^2, n \in \mathbb{N}\}$, $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1/4\} \cup \{(x, 0) \in \mathbb{R}^2 \mid 1/2 \leq x \leq 1\}$ and $X = Y \cup A$ with $a = (1, 0)$ as the base point (see Figure 2). Then X is connected, locally simply connected metric space and A is compact, connected and locally path connected subset of X , but obviously, X is not locally path connected. Since for every open neighborhood $U \subseteq X/A$ of $*$ there exists $N_0 \in \mathbb{N}$ such that for $n > N_0$, $p^{-1}(U)$ contains circles with radius $1/2 + 1/n$, X/A is the union of a null sequence of simple closed curves meeting in a common point that is homeomorphic to the Hawaiian earring space. Hence X/A is not locally simply connected.

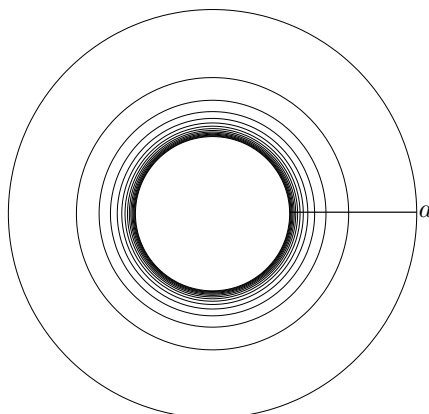


Figure 2:

Example 4.8. Let $X = \{(x, y) \in \mathbb{R}^2 \mid 1/4 \leq x^2 + y^2 \leq 1\} \setminus \{(-1/2 - 1/n, 0) \in \mathbb{R}^2 \mid n \in \mathbb{N}\}$ and $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1/4\} \cup \{(x, 0) \in \mathbb{R}^2 \mid 1/2 \leq x \leq 1\}$. Then X is a connected, locally path connected metric space that is not locally simply connected and A is a compact, connected and locally path connected subset of X . Similar to Example 4.7, it can be proved that X/A is homeomorphic to $Y = D^1 \setminus \{(-1/n, 0) \mid n \in \mathbb{N}\}$ which is not locally simply connected.

5. Some applications to fundamental groups

In this section, first we give a short proof of Theorem B and then prove some properties of the fundamental group of the quotient space $X/(A_1, A_2, \dots, A_n)$.

Proof of Theorem B. By Theorem 2.4, it suffices to prove that $X/(A_1, A_2, \dots, A_n)$ is semi-locally simply connected. But by Theorem A, we have $X/(A_1, A_2, \dots, A_n)$ is locally 1-connected which is more.

Remark 5.1. In Examples 4.7 and 4.8, p_* is not onto since $\pi_1(X, a)$ is a free group on countably many generators and $\pi_1(X/A, *)$ is uncountable. This shows that the hypothesis locally 1-connectedness is necessary for Theorem B.

We know that any homomorphic image of a finitely generated group is also finitely generated. Hence we have the following consequence of Theorem B.

Corollary 5.2. *Let X be connected, locally 1-connected metric space and A_1, A_2, \dots, A_n be compact, connected, locally path connected subsets of X . If $\pi_1(X, a)$ is finitely generated, then so is $\pi_1(X/(A_1, A_2, \dots, A_n), *)$.*

By a theorem of Shelah [12], if X is a compact, connected, locally path connected metric space and $\pi_1(X, x)$ is countable, then $\pi_1(X, x)$ is finitely generated. Using this and Theorem B, we have

Corollary 5.3. *Let X be compact, connected, locally 1-connected, metric space and A_1, A_2, \dots, A_n be closed, connected, locally path connected subsets of X . If $\pi_1(X/(A_1, A_2, \dots, A_n), *)$ is not finitely generated, then $\pi_1(X, a)$ is uncountable.*

J. Dydak and Z. Virk [7] proved that if X is a connected, locally path connected metric space and $\pi_1(X, x)$ is countable, then $\pi_1(X, x)$ is a finitely presented group. Note that a homomorphic image of a finitely presented group is not necessarily a finitely presented group.

Corollary 5.4. *Let X be connected, locally 1-connected metric space and A_1, A_2, \dots, A_n be compact, connected, locally path connected subsets of X . If $\pi_1(X, a)$ is countable, then $\pi_1(X/(A_1, A_2, \dots, A_n), *)$ is finitely presented.*

J.W. Cannon and G.R. Conner [6] proved that if X is a one-dimensional metric space which is connected, locally path connected and semi-locally simply connected (or equivalently if X admits a universal covering space), then $\pi_1(X, x)$ is a free group. Therefore by Theorem A we have

Corollary 5.5. *If X is a one-dimensional metric space which is connected, locally 1-connected, and A_1, A_2, \dots, A_n are compact, connected, locally path connected subsets of X , then $\pi_1(X/(A_1, A_2, \dots, A_n), *)$ is a free group.*

Theorem 5.6. *If X is a separable, connected, locally 1-connected metric space and A_1, A_2, \dots, A_n are compact, connected, locally path connected subsets of X , then $\pi_1(X/(A_1, A_2, \dots, A_n), *)$ is countable. Moreover, $\pi_1(X/(A_1, A_2, \dots, A_n), *)$ is finitely presented.*

Proof. Using Theorem A and [6, Lemma 5.6], we have $\pi_1(X/(A_1, A_2, \dots, A_n), *)$ is countable and by [7], $\pi_1(X/(A_1, A_2, \dots, A_n), *)$ is finitely presented. \square

J. Pawlikowski [11] proved that compact, connected, locally path connected metric spaces with countable fundamental groups are semi-locally simply connected. Therefore we have

Corollary 5.7. *If X is a compact, connected, locally path connected metric space and A_1, A_2, \dots, A_n are connected, locally path connected closed subsets of X , then $p_* : \pi_1(X, a) \rightarrow \pi_1(X/(A_1, \dots, A_n), *)$ is an epimorphism if one of the following conditions holds*

- (i) $\pi_1(X/(A_1, A_2, \dots, A_n), *)$ is countable,
- (ii) $\pi_1(X/(A_1, A_2, \dots, A_n), *)$ is finitely generated,
- (iii) $\pi_1(X/(A_1, A_2, \dots, A_n), *)$ is finitely presented,
- (iv) $X/(A_1, \dots, A_n)$ has universal covering.

Proof. By Assumptions, Remark 4.5 and continuity of $p : X \rightarrow X/(A_1, A_2, \dots, A_n)$, $X/(A_1, A_2, \dots, A_n)$ is a compact, connected, locally path connected metric space and the result follows from Theorem 2.4. \square

Remark 5.8. J. Brazas [4] introduced a new topology on $\pi_1(X, x)$ coarser than the topology of $\pi_1^{qtop}(X, x)$ and proved that fundamental groups by this new topology are topological groups, denoted by $\pi_1^\tau(X, x)$. Since the topology of $\pi_1^\tau(X, x)$ is coarser than $\pi_1^{qtop}(X, x)$, the results of [10] remain true and we have a similar result to Theorem B for $\pi_1^\tau(X, x)$ and $\pi_1^\tau(X/(A_1, \dots, A_n), *)$.

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