Filomat 28:1 (2014), 49–63
DOI 10.2298/FIL1401049O

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Nyström Methods for Fredholm Integral Equations Using Equispaced Points

Donatella Occorsio[a], Maria Grazia Russo[a]

[a]*Department of Mathematics, Computer Science and Economics, University of Basilicata*
viale dell'Ateneo Lucano 10, 85100 Potenza, Italy

Abstract.
In this paper we investigate some Nyström methods for Fredholm integral equations in the interval $[0, 1]$. We give an overview of the order of convergence, which depends on the smoothness of the involved functions. In particular, we consider the Nyström methods based on the so called Generalized Bernstein quadrature rule, on a Romberg scheme and on the so-called IMT rule. We prove that the proposed methods are convergent, stable and well conditioned. Also, we give several numerical tests for comparing these three methods.

1. Introduction

In the present paper we consider the Fredholm integral equations of the second kind

$$f(x) - \mu \int_0^1 f(t)k(x,t)dt = g(x) \tag{1}$$

where $\mu \in \mathbb{R}$, the kernel k and the right–hand side g are given and f is the unknown function.

Many problems in engineering and mathematical physics can be modeled by one dimensional Fredholm integral equations of the type (1). Take, for instance, the Love's equation arising in the electrostatic problem of a circular plate condenser in an unbounded perfect fluid (see [15]).

There is a wide literature about the numerical methods for solving this kind of equations (see for instance [2] and the related bibliography). Projection and Nyström methods based on Gauss-Jacobi quadrature rules were deeply investigated in weighted spaces of functions (see for instance [10] and the references therein). Since in some applications only the pointwise values of the kernels and/or the right-hand sides can be known (generally on equally spaced knots), the above mentioned methods cannot be applied. On the other hand the methods based on piecewise polynomials usually produce a low degree of approximation, or more in general show saturation phenomena. Therefore, in order to obtain"faster methods", a possible solution is to introduce suitable extrapolation techniques (see for instance [23]).

Here, first of all, we propose a Nyström method based on a quadrature rule obtained by means of the sequence $\{B_{m,s}(f)\}_m$ of the so called *Generalized Bernstein polynomials* introduced in [16] (see also [17]).

2010 *Mathematics Subject Classification*. 65D32; 65R20, 41A10
Keywords. Fredholm integral equations, Nyström method, Generalized Bernstein polynomials
Received: 26 February 2013; Accepted: 20 July 2013
Communicated by Gradimir Milovanović
Email addresses: donatella.occorsio@unibas.it (Donatella Occorsio), mariagrazia.russo@unibas.it (Maria Grazia Russo)

Such a kind of formula is of interest since it is based on equally spaced knots in $[0,1]$ and its convergence order increases for smoother integrand functions, differently from the most popular rules based on piecewise polynomials approximation. Moreover there exists a numerical evidence proving that the speed of convergence of the formula increases for higher values of the parameter s (and for fixed m).

Hence we consider two more methods, the first one based on the Romberg integration scheme (see for instance [9]) which also use equispaced nodes and the Iri, Moriguti, Takasawa (IMT) formula [14] which use the equispaced points in order to compute the nodes. These methods have both a speed of convergence depending on the smoothness of the involved functions.

We outline that all the proposed Nyström methods are stable and convergent in the space of continuous functions and we give an error estimate in a suitable subspace of Sobolev type in the case of the method based on the Generalized Bernstein polynomials.

Moreover the linear systems equivalent to the Nyström methods are uniquely solvable and well-conditioned too. The numerical experiments show that the three methods are almost equivalent.

Finally we focus our attention on *centrosymmetric kernels* (i.e. $k(x,t) = k(1-x,1-t)$, or *skew-centrosymmetric kernels* (i.e. $k(x,t) = -k(1-x,1-t)$. For instance the previously described Love's equation and the boundary integral equation for a plane interior Dirichlet problem for an ellipse (see [3]) have centrosymmetric kernels. For these choices of kernels, due to the particular structure of the matrix of the linear system, the Nyström interpolant can be computed with at least a 75% reduction of the time complexity, if the Gaussian elimination method is used.

The outline of the paper is as follows. Section 2 is devoted to some notations and preliminary results. In Section 3 we study the quadrature rule based on the Generalized Bernstein polynomials and summarize the most important properties of the Romberg and IMT rules. Section 4 is devoted to the Nyström methods. Finally, Section 5 contains computational details about the construction of the approximating solutions and some numerical tests which confirm the theoretical results and the performance of three procedures.

2. Notations and preliminary results

In the sequel we will denote by C any positive constant which can be different in different formulas. Moreover $C \neq C(a,b,..)$ will be used meaning that the constant C is independent of $a,b,..$, while $C = C(a,b,\ldots)$ will outline the dependence on $a,b,\ldots$.

Now denote by $C^0([0,1])$ the space of continuous functions, equipped with the uniform norm $\|f\|_\infty = \max_{x\in[0,1]} |f(x)|$. In the space $C^0([0,1])$ it is possible to define the following modulus of smoothness of Ditzian and Totik [12]

$$\begin{aligned}\omega^k_\varphi(f,t) &= \sup_{h\le t} \max_{x\in[4h^2k^2,1-4h^2k^2]} |\Delta^k_\varphi hf(x)| + \inf_{P\in\mathbb{P}_{k-1}} \max_{x\in[0,4t^2k^2]} |f(x)-P(x)| \\ &+ \inf_{P\in\mathbb{P}_{k-1}} \max_{x\in[1-4t^2k^2,1]} |f(x)-P(x)|,\end{aligned}$$

where $\varphi(x) = \sqrt{x(1-x)}$,

$$\Delta^k_\varphi hf(x) = \sum_{i=0}^{k} (-1)^{i-1} \binom{k}{i} f\left(x + \frac{kh}{2}\varphi(x) - ih\varphi(x)\right)$$

and $\mathbb{P}_n$ denotes the space of all algebraic polynomials of degree n. It is well known that if we define by $E_m(f) = \inf_{P\in\mathbb{P}_m} \|f-P\|_\infty$ the error of best polynomial approximation of f in $C^0([0,1])$, the following Jackson and Stechkin inequalities hold true

$$E_m(f) \le C\omega^k_\varphi\left(f, \frac{1}{m}\right), \quad C \neq C(m,f) \tag{2}$$

and

$$\omega_\varphi^k(f,t) \le Ct^k \sum_{0\le i\le 1/t} (1+i)^{k-1} E_i(f), \quad C \ne C(t,f).$$

If $\mathcal{AC}(0,1)$ denotes the space of the absolutely continuous functions on $(0,1)$, let W_r, $r \ge 1$, be the following Sobolev–type space

$$W_r = \left\{ f \in C^0([0,1]) : f^{(r-1)} \in \mathcal{AC}(0,1), \|f^{(r)}\varphi^r\|_\infty < \infty \right\},$$

equipped with the norm $\|f\|_{W_r} = \|f\|_\infty + \|f^{(r)}\varphi^r\|_\infty$.

Obviously if $f \in C^r([0,1])$, i.e. f has r continuous derivatives in $[0,1]$, then $f \in W_r$ too.

Finally it is possible to prove that [12] if $f \in W_r$ then

$$\omega_\varphi^k(f,t) \le Ct^r \|f^{(r)}\varphi^r\|_\infty, \quad k \ge r, \quad C \ne C(t,f).$$

Therefore by (2) it immediately follows, for any $f \in W_r$,

$$E_m(f) \le \frac{C}{m^r} \|f^{(r)}\varphi^r\|_\infty, \quad C \ne C(m,f). \tag{3}$$

3. Quadrature rules

In this section we describe the three quadrature formulas on which we will construct the corresponding Nyström methods.

3.1. The GB quadrature formula

This formula was introduced in [16] and recently investigated in [20]. The quadrature rule is strictly connected with the so-called Generalized Bernstein polynomials.

Denote by $B_m(f)$ the m-th Bernstein polynomial approximating a given function $f \in C^0([0,1])$

$$B_m(f;x) = \sum_{k=0}^m p_{m,k}(x) f(t_k), \qquad t_k = \frac{k}{m}, \quad k = 0,1,\dots,m,$$

where the polynomials $p_{m,k}$ are defined as

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}, \quad k = 0,1,\dots,m,$$

and satisfy the following recurrence relation

$$p_{m,k}(x) = (1-x)p_{m-1,k}(x) + xp_{m-1,k-1}(x). \tag{4}$$

We recall the definition of the Generalized Bernstein operator $B_{m,s}$, $s \in \mathbb{N}$, introduced in [16] as the s-th iterated Boolean sum of B_m and defined as

$$B_{m,s} = I - (I - B_m)^s, \quad B_m^1 \equiv B_m, \quad B_m^i = B_m(B_m^{i-1}), \quad i = 2,\dots,s.$$

Using the definition of the Bernstein polynomials it is possible to give the following explicit expression of the polynomial $B_{m,s}(f)$:

$$B_{m,s}(f;x) = \sum_{j=0}^m p_{m,j}^{(s)}(x) f(t_j), \tag{5}$$

where

$$p_{m,j}^{(s)}(x) = \sum_{i=1}^{s} \binom{s}{i} (-1)^{i-1} B_m^{i-1}(p_{m,j}; x). \tag{6}$$

Starting by the formulation (5) of $B_{m,s}(f)$, we get the Generalized Bernstein (GB) quadrature rule

$$\int_0^1 f(t)dt = \int_0^1 B_{m,s}(f;t)dt + R_m^{(s)}(f) =: \Sigma_m^{(s)}(f) + R_m^{(s)}(f), \tag{7}$$

where $R_m^{(s)}(f)$ denotes the quadrature error and

$$\Sigma_m^{(s)}(f) = \sum_{j=0}^{m} D_j^{(s)} f(t_j) := \sum_{j=0}^{m} f(t_j) \int_0^1 p_{m,j}^{(s)}(t)\, dt\,. \tag{8}$$

In [20] it was proved that (8) is stable and convergent and its order of accuracy increases faster as smoother is the function f. More precise estimates can be deduced from a result in [13], showing also the dependance on the parameter s.

Proposition 3.1. *Let $f \in C^0([0,1])$. Then, for any fixed $s \geq 1$, there holds*

$$|R_m^{(s)}(f)| \leq C\left(\omega_\varphi^{2s}\left(f, \frac{1}{\sqrt{m}}\right) + \frac{\|f\|_\infty}{m^s}\right), \qquad C \neq C(f,m). \tag{9}$$

Moreover if $f \in W_r$, $r \geq 1$, there holds, for sufficiently large m, and for $s \geq \frac{r}{2}$,

$$|R_m^{(s)}(f)| \leq C\left(\frac{\|f^{(r)}\varphi^r\|_\infty}{\sqrt{m^r}} + \frac{\|f\|_\infty}{m^s}\right), \tag{10}$$

where C is a positive constant depending on s and independent of f and m.

Now we give some details on the practical implementation of the formula (8).

Let us start with giving a simpler expression of the basis polynomials $\{p_{m,j}^{(s)}(x)\}_{i=0}^m$ defined in (6). Setting

$$\mathbf{p}_m^{(s)}(x) := [p_{m,0}^{(s)}(x), p_{m,1}^{(s)}(x), \ldots, p_{m,m}^{(s)}(x)]^T,$$

and

$$\mathbf{p}_m(x) := [p_{m,0}(x), \ldots, p_{m,m}(x)]^T,$$

in [19] the following vector expression for the basis was proved $\{p_{m,i}^{(s)}\}_{i=0}^m$,

$$\mathbf{p}_m^{(s)}(x)^T = \mathbf{p}_m^T(x) C_{m,s}$$

where $C_{m,s} \in \mathbb{R}^{(m+1)\times(m+1)}$ is defined as

$$C_{m,s} \quad = \quad I + (I - \mathbf{A}_m) + \ldots + (I - \mathbf{A}_m)^{s-1}, \tag{11}$$

I being the identity matrix of order $m+1$ and the entries of $\mathbf{A}_m \in \mathbb{R}^{(m+1)\times(m+1)}$ given by

$$(\mathbf{A}_m)_{i,j} = p_{m,j}(t_i), \quad i = 0,1,\ldots,m, \quad j = 0,1,2,\ldots m.$$

We remark that the matrix $C_{m,s}$ is non-singular (see [19], [20]). Therefore, setting $\mathbf{f} = [f_0, f_1, \ldots, f_m]^T$, $f_i = f\left(\frac{i}{m}\right)$, the polynomial $B_{m,s}(f)$ can be represented as

$$B_{m,s}(f;x) = \mathbf{p}_m^T(x) C_{m,s} \mathbf{f}. \tag{12}$$

The relation (12) allows us to say that the polynomial $B_{m,s}(f)$ is the Bernstein approximant of a function G s.t. $G(t_i) = (C_{m,s}\mathbf{f})_i$, $i = 0, 1, \ldots, m$.

Now, taking into account that

$$\int_0^1 p_{m,i}(t)dt = \frac{1}{m+1}, \quad i = 0, 1, \ldots, m,$$

we easily deduce

$$\Sigma_m^{(s)}(f) = \sum_{j=0}^{m} D_j^{(s)} f(t_j) = \frac{1}{m+1} \sum_{j=0}^{m} \left(\sum_{i=0}^{m} (C_{m,s})_{i,j} \right) f(t_j). \tag{13}$$

By (13) it follows that the computational cost of the quadrature rule is essentially that of the construction of $C_{m,s}$.

In addition we remark that the matrix $\mathbf{A}_m$ is *centrosymmetric*, i.e. $\mathbf{A}_m = J\mathbf{A}_m J$, where J is the *counteridentity* matrix, the entries of which are defined as $J_{i,j} = \delta_{i,m-j+1}$, $\delta_{h,k}$ being the Kronecher delta. Therefore, also in view of (4), the construction of $\mathbf{A}_m$ can be arranged in $\frac{m^3}{4}$ flops. Moreover, since the product of two centrosymmetric matrices can be performed in $O(m^3/4)$ flops (see for instance [22],[5]), the computation of $C_{m,s}$ requires $(s-1)m^3/4$ flops.

A significant reduction in computing $C_{m,s}$ can be realized by choosing $s = 2^p$, since in this case

$$C_{m,2^p} = C_{m,2^{p-1}} + (I - \mathbf{A}_m)^{2^{p-1}} C_{m,2^{p-1}}, \tag{14}$$

and then the construction of $C_{m,s}$ requires $2(p-1)$ products of centrosymmetric matrices that means about $\frac{m^3}{2}(p-1)$ flops.

For instance, for $s = 256$, the cost is $3.5m^3$ instead of $63.75m^3$ needed by using the Horner's algorithm applied to the matrix polynomial in (11).

3.2. The Romberg rule

Now we recall the well known Romberg rule

$$\int_0^1 f(t)dt = T_{N,N}(f) + e_N(f),$$

where $e_N(f)$ denotes the quadrature error and $T_{N,N}(f)$ is obtained by the triangular scheme

$$T_{j,k}(f) = \frac{4^k T_{j-1,k-1}(f) - T_{j,k-1}(f)}{4^k - 1}, \quad 1 \le k \le j \le N, \tag{15}$$

being $T_{j,0}(f)$ the trapezoidal rules in $[0, 1]$ associated with the steps 2^{-j}, $j = 0, 1, \ldots, N$, respectively, i.e.

$$T_{j,0}(f) = \frac{1}{2^j} \left[\frac{f(0) + f(1)}{2} + \sum_{h=1}^{2^j-1} f\left(\frac{h}{2^j}\right) \right]. \tag{16}$$

Since our final aim is to consider a Nyström method based on this quadrature rule, it is necessary to rewrite it as

$$\int_0^1 f(t)dt = \sum_{i=0}^{2^N} \sigma_i f\left(\frac{i}{2^N}\right) + e_N(f). \tag{17}$$

By writing down explicitly the coefficients of the trapezoidal rules, using (15) and collecting all the coefficients of any $f\left(\frac{i}{2^N}\right)$, $i = 0, 1, \ldots, 2^N$, we found that (see also [6]),

$$\sigma_0 = \sigma_{2^N} = \frac{1}{2}\sum_{k=0}^{N}\beta_k, \quad \sigma_2 = \beta_N + \beta_{N-1}$$

$$\sigma_{2^p} = \sigma_{2^{p-1}} + \beta_{N-p}, \qquad p = 2, 3, \ldots, N-1$$

$$\sigma_{2^p+k2^{p+1}} = \sigma_{2^p}, \qquad k = 1, 2, \ldots, 2^{N-p-1}-1, \quad p = 1, 2, \ldots, N-2,$$

$$\sigma_{2i-1} = \beta_N, \qquad i = 1, 2, .., 2^{N-1}$$

with $\beta_j = \frac{\alpha_j}{2^j}$, $\alpha_j = (-1)^N \prod_{\substack{i=0 \\ i\neq j}}^{N} \frac{1}{4^{i-j}-1}$, $j = 0, 1, \ldots, N$.

About the convergence of the formula (17) we recall two results.

The first one is the classical estimate holding for very smooth functions [9]. For any function having the $(2N+2)$-th continuous derivative in $[0, 1]$, it results

$$e_N(f) \leq \frac{|B_{2N+2}|}{(2N+2)!}\frac{\|f^{(2N+2)}\|_\infty}{2^{N(N+1)}}, \tag{18}$$

where B_{2N+2} is the $(2N+2)$-th Bernoulli number.

The second result completes the previous one in the case of functions having a continuous derivative of fixed order $\mu \geq 1$ in $[0, 1]$. It can be found in [7]:

$$e_N(f) \leq C(N, \mu)\frac{2^{(\mu-1)^2/4}}{(2\pi)^\mu}\frac{\|f^{(\mu)}\|_\infty}{2^{N\mu}}, \quad \forall N \geq \frac{\mu-1}{2}, \tag{19}$$

where $C(N, \mu) \in [1.5, 3.1]$.

3.3. The IMT rule

The IMT quadrature formula [14] is based on the idea of making a change of variables in order to obtain a new integrand function, vanishing, with its derivatives, at both the endpoints of the interval $[0, 1]$. Hence the trapezoidal rule is applied to the new integrand, since for this class of functions this formula has a high degree of accuracy.

Indeed let G be a function having the $(2l+1)$-th continuous derivative on $[0, 1]$, $l \geq 1$, and such that $G'(0) = G'(1)$, $G^{(3)}(0) = G^{(3)}(1)$, …, $G^{(2l-1)}(0) = G^{(2l-1)}(1)$. If we consider the trapezoidal rule $T_n(G)$ on $n+1$ knots (replace in (16) f with G and 2^j with n), the following result is known [9, p.137]

$$\left|\int_0^1 G(t)dt - T_n(G)\right| \leq C\frac{\|G^{(2l+1)}\|_\infty}{n^{2l+1}}, \tag{20}$$

where $C = \frac{2\zeta(2l+1)}{(2\pi)^{2l+1}}$, and $\zeta(l) = \sum_{j=1}^{\infty} j^{-l}$ denotes the Riemann zeta function.

So the proposed transformation ψ in [14] (see also [9]) for the so-called IMT formula, is given by the following definition

$$\phi(\tau) = e^{-a\tau^{-p}-b(1-\tau)^{-q}}, \quad p, q \geq 1, \quad a, b > 0,$$

$$\psi(x) = \frac{1}{L}\int_0^x \phi(\tau)d\tau, \quad L = \int_0^1 \phi(\tau)d\tau.$$

The function ψ is an increasing one-to-one transformation of $[0,1]$ onto itself. Therefore

$$\int_0^1 f(t)dt = \frac{1}{L}\int_0^1 f(\psi(x))\phi(x)dx =: \int_0^1 G(x)dx.$$

Since all the derivatives of ϕ vanish at both the endpoints, then if f has r continuous derivatives on $[0,1]$, also G has at least r continuous derivatives and $G^{(j)}(0) = G^{(j)}(1) = 0$, for $j = 0, 1, \ldots, r$.

Applying the trapezoidal rule T_n, and taking into account that $\phi(0) = \phi(1) = 0$, we get

$$\int_0^1 f(t)dt = \frac{1}{nL}\sum_{j=1}^{n-1} f(\psi(t_j))\phi(t_j) + \varepsilon_n(f) =: \sum_{j=1}^{n-1} W_j f(x_j) + \varepsilon_n(f), \tag{21}$$

where $\varepsilon_n(f)$ denotes the error of the quadrature formula and

$$t_j = \frac{j}{n}, \quad x_j = \psi(t_j), \quad W_j = \frac{1}{nL}\phi(t_j), \quad j = 1, \ldots, n-1.$$

Therefore the equispaced points are not used directly in the formula, but they appear in the definition of the knots x_j of the quadrature formula.

In [9] it is suggested to take $p = q = 1$ or $p = q = 2$ and $a = b$. In these cases the rule is symmetric about $x = \frac{1}{2}$.

Of course the knots x_j, $j = 1, \ldots, n-1$, need to be computed by a suitable quadrature formula. We found that a good choice in order to perform the x_j in double precision is the Gauss-Legendre quadrature rule on 128 nodes (see also [4, p.117]). Hence, in order to use (21), $(n-1)$ evaluations of the function f and $128(n+1)$ evaluations of the function ϕ are required.

Finally we remark that, to our knowledge, there is no convergence estimate, in terms of the smoothness of the function f, available in the literature.

4. The Nyström method

Consider the Fredholm integral equation

$$f(x) - \mu\int_0^1 f(t)k(x,t)dt = g(x), \quad x \in [0,1]. \tag{22}$$

Denoting by

$$Kf(x) = \mu\int_0^1 k(x,t)f(t)\,dt$$

the equation (22) can be rewritten as

$$(I - K)f = g, \tag{23}$$

where I is the identity operator on $C^0([0,1])$. Here and in the sequel we will use k_t (respectively k_x) in order to denote the bivariate function $k(x,t)$ as a function of the single variable x (respectively t).

It is known [2] that if the kernel $k(x,t)$ is continuous, then the operator $K : C^0([0,1]) \to C^0([0,1])$ is compact. Consequently the Fredholm Alternative holds true for (23) in $C^0([0,1])$. Moreover under the assumption

$$\sup_{t\in[0,1]} \|k_t\|_{W_r} < +\infty, \quad r \geq 1, \tag{24}$$

it immediately results that $Kf \in W_r$, for any $f \in C^0([0,1])$. Therefore if in addition to (24) we assume that $g \in W_r$, and $\ker(I-K) = \{0\}$ in $C^0([0,1])$, then the solution f of (23) is in W_r too.

Let us consider now the Nyström method based on a quadrature formula of the type

$$\int_0^1 f(t)\,dt = \sum_{i=0}^{M} w_i f(\tau_i) + R_M(f), \tag{25}$$

where w_i, τ_i, $i = 0,1,\dots,M$, denote the coefficients and the knots of the formula respectively, while $R_M(f)$ is the quadrature error.

Then define the discrete operator

$$K_M f(x) = \mu \sum_{i=0}^{M} w_i k(x,\tau_i) f(\tau_i),$$

and consider the operator equation

$$(I - K_M) f_M = g, \tag{26}$$

where f_M is unknown. Collocating the equation (26) on the knots τ_h, $h = 0,\dots,M$, and setting $\alpha_i = f(\tau_i)$, $i = 0,1,\dots,M$, the quantities $\{\alpha_i\}_{i=0}^M$ turns to be the unknowns of the linear system

$$\alpha_h - \mu \sum_{i=0}^{M} w_i k(\tau_h,\tau_i)\alpha_i = g(\tau_h), \qquad h = 0,1,\dots,M. \tag{27}$$

By means of the solution $\{\alpha_i^*\}_{i=0}^M$ of the system, if it exists, it is possible to construct the Nyström interpolant of the solution f

$$f_M(x) = \mu \sum_{i=0}^{M} w_i k(x,\tau_i)\alpha_i^* + g(x). \tag{28}$$

Now denote by $\mathbf{H}_M$ the coefficient matrix of the system (27) of order $M+1$ and by $\mathbf{cond}(\mathbf{H}_M) = \|\mathbf{H}_M\|_\infty \|\mathbf{H}_M^{-1}\|_\infty$ the condition number of the linear system (27) w.r.t. the infinity norm.

The following theorem about the stability and the convergence of the proposed Nyström method is well known [2, Th.4.1.2].

Theorem 4.1. *Assume that k is continuous on $[0,1]^2$ and that $\ker(I-K) = \{0\}$ in $C^0([0,1])$. Denote by f^* the unique solution of (23) in $C^0([0,1])$ for a given $g \in C^0([0,1])$. Then, if (25) converges for any $f \in C^0([0,1])$ then the system (27) is uniquely solvable and well conditioned, i.e.*

$$\mathbf{cond}(\mathbf{H}_M) \leq C, \qquad C \neq C(M).$$

Moreover the sequence (28) is convergent and it results

$$\|f^* - f_M\|_\infty \sim \|Kf^* - K_M f^*\|_\infty. \tag{29}$$

Now we state a corollary about the Nyström method based on the GB quadrature rule. This means that we use (25) defined by (7)-(8). Hence $M \equiv m$, $\tau_i \equiv t_i$ and $w_i \equiv D_i^{(s)}$.

Corollary 4.2. *If for some $r \in \mathbb{N}, r \geq 1$, $g \in W_r$, the kernel k satisfies (24) and*

$$\sup_{x\in[0,1]} \|k_x\|_{W_r} < +\infty, \tag{30}$$

then, for any $s \geq \frac{r}{2}$ it results

$$\|f^* - f_m\|_\infty \leq C\frac{\|f^*\|_{W_r}}{\sqrt{m^r}}, \tag{31}$$

where $C \neq C(m, f^*)$ and $C = C(s)$.

Proof. Holding (9), the Nyström method (26) is based on a quadrature formula which is convergent for any continuous function. Hence, by Theorem 4.1 the method is stable and convergent in $C^0([0,1])$ and well conditioned too.

Now in view of (10), we get, since $s \geq \frac{r}{2}$,

$$\begin{aligned}\|Kf^* - K_m f^*\|_\infty &\leq C\left\{\frac{\|f^* k_x\|_\infty}{m^s} + \frac{\|(f^* k_x)^{(r)}\varphi^r\|_\infty}{\sqrt{m^r}}\right\} \\ &\leq \frac{C}{\sqrt{m^r}}\sum_{j=0}^{r}\binom{r}{j}\|f^{*(j)}k_x^{(r-j)}\varphi^r\|_\infty \leq \frac{C}{\sqrt{m^r}}\sum_{j=0}^{r} a_j\|f^{*(j)}\varphi^j\|_\infty\end{aligned}$$

where $C \neq C(m, f^*)$ and $a_j = \binom{r}{j}\sup_{x\geq 0}\|k_x^{(r-j)}\varphi^{r-j}\|_\infty$. Hence, since by (30) it follows that a_j, $j = 0, \ldots, r$, are bounded, using [11, p. 310, Lemma 2.1]

$$\sum_{j=0}^{r} a_j\|f^{*(j)}\varphi^j\|_\infty \leq C(\|f^*\|_\infty + \|f^{*(r)}\varphi^r\|_\infty),$$

with $C \neq C(f^*)$, and then (31) follows by (29).

□

Remark 1. *The convergence estimate (31) says that if the known functions in equation (22) are in W_r, then the order of convergence is $O(m^{-\frac{r}{2}})$. Comparing it with (3) this means that in the class W_r the method converges with an order that is the half of the order of the best polynomial approximation. Nevertheless we outline that, as showed in the numerical tests, there is a numerical evidence that, for fixed m, the convergence "improves" for large value of the parameter s defining the GB quadrature formula. This behavior seems to be independent of the smoothness of the involved functions. Unfortunately at the moment we are not able to prove this formally.*

Remark 2. *Finally we remark that also the Romberg and IMT quadrature rules, shown in the previous Section, are both convergent for continuous functions. Therefore Theorem 4.1 can be applied to the Nyström methods based on these formulas. This means that in the Romberg case (25) is defined by (17), whence we have $M = 2^N$, $\tau_i = t_i$ and $w_i = \sigma_i$. In the case of the IMT rule (25) is defined by (21) and therefore $M = n$, $\tau_i = x_i$ and $w_i = W_i$.*

The convergence estimates of the Nyström methods will essentially agree with the convergence estimates of the corresponding quadrature formulas.

5. Numerical tests

In this section we will compare the results obtained implementing the Nyström method proposed in the previous section. We remark that the order of convergence of the Nyström methods based on the GB rule is theoretically half the order of the other two methods, for functions belonging to the same class. However, as the numerical evidence shows, the GB method is competitive with the other two, for suitable values of the parameter s.

In all tests we approximate the exact solution f by the Nyström interpolant f_M given in (28). In the tables we specify the values of M for each method, showing the corresponding maximum relative error attained in the computation of f_M at the equally spaced points $[0 : 0.1 : 1]$. These relative errors were performed by using, as "correct" values, those obtained by applying the Nyström method based on the Gauss-Legendre rule with a linear system of suitable dimension. In all the table the bold characters denote that the machine precision is attained.

In the case of the GB method we show the different errors obtained for different values of the parameter s. In the case of the method based on the IMT rule we have chosen $a = b = 4$ (but analogous results are attained for instance with $a = b = 1$).

Somewhere we will denote by $C(m, 2^p)$ the computational cost of the method based on the GB quadrature rule $\Sigma_m^{(s)}$ with $s = 2^p$. In view of (14) one has, in the general case,

$$C(m, 2^p) = \frac{m^3}{2}(p-1) + \frac{m^3}{3}.$$

The involved linear systems were solved by the Gaussian elimination and all the computations were performed in 16−digits precision.

Example 1. *We consider the following equation*

$$f(x) - \int_0^1 \frac{f(t)}{tx^2 + xt^2 + 25}\, dt = \sin(x).$$

Here the kernel $k(x,t) = \dfrac{1}{tx^2 + xt^2 + 25}$ *and the known function* $g(x) = \sin(x)$ *are very smooth functions. That means all the considered Nyström methods have very fast convergence. The numerical results are the following*

m	$s = 8$	$s = 16$	$s = 32$
16	0.16×10^{-8}	0.38×10^{-10}	0.10×10^{-11}
32	0.34×10^{-12}	0.38×10^{-14}	$\mathbf{0.23 \times 10^{-15}}$
64	0.79×10^{-14}	$\mathbf{0.91 \times 10^{-15}}$	
128	$\mathbf{0.54 \times 10^{-15}}$		

M	*Romberg*	n	*IMT*
16	0.31×10^{-13}	16	0.14×10^{-7}
32	$\mathbf{0.54 \times 10^{-15}}$	32	0.16×10^{-12}
		64	$\mathbf{0.36 \times 10^{-15}}$

As we can see the Romberg and the GB methods are equivalent since they reach the machine precision with the solution of a linear system of order 33*, if in the GB rule we take* $s = 32$*.*

Example 2. *We consider the following equation*

$$f(x) - 0.2 \int_0^1 f(t) \frac{|x-t|^{12.5}}{x^2 + t^2 + 15}\, dt = e^{-x}(1+x).$$

Here $\mu = 0.2$*, the kernel* $k(x,t) = \dfrac{|x-t|^{12.5}}{x^2 + t^2 + 15}$ *is in* W_{12}*, w.r.t. both the variables, and* $g(x) = e^{-x}(1+x)$ *is analytic. The convergence results say that our method has a convergence order* $O(m^{-6})$*, for* $s \geq 6$*. The Romberg method,*

according with (18), has a convergence order $O(M^{-12})$ *with* $M \geq 129$*, while the IMT method according to (20) has at least a convergence order* $O(n^{-11})$*. The numerical results are the following*

m	$s = 16$	$s = 64$	$s = 128$
16	0.31×10^{-6}	0.65×10^{-12}	$\mathbf{0.44 \times 10^{-15}}$
32	0.58×10^{-9}	0.33×10^{-13}	
64	0.65×10^{-12}	0.16×10^{-14}	
128	$\mathbf{0.44 \times 10^{-15}}$	$\mathbf{0.14 \times 10^{-15}}$	

M	*Romberg*	n	*IMT*
16	0.19×10^{-7}	16	0.56×10^{-6}
32	0.34×10^{-10}	32	0.59×10^{-13}
64	0.79×10^{-13}	64	$\mathbf{0.12 \times 10^{-15}}$
128	$\mathbf{0.12 \times 10^{-15}}$		

There is a numerical evidence showing that the GB method goes faster than the theoretical speed of convergence. As the table show the performance of the method is comparable with those of the other two methods. Moreover for a suitable value of $s = 128$ *the machine precision is attained solving a linear system of degree* 17 *(*$m = 16$*) and a global computational cost* $C(16, 2^7) = 1.4 \times 10^4$*. The same precision is attained with a computational cost of* 7.1×10^5 *flops in the case of the Romberg method, and with* 8.7×10^4 *flops in order to solve the linear system, plus the computational effort in order to compute the knots of the IMT quadrature formula (via the Gaussian rule), for the IMT method.*

5.1. The case of centrosymmetric and skew-centrosymmetric kernels

Let A, B two matrices of order n and let x be a vector of length n. Denote by J the counteridentity matrix of order n. First we recall some definitions and results which be useful hereinafter:

- A vector x is said *symmetric* if $Jx = x$, *skew-symmetric* if $Jx = -x$;
- A matrix B is said *skew-centrosymmetric* if $JBJ = -B$;
- If A is a centrosymmetric matrix and B is a skew-centrosymmetric matrix, the product AB is a skew-centrosymmetric;
- If A is a skew-centrosymmetric (resp. centrosymmetric) matrix and x is a skew-symmetric vector, the product Ax is a skew-symmetric (resp. symmetric);
- If A is a centrosymmetric matrix and x is a symmetric vector, the product Ax is symmetric.

Now we discuss on how to reduce the computational cost in computing the solution of the linear system of the GB method in the case of particular kinds of kernels.

We will consider the following two cases:

1) the centrosymmetric kernels, i.e. $k(x, t) = k(1 - x, 1 - t)$,
2) the skew-centrosymmetric kernels, i.e. $k(x, t) = -k(1 - x, 1 - t)$.

In the first case the matrix $\mathbf{H}_m$ of the linear system is centrosymmetric, since by $D_j^{(s)} = D_{m-j+1}^{(s)}$, $j = 0, 1, \ldots, m + 1$, $\mathbf{D^{(s)}}$ is centrosymmetric too. Therefore $\mathbf{H}_m$ is orthogonally similar to a block diagonal matrix (see [5]). Let us start from the representation of the $(m + 1)$ order centrosymmetric matrix $\mathbf{H_m}$ in the case $(m + 1)$ even:

$$\mathbf{H}_m = \begin{pmatrix} F & JGJ \\ G & JFJ \end{pmatrix}.$$

One has

$$D_1 = Q_1^T \mathbf{H}_m Q_1,$$

being

$$Q_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -J & J \end{pmatrix}, \qquad D_1 = \begin{pmatrix} F - JG & 0 \\ 0 & F + JG \end{pmatrix}.$$

Similarly we can proceed in the case $(m + 1)$ odd. For the sake of brevity we omit the details that the interested reader can find in [5] (see also [8],[22]).

Therefore the solution of the linear system is obtained by solving two half-order systems, and, using for instance the Gaussian elimination method, the global computational effort of the method GB is

$$C(m, 2^p) = \frac{m^3}{2}(p - 1) + \frac{m^3}{12} \tag{32}$$

and therefore the time complexity is reduced of 75%. If in addition the known function g is even w.r.t. the direction $x = \frac{1}{2}$, also the solution $f_m(x)$ will be even. This means that only one system of half-order is needed in order to construct the solution f_m, reducing in this way the time complexity of 87.5%, since in this case the computational cost $C(m, 2^p)$ is

$$C(m, 2^p) = \frac{m^3}{2}(p - 1) + \frac{m^3}{24}. \tag{33}$$

If the kernel is skew-centrosymmetric, and considering $(m+1)$ even, the matrix of the system $\mathbf{H}_m$ is skew-centrosymmetric. The case $(m + 1)$ odd cannot be considered, since odd skew-centrosymmetric matrices are singular. Setting

$$\mathbf{E}_m = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}, \quad I \in \mathbb{R}^{\frac{m+1}{2} \times \frac{m+1}{2}}$$

the matrix $\mathbf{E}_m \mathbf{H}_m$ is centrosymmetric [5].

Therefore the case of skew-centrosymmetric kernels is reduced to the case of centrosymmetric kernels. If in addition the known function g is odd w.r.t. the direction $x = \frac{1}{2}$, the solution f_m will be an odd function w.r.t. $x = \frac{1}{2}$.

Also in this case, following the same arguments used above, we can save 90% of the time complexity in the solution of the linear system.

Finally in what follows we take into account that also the linear systems in the Romberg and IMT Nyström methods are centrosymmetric when the kernel is centrosymmetric.

Example 3. *We consider the following equation*

$$f(x) - 0.2 \int_0^1 f(t)|x - t|^{7.5}\, dt = |\arctan(x - 0.5)|^{10.4}.$$

Here $\mu = 0.2$, *the kernel* $k(x, t) = |x - t|^{7.5} \in W_7$, *w.r.t. both the variables, is centrosymmetric and* $g(x) = |\arctan(x - 0.5)|^{10.4} \in W_{10}$ *is an even function w.r.t.* $x = \frac{1}{2}$.

m	$s = 16$	$s = 32$	$s = 64$
16	0.71×10^{-2}	0.34×10^{-2}	0.16×10^{-2}
32	0.58×10^{-4}	0.43×10^{-5}	0.37×10^{-6}
64	0.41×10^{-7}	0.15×10^{-9}	0.15×10^{-10}
128	0.18×10^{-11}	0.27×10^{-14}	0.24×10^{-14}
256	0.12×10^{-14}	$\mathbf{0.33 \times 10^{-15}}$	$\mathbf{0.69 \times 10^{-15}}$

M	*Romberg*	n	*IMT*
16	0.22×10^{-2}	16	0.24×10^{-2}
32	0.41×10^{-5}	32	0.49×10^{-7}
64	0.26×10^{-8}	64	0.25×10^{-12}
128	0.43×10^{-12}	128	0.19×10^{-13}
256	$\mathbf{0.35 \times 10^{-15}}$	256	0.21×10^{-13}

The theoretical error in our method behaves like $O(m^{-7/2})$ for $s > 3$, while the Romberg method has an order $O(M^{-7})$, for $M \geq 8$ and the IMT method an order $O(n^{-7})$, at least.

Also in this case the numerical experience shows that the GB method goes faster than the attended speed of convergence. The machine precision is attained for $m = 256$, $s = 32$ and hence with a computational cost, according with (33), $C(256, 32) = 3.42 \times 10^7$. The Romberg method has a cost of 7.1×10^5 flops.

Example 4. *This example is known as Love's equation. It arises in the electrostatic problem of a circular plate condenser, for a pair of identical coaxial discs, being b their distance, in an unbounded perfect fluid [15].The equation is*

$$f(z) - \frac{b}{\pi} \int_{-1}^{1} \frac{f(y)}{b^2 + (z-y)^2}\, dy = 1, \quad z \in [-1, 1].$$

By a linear transformation, we obtain the equivalent equation

$$\bar{f}(x) - \frac{2b}{\pi} \int_{0}^{1} \frac{\bar{f}(t)}{b^2 + 4(x-t)^2}\, dt = 1, \quad x \in [0, 1]$$

with $\mu = \dfrac{2b}{\pi}$, $k(x,t) = \dfrac{1}{b^2 + 4(x-t)^2}$, $g(x) = 1$ and $\bar{f}(x) = f(2x-1)$.

Here we consider $b = 1$, since as b decreases the closeness of the complex poles of the kernel function to the real axis leads to harder difficulties in the numerical treatment. In this case, other procedures can be successfully implemented (see [18],[21]).

m	$s = 8$	$s = 32$	$s = 64$
16	0.53×10^{-5}	0.31×10^{-5}	0.15×10^{-5}
32	0.84×10^{-7}	0.14×10^{-7}	0.58×10^{-8}
64	0.45×10^{-9}	0.11×10^{-10}	0.61×10^{-11}
128	0.19×10^{-11}	0.33×10^{-14}	0.25×10^{-14}
256	0.44×10^{-14}	$\mathbf{0.71 \times 10^{-15}}$	$\mathbf{0.61 \times 10^{-15}}$

M	*Romberg*	n	*IMT*
16	0.20×10^{-5}	16	0.15×10^{-3}
32	0.92×10^{-8}	32	0.22×10^{-7}
64	0.20×10^{-10}	64	$\mathbf{0.81 \times 10^{-15}}$
128	0.16×10^{-13}		
256	$\mathbf{0.83 \times 10^{-15}}$		

Also in this case the kernel is centrosymmetric and moreover the known function g is even. The numerical test shows that the GB method and the Romberg one have the same behavior. In this case the IMT goes faster than the other two.

Example 5. *We consider the following equation*

$$f(x) - 0.4 \int_{0}^{1} f(t) \sqrt{(x(1-x)t(1-t))^{11}}\, dt = \sin(x).$$

Here $\mu = 0.4$, the kernel $k(x,t) = \sqrt{(x(1-x)t(1-t))^{11}}$ and $g(x) = \sin(x)$.

m	$s = 8$	$s = 16$	$s = 32$
16	0.17×10^{-10}	0.46×10^{-11}	0.14×10^{-10}
32	0.28×10^{-12}	0.34×10^{-13}	0.46×10^{-13}
64	0.32×10^{-14}	$\mathbf{0.57 \times 10^{-15}}$	$\mathbf{0.27 \times 10^{-15}}$
128	$\mathbf{0.27 \times 10^{-15}}$		

M	*Romberg*	n	*IMT*
16	0.18×10^{-10}	16	0.41×10^{-9}
32	0.55×10^{-13}	32	$\mathbf{0.28 \times 10^{-15}}$
64	$\mathbf{0.27 \times 10^{-15}}$		

Due to the particular definition of the kernel it belongs to W_{11}, while is only in C^5 w.r.t. both the variables. Therefore the error in the GB procedure behaves like $O(m^{-11/2})$, while the Romberg method, according to (19), has a rate of convergence $O(M^{-5})$. A similar behavior holds true for the IMT method. Nevertheless all the methods are comparable and very fast (the real speed of convergence seem to be $O(M^{-10})$.

Example 6. *This example deals with skew-centrosymmetric kernels. In addition to the Romberg and IMT methods, we compare our results with those obtained by using the spline quasi-interpolant method proposed in [1]. By a change of variable in Example 3 in [1], we get the following equation*

$$f(x) - \pi \int_0^1 f(t) \sin(\pi(x-t))\, dt = \frac{(2+\pi^2)\cos(\pi x) + \pi \sin(\pi x)}{2(4+\pi^2)}.$$

Here $\mu = \pi$ and $k(x,t) = \sin(\pi(x-t)) = -\sin(\pi(x-t))$.

In order to compare our results with those obtained in [1] we produce here the absolute errors. On the right of the table we set the number n of subintervals in the Right Approximation of the kernel by Quasi Interpolant (RQI) and the corresponding (absolute) errors.

m	$s = 8$	$s = 16$	$s = 32$	n	*RQI*
16	0.36×10^{-5}	0.16×10^{-5}	0.77×10^{-7}	*16*	0.15×10^{-6}
32	0.29×10^{-7}	0.18×10^{-8}	0.11×10^{-9}	*32*	0.28×10^{-8}
64	0.20×10^{-9}	0.94×10^{-13}	0.43×10^{-14}	*64*	0.48×10^{-10}
128	0.50×10^{-12}	$\mathbf{0.38 \times 10^{-15}}$	$\mathbf{0.38 \times 10^{-15}}$	*128*	0.78×10^{-12}
256	0.11×10^{-14}				

N	*Romberg*	n	*IMT*
16	0.28×10^{-6}	16	0.45×10^{-4}
32	0.27×10^{-9}	32	0.23×10^{-11}
64	0.66×10^{-13}	64	$\mathbf{0.30 \times 10^{-15}}$
128	$\mathbf{0.22 \times 10^{-15}}$		

The GB method with $s = 8$ seems to have a behavior similar to that of the RQI method. Since the kernel is skew-centrosymmetric, the computational cost of the GB method is as in (32). The method RQI requires $n^3/3$ flops for the solution of the linear system and in addition the evaluation of $(n+2)(n+3)$ simple integrals. Moreover for $s = 16$ the GB method is more precise than the RQI and is comparable to the Romberg one.

References

[1] C. Allouch , P. Sablonniére, D. Sbibih, Solving Fredholm integral equations by approximating kernels by spline quasi-interpolants, Numer. Algorithms 56 (2011) 437–453.

[2] K. E. Atkinson The Numerical Solution of Integral Equations of the second kind, Cambridge Monographs on Applied and Computational Mathematics, 4. Cambridge University Press, Cambridge, 1997.

[3] K. E. Atkinson, L. F. Shampine, Algorithm 876: solving Fredholm integral equations of the second kind in Matlab, ACM Trans. Math. Software 34 (4), Art. 21. (2008).
[4] K. E. Atkinson, D. D. K. Chien , J. Seol, Numerical Analysis of the radiosity equation using the collocation method, Electron. Trans. Numer. Anal. 11 (2000) 94–120.
[5] I. T. Abu-Jeib, Algorithms for Centrosymmetric and Skew-Centrosymmetric Matrices, Missouri J. Math. Sci. 18 (1) (2006) 1–8.
[6] F. L. Bauer, H. Rutishauser, E. Stiefel, New aspects in numerical quadrature, in: Proc. Sympos. Appl. Math., Vol. XV, Amer. Math. Soc., Providence, R.I., (1963) 199–218.
[7] H. Brass, J. W. Fischer, Error bounds for Romberg quadrature, Numer. Math. 82 (3) (1999) 389–408.
[8] L. Datta, S. D. Morgera, On the reducibility of centrosymmetric matricesapplications in engineering problems, Circuits Systems Signal Process. 8 (1)(1989) 71-96.
[9] P.J. Davis, P. Rabinowitz, Methods of numerical integration, Computer Science and Applied Mathematics, Academic Press Inc., Orlando, FL, 1984.
[10] M. C. De Bonis, G. Mastroianni, Projection methods and condition numbers in uniform norm for Fredholm and Cauchy singular integral equations, SIAM J. Numer. Anal. 44 (4) (2006) 1351–1374.
[11] Z. Ditzian, On interpolation of $L_p[a, b]$ and weighted Sobolev spaces, Pacific Journal of Mathematics 90 (2) (1980) 307–323.
[12] Z. Ditzian, V. Totik, Moduli of smoothness, Springer Series in Computational Mathematics 9, Springer-Verlag, New York, 1987.
[13] H. H. Gonska, X. L. Zhou, Approximation theorems for the iterated Boolean sums of Bernstein operators, J. Comput. Appl. Math. 53 (1994) 21–31.
[14] M. Iri, S. Moriguti, Y. Takasawa, On a certain quadrature formula (in Japanese), Kokyuroku of Res. Inst. for Math. Sci., Kyoto Univ. 91 (1970) 82-118.
[15] P. A. Martin, L. Farina, Radiation of water waves by a heaving submerged horizontal disc, J. Fluid. Mech. 337 (1997) 365–379.
[16] G. Mastroianni, M. R. Occorsio, A generalization of the Bernstein operator. (Italian), Rend. Accad. Sci. Fis. Mat. Napoli 44 (4) (1977) 151–169.
[17] C. A. Micchelli, The saturation class and iterates of the Bernstein polynomials, Collection of articles dedicated to Isaac Jacob Schoenberg on his 70th birthday, I., J. Approx. Theory 8 (1973) 1-18.
[18] G. Monegato, A. Palamara Orsi, Product formulas for Fredholm integral equations with rational kernel functions, in: Numerical Integration, III (Oberwolfach, 1987) Internat. Schriftenreihe Numer. Math., 85, Birkhuser, Basel, (1988) 140-156.
[19] D. Occorsio, A. C. Simoncelli, How to go from Bézier to Lagrange curves by means of generalized Bézier curves, Facta Univ. Ser. Math. Inform. (Niš) 11 (1996) 101–111.
[20] D. Occorsio, Some new properties of Generalized Bernstein polynomials, Stud. Univ. Babeş-Bolyai Math. 56 (3) (2011) 147–160.
[21] P. Pastore, The numerical treatment of Love's integral equation having very small parameter, J. Comput. Appl. Math. 236 (6) (2011) 1267-1281.
[22] Z. Peng, X. Hu, L. Zhang, The inverse problem of centrosymmetric matrices with a submatrix constraint, J. Comput. Math. 22 (4) (2004) 535-544.
[23] A. Sidi, Comparison of some numerical quadrature formulas for weakly singular periodic Fredholm integral equations, Computing 43 (2) (1989) 159-170.