

The Residual Spectrum and the Continuous Spectrum of Upper Triangular Operator Matrices

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Abstract. Let \mathcal{H} and \mathcal{K} be separable infinite dimensional Hilbert spaces. We denote by M_C the 2×2 upper triangular operator matrix acting on $\mathcal{H} \oplus \mathcal{K}$ of the form $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. For given operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, the sets $\bigcup_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_r(M_C)$ and $\bigcup_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_c(M_C)$ are characterized, where $\sigma_r(\cdot)$ and $\sigma_c(\cdot)$ denote the residual spectrum and the continuous spectrum, respectively.

1. Introduction

The study of operator matrices arises naturally from the following fact: if \mathcal{X} is a Hilbert space and we decompose \mathcal{X} as a direct sum of two closed subspaces \mathcal{H} and \mathcal{K} , each bounded linear operator $T : \mathcal{X} \rightarrow \mathcal{X}$ can be expressed as the operator matrix form

$$T = \begin{pmatrix} A & C \\ D & B \end{pmatrix}$$

with respect to the space decomposition $\mathcal{X} = \mathcal{H} \oplus \mathcal{K}$, where $A : \mathcal{H} \rightarrow \mathcal{H}$, $B : \mathcal{K} \rightarrow \mathcal{K}$, $C : \mathcal{K} \rightarrow \mathcal{H}$ and $D : \mathcal{H} \rightarrow \mathcal{K}$ are bounded linear operators, respectively. In particular, if \mathcal{H} is an invariant subspace for T then $D = 0$, and so T has an upper triangular operator matrix form. One way to study operators is to see them as being composed of simpler operators. The operator matrices have been studied by numerous authors [1–5, 7–11]. This paper is concerned with the residual spectrum and the continuous spectrum of operator matrices.

In this paper, \mathcal{H} and \mathcal{K} are separable infinite dimensional Hilbert spaces. Let $\mathcal{B}(\mathcal{H}, \mathcal{K})$ denote the set of bounded linear operators from \mathcal{H} into \mathcal{K} . When $\mathcal{H} = \mathcal{K}$ we write $\mathcal{B}(\mathcal{H}, \mathcal{H}) = \mathcal{B}(\mathcal{H})$. If $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, we use $\mathcal{R}(T)$, $\mathcal{N}(T)$ and T^* to denote the range space, the null space and the adjoint of T . For a linear subspace $\mathcal{M} \subseteq \mathcal{H}$, its closure and orthogonal complement are denoted by $\overline{\mathcal{M}}$ and \mathcal{M}^\perp . Write $P_{\overline{\mathcal{M}}}$ for the orthogonal projection onto $\overline{\mathcal{M}}$ along \mathcal{M}^\perp and $T|_{\mathcal{M}}$ for the restriction of T to \mathcal{M} . If $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, write $n(T)$ for the nullity of T , i.e., $n(T) = \dim \mathcal{N}(T)$, and write $d(T)$ for the deficiency of T , i.e., $d(T) = \dim \mathcal{R}(T)^\perp$.

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For $T \in \mathcal{B}(\mathcal{H})$, the resolvent set $\rho(T)$ and the spectrum $\sigma(T)$ of T are defined by

$$\begin{aligned} \rho(T) &= \{\lambda \in \mathbb{C} : \mathcal{N}(T - \lambda I) = \{0\}, \mathcal{R}(T - \lambda I) = \mathcal{H}\}, \\ \sigma(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}. \end{aligned}$$

Furthermore, the spectrum $\sigma(T)$ is classified by the point spectrum $\sigma_p(T)$, the residual spectrum $\sigma_r(T)$ and the continuous spectrum $\sigma_c(T)$, and we define them by

$$\begin{aligned} \sigma_p(T) &= \{\lambda \in \mathbb{C} : \mathcal{N}(T - \lambda I) \neq \{0\}\}, \\ \sigma_r(T) &= \{\lambda \in \mathbb{C} : \mathcal{N}(T - \lambda I) = \{0\}, \overline{\mathcal{R}(T - \lambda I)} \neq \mathcal{H}\}, \\ \sigma_c(T) &= \{\lambda \in \mathbb{C} : \mathcal{N}(T - \lambda I) = \{0\}, \mathcal{R}(T - \lambda I) \neq \overline{\mathcal{R}(T - \lambda I)} = \mathcal{H}\}. \end{aligned}$$

It is easy to find (see [9, p. 92]) that $\sigma_p(T)$, $\sigma_r(T)$ and $\sigma_c(T)$ are mutually disjoint and

$$\sigma_p(T) \cup \sigma_r(T) \cup \sigma_c(T) = \sigma(T) = \mathbb{C} \setminus \rho(T).$$

Moreover, $\lambda \in \sigma_c(T)$ if and only if $\bar{\lambda} \in \sigma_c(T^*)$.

When $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ are given operators, we denote by M_C an operator on $\mathcal{H} \oplus \mathcal{K}$ of the form

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

for $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. The set $\bigcap_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma(M_C)$ was determined in [4]. For the approximate point spectrum, the essential spectrum, the Weyl spectrum, the point spectrum, the continuous spectrum and the residual spectrum of M_C , analogous results have been obtained in many literatures (see [1, 2, 5, 10]). The sets $\bigcup_{C \in \text{Inv}(\mathcal{K}, \mathcal{H})} \sigma_{ae}(M_C)$ and $\bigcup_{C \in \text{Inv}(\mathcal{K}, \mathcal{H})} \sigma_l(M_C)$ were discussed in [7, 8], where $\sigma_{ae}(\cdot)$ and $\sigma_l(\cdot)$ denote the essential approximate point spectrum and the left spectrum, and $\text{Inv}(\mathcal{K}, \mathcal{H})$ denotes the set of all invertible operators from \mathcal{K} into \mathcal{H} . In this paper, the sets $\bigcup_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_r(M_C)$ and $\bigcup_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_c(M_C)$ are characterized.

2. Main results

For the proof of our main results, we need some auxiliary lemmas.

Lemma 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be given operators. Then $n(M_C) \leq n(A) + n(B)$ for every $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.*

Proof. For every $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, it is easy to see that

$$\mathcal{N}(M_C) = \begin{pmatrix} \mathcal{N}(A) \\ 0 \end{pmatrix} \oplus \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : Ax + Cy = 0, x \in \mathcal{N}(A)^\perp, y \in \mathcal{N}(B) \right\}, \tag{1}$$

and hence one can see that $n(M_C) \leq n(A) + n(B)$. \square

Lemma 2.2. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be given operators, and let $\mathcal{R}(A)$ be closed.*

- (i) *If $n(B) > d(A)$, then M_C is not injective for every $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.*
- (ii) *If $n(B) = d(A) < \infty$, then for every $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ we have that M_C is not injective or $C_3 = P_{\mathcal{R}(A)^\perp} C|_{\mathcal{N}(B)} : \mathcal{N}(B) \rightarrow \mathcal{R}(A)^\perp$ is invertible.*

Proof. It follows from the closedness of $\mathcal{R}(A)$ that M_C has an operator matrix representation

$$M_C = \begin{pmatrix} A_1 & C_1 & C_2 \\ 0 & C_3 & C_4 \\ 0 & 0 & B_1 \end{pmatrix} : \mathcal{H} \oplus \mathcal{N}(B) \oplus \mathcal{N}(B)^\perp \longrightarrow \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \oplus \mathcal{K}$$

and $A_1 : \mathcal{H} \longrightarrow \mathcal{R}(A)$ is right invertible. Let $A_1^r : \mathcal{R}(A) \longrightarrow \mathcal{H}$ be a right inverse of A_1 . Then there is an invertible operator

$$U = \begin{pmatrix} I_{\mathcal{H}} & -A_1^r C_1 & -A_1^r C_2 \\ 0 & I_{\mathcal{N}(B)} & 0 \\ 0 & 0 & I_{\mathcal{N}(B)^\perp} \end{pmatrix} : \mathcal{H} \oplus \mathcal{N}(B) \oplus \mathcal{N}(B)^\perp \longrightarrow \mathcal{H} \oplus \mathcal{N}(B) \oplus \mathcal{N}(B)^\perp$$

such that

$$\begin{pmatrix} A_1 & C_1 & C_2 \\ 0 & C_3 & C_4 \\ 0 & 0 & B_1 \end{pmatrix} U = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & C_3 & C_4 \\ 0 & 0 & B_1 \end{pmatrix}. \tag{2}$$

(i) If $n(B) > d(A)$, then for every $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ we have that $C_3 : \mathcal{N}(B) \longrightarrow \mathcal{R}(A)^\perp$ is not injective. This implies that M_C is not injective for every $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.

(ii) If $n(B) = d(A) < \infty$, then for every $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ we have that M_C is not injective or M_C is injective. When M_C is injective, it follows from (2) that $C_3 : \mathcal{N}(B) \longrightarrow \mathcal{R}(A)^\perp$ is injective. Therefore $C_3 : \mathcal{N}(B) \longrightarrow \mathcal{R}(A)^\perp$ is invertible by $n(B) = d(A) < \infty$. \square

Lemma 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$, and let $\mathcal{R}(T)$ be not closed. Then there is an infinite dimensional closed subspace \mathcal{M} of $\overline{\mathcal{R}(T)}$ such that $\mathcal{M} \cap \mathcal{R}(T) = \{0\}$.*

Proof. Since $\mathcal{R}(T)$ is not closed, it follows that $\overline{\mathcal{R}(T)}$ is an infinite dimensional closed subspace of \mathcal{H} . Note that \mathcal{H} is a separable infinite dimensional Hilbert space, and hence there exists an invertible operator $J : \overline{\mathcal{R}(T)} \longrightarrow \mathcal{H}$. Then $\tilde{T} = P_{\overline{\mathcal{R}(T)}} T J : \overline{\mathcal{R}(T)} \longrightarrow \overline{\mathcal{R}(T)}$ is a bounded linear operator and $\mathcal{R}(T) = \mathcal{R}(\tilde{T})$. Now, applying [6, Lemma 2.1] to \tilde{T} we get that there is an infinite dimensional closed subspace $\mathcal{M} \subset \overline{\mathcal{R}(T)}$ such that $\mathcal{M} \cap \mathcal{R}(\tilde{T}) = \{0\}$, which implies that $\mathcal{M} \cap \mathcal{R}(T) = \{0\}$. \square

Our main results are the following theorems.

Theorem 2.4. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be given operators. Then*

$$\begin{aligned} \bigcup_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_r(M_C) = & \{ \lambda \in \mathbb{C} : \mathcal{R}(A - \lambda I) \text{ is not closed, } d(A - \lambda I) + d(B - \lambda I) > 0 \} \\ & \cup \{ \lambda \in \mathbb{C} : n(B - \lambda I) \leq d(A - \lambda I), n(B - \lambda I) < d(A - \lambda I) + d(B - \lambda I) \} \\ & \cup \{ \lambda \in \mathbb{C} : n(B - \lambda I) = d(A - \lambda I) = \infty \} \setminus \sigma_p(A). \end{aligned}$$

Proof. First we prove that the right side of the above equality includes the left side. Suppose that there exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $\lambda \in \sigma_r(M_C)$. Without loss of generality we assume that $\lambda = 0$. Then M_C is injective and $\overline{\mathcal{R}(M_C)} \neq \mathcal{H} \oplus \mathcal{K}$. Since M_C is injective, it follows that A is injective. Now we consider two cases.

Case I. Suppose that $\mathcal{R}(A)$ is not closed. Since $\overline{\mathcal{R}(M_C)} \neq \mathcal{H} \oplus \mathcal{K}$, it follows from Lemma 2.1 that $n(A^*) + n(B^*) \geq n(M_C^*) > 0$. Therefore $d(A) + d(B) > 0$.

Case II. Suppose that $\mathcal{R}(A)$ is closed. Then by Lemma 2.2 (i) and the injectivity of M_C we can infer that $n(B) \leq d(A)$. If $n(B) = \infty$ then $d(A) = \infty$. Therefore $n(B) = d(A) = \infty$. If $n(B) < \infty$, one can show that $n(B) < d(A) + d(B)$. Assume to the contrary that $n(B) \geq d(A) + d(B)$. This, together with $n(B) \leq d(A)$, implies that $d(B) = 0$ and $n(B) = d(A) < \infty$. From Lemma 2.2 (ii) it follows that M_C is not injective or

$C_3 : P_{\mathcal{R}(A)^\perp} C|_{\mathcal{N}(B)} : \mathcal{N}(B) \rightarrow \mathcal{R}(A)^\perp$ is invertible for every $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. When C_3 is invertible, we derive from $d(B) = 0$ that $\overline{\mathcal{R}(M_C)} = \mathcal{H} \oplus \mathcal{K}$. Indeed, since $\mathcal{R}(A)$ is closed, it follows that M_C has an operator matrix representation

$$M_C = \begin{pmatrix} A_1 & C_1 & C_2 \\ 0 & C_3 & C_4 \\ 0 & 0 & B_1 \end{pmatrix} : \mathcal{H} \oplus \mathcal{N}(B) \oplus \mathcal{N}(B)^\perp \rightarrow \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \oplus \mathcal{K}$$

and $A_1 : \mathcal{H} \rightarrow \mathcal{R}(A)$ is right invertible. Let $A_1^r : \mathcal{R}(A) \rightarrow \mathcal{H}$ be a right inverse of A_1 . Then there is an invertible operator

$$V = \begin{pmatrix} I_{\mathcal{H}} & -A_1^r C_1 & -A_1^r C_2 + A_1^r C_1 C_3^{-1} C_4 \\ 0 & I_{\mathcal{N}(B)} & -C_3^{-1} C_4 \\ 0 & 0 & I_{\mathcal{N}(B)^\perp} \end{pmatrix} : \mathcal{H} \oplus \mathcal{N}(B) \oplus \mathcal{N}(B)^\perp \rightarrow \mathcal{H} \oplus \mathcal{N}(B) \oplus \mathcal{N}(B)^\perp$$

such that

$$\begin{pmatrix} A_1 & C_1 & C_2 \\ 0 & C_3 & C_4 \\ 0 & 0 & B_1 \end{pmatrix} V = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & C_3 & 0 \\ 0 & 0 & B_1 \end{pmatrix}.$$

Note that $d(B) = 0$, and hence $\overline{\mathcal{R}(M_C V)} = \mathcal{H} \oplus \mathcal{K}$. This, together with the invertibility of V , shows that $\overline{\mathcal{R}(M_C)} = \mathcal{H} \oplus \mathcal{K}$. This is a contradiction.

Next we show that the converse inclusion is also true. Without loss of generality, assume that $\lambda = 0$. We consider three cases:

Case I. Suppose that A is injective, $\mathcal{R}(A)$ is not closed and $d(A) + d(B) > 0$. Since $\mathcal{R}(A)$ is not closed, it follows from Lemma 2.3 that there exists an infinite dimensional closed subspace $\mathcal{M} \subset \overline{\mathcal{R}(A)}$ such that $\mathcal{M} \cap \mathcal{R}(A) = \{0\}$. Hence there is an operator $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $\mathcal{N}(B) = \mathcal{N}(C)^\perp$ and $\mathcal{R}(C) \subset \mathcal{M}$. Then clearly $\mathcal{R}(C) \cap \mathcal{R}(A) = \{0\}$. This, together with (1) and the injectivity of A , shows that M_C is injective. On the other hand, since M_C has an operator matrix representation

$$M_C = \begin{pmatrix} A_1 & C_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_1 \end{pmatrix} : \mathcal{H} \oplus \mathcal{N}(B) \oplus \mathcal{N}(B)^\perp \rightarrow \overline{\mathcal{R}(A)} \oplus \mathcal{R}(A)^\perp \oplus \mathcal{K},$$

it follows from $d(A) + d(B) > 0$ that $\overline{\mathcal{R}(M_C)} \neq \mathcal{H} \oplus \mathcal{K}$. Hence $0 \in \sigma_r(M_C)$.

Case II. Suppose that A is injective, $n(B) \leq d(A)$ and $n(B) < d(A) + d(B)$. Since $n(B) \leq d(A)$, there is an operator $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $\mathcal{N}(C)^\perp = \mathcal{N}(B)$ and $\mathcal{R}(C) \subset \mathcal{R}(A)^\perp$. Then M_C has an operator matrix representation

$$M_C = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & C_3 & 0 \\ 0 & 0 & B_1 \end{pmatrix} : \mathcal{H} \oplus \mathcal{N}(B) \oplus \mathcal{N}(B)^\perp \rightarrow \overline{\mathcal{R}(A)} \oplus \mathcal{R}(A)^\perp \oplus \mathcal{K}. \tag{3}$$

Note that A is injective and $\mathcal{N}(C)^\perp = \mathcal{N}(B)$, and hence M_C is injective. On the other hand, if $d(B) > 0$ then clearly $\overline{\mathcal{R}(M_C)} \neq \mathcal{H} \oplus \mathcal{K}$; if $d(B) = 0$, it follows from $d(A) + d(B) > n(B)$ that $d(A) > n(B)$, and hence $\overline{\mathcal{R}(C_3)} \neq \mathcal{R}(A)^\perp$, which implies that $\overline{\mathcal{R}(M_C)} \neq \mathcal{H} \oplus \mathcal{K}$. Therefore $0 \in \sigma_r(M_C)$.

Case III. Suppose that A is injective and $n(B) = d(A) = \infty$. Since $n(B) = d(A) = \infty$, there exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $\mathcal{N}(C)^\perp = \mathcal{N}(B)$, $\mathcal{R}(C) \subset \mathcal{R}(A)^\perp$ and $\overline{\mathcal{R}(C)} \neq \mathcal{R}(A)^\perp$. Then M_C has an operator matrix representation (3). In a similar way as above, we obtain that M_C is injective. Also, since $\overline{\mathcal{R}(C_3)} \neq \mathcal{R}(A)^\perp$, it follows that $\overline{\mathcal{R}(M_C)} \neq \mathcal{H} \oplus \mathcal{K}$. Thus $0 \in \sigma_r(M_C)$. \square

For the continuous spectrum, we also have the following result.

Theorem 2.5. Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be given operators. Then

$$\begin{aligned} \bigcup_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_c(M_C) = & \{ \lambda \in \mathbb{C} : \mathcal{R}(A) \text{ is not closed, } d(A - \lambda I) \leq n(B - \lambda I) \} \\ & \cup \{ \lambda \in \mathbb{C} : \mathcal{R}(B) \text{ is not closed, } d(A - \lambda I) \geq n(B - \lambda I) \} \\ & \cup \{ \lambda \in \mathbb{C} : d(A - \lambda I) = n(B - \lambda I) = \infty \} \setminus [\sigma_p(A) \cup \sigma_p(B^*)]. \end{aligned}$$

Proof. We first show that the right side of the above equality includes the left side. Suppose that there exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $\lambda \in \sigma_c(M_C)$. Without loss of generality, assume that $\lambda = 0$. Then M_C is injective and $\mathcal{R}(M_C) \neq \overline{\mathcal{R}(M_C)} = \mathcal{H} \oplus \mathcal{K}$. Since M_C is injective, it follows that A is injective. On the other hand, note that $\overline{\mathcal{R}(M_C)} = \mathcal{H} \oplus \mathcal{K}$ implies that M_C^* is injective, and hence B^* is injective. Therefore $0 \notin \sigma_p(A) \cup \sigma_p(B^*)$. We consider two cases.

Case I. Suppose that $\mathcal{R}(A)$ is not closed or $\mathcal{R}(B)$ is not closed. If $\mathcal{R}(B)$ is not closed, then $\mathcal{R}(A)$ is not closed or $d(A) \geq n(B)$. In fact, if we assume that $\mathcal{R}(A)$ is closed and $d(A) < n(B)$, then M_C is not injective for every $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ by Lemma 2.2 (i), which is a contradiction. If $\mathcal{R}(A)$ is not closed, we can also show by applying the above argument to the adjoint M_C^* that $\mathcal{R}(B)$ is not closed or $n(B) \geq d(A)$.

Case II. Suppose that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed. Then M_C as an operator from $\mathcal{H} \oplus \mathcal{N}(B) \oplus \mathcal{N}(B)^\perp$ into $\mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \oplus \mathcal{K}$ has the following operator matrix representation

$$M_C = \begin{pmatrix} A_1 & C_1 & C_2 \\ 0 & C_3 & C_4 \\ 0 & 0 & B_1 \end{pmatrix}.$$

Note that A and B^* are injective, and hence $A_1 : \mathcal{H} \rightarrow \mathcal{R}(A)$ and $B_1 : \mathcal{N}(B)^\perp \rightarrow \mathcal{K}$ are invertible. Then there exist invertible operators

$$U = \begin{pmatrix} I & -A_1^{-1}C_1 & -A_1^{-1}C_2 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} : \mathcal{H} \oplus \mathcal{N}(B) \oplus \mathcal{N}(B)^\perp \rightarrow \mathcal{H} \oplus \mathcal{N}(B) \oplus \mathcal{N}(B)^\perp$$

and

$$V = \begin{pmatrix} I & 0 & 0 \\ 0 & I & -C_4B_1^{-1} \\ 0 & 0 & I \end{pmatrix} : \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \oplus \mathcal{K} \rightarrow \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \oplus \mathcal{K}$$

such that

$$VM_CU = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & C_3 & 0 \\ 0 & 0 & B_1 \end{pmatrix}.$$

Since $\mathcal{R}(M_C)$ is not closed, it follows from the invertibility of U and V that $\mathcal{R}(VM_CU)$ is not closed. This, together with the closedness of $\mathcal{R}(A)$ and $\mathcal{R}(B)$, shows that $\mathcal{R}(C_3)$ is not closed. Thus $d(A) = n(B) = \infty$.

Next we show that the converse inclusion is also true. Without loss of generality, assume that $\lambda = 0$. We consider three cases:

Case I. Suppose that A is injective, $\overline{\mathcal{R}(B)} = \mathcal{K}$, $\mathcal{R}(A)$ is not closed and $d(A) \leq n(B)$. Since $\mathcal{R}(A)$ is not closed, it follows from Lemma 2.3 that there exists an infinite dimensional closed subspace $\mathcal{M} \subset \overline{\mathcal{R}(A)}$ such that $\mathcal{M} \cap \mathcal{R}(A) = \{0\}$. On the other hand, note that $d(A) \leq n(B)$, and hence there is a closed subspace $\mathcal{G}_2 \subset \mathcal{N}(B)$ such that $\dim \mathcal{G}_2 = d(A)$. Let $\mathcal{N}(B) = \mathcal{G}_1 \oplus \mathcal{G}_2$. By $\dim \mathcal{G}_2 = d(A)$ and $\dim \mathcal{M} = \infty$ one can show that there exist bounded linear operators $C_2 : \mathcal{G}_2 \rightarrow \mathcal{R}(A)^\perp$ and $C_1 : \mathcal{G}_1 \rightarrow \mathcal{M}$ such that C_2 is invertible and C_1 is injective. Define $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ by

$$C = \begin{pmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \end{pmatrix} : \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \mathcal{N}(B)^\perp \rightarrow \overline{\mathcal{R}(A)} \oplus \mathcal{R}(A)^\perp.$$

Then M_C has an operator matrix representation

$$M_C = \begin{pmatrix} A_1 & C_1 & 0 & 0 \\ 0 & 0 & C_2 & 0 \\ 0 & 0 & 0 & B_1 \end{pmatrix} : \mathcal{H} \oplus \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \mathcal{N}(B)^\perp \longrightarrow \overline{\mathcal{R}(A)} \oplus \mathcal{R}(A)^\perp \oplus \mathcal{K}.$$

Clearly, A_1, B_1, C_1 and C_2 are injective, and so by $\mathcal{R}(A) \cap \mathcal{M} = \{0\}$ one can see that M_C is injective. On the other hand, from $\overline{\mathcal{R}(B)} = \mathcal{K}$ and $\mathcal{R}(C_2) = \mathcal{R}(A)^\perp$ we have that $\overline{\mathcal{R}(M_C)} = \mathcal{H} \oplus \mathcal{K}$. Now $\mathcal{R}(M_C) \neq \mathcal{H} \oplus \mathcal{K}$ follows from the fact that $\mathcal{R}(A)$ is not closed. Therefore $0 \in \sigma_c(M_C)$.

Case II. Suppose that A is injective, $\overline{\mathcal{R}(B)} = \mathcal{K}$, $\mathcal{R}(B)$ is not closed and $d(A) \geq n(B)$. Then clearly B^* is injective, $\overline{\mathcal{R}(A^*)} = \mathcal{H}$, $\mathcal{R}(B^*)$ is not closed and $n(A^*) \geq d(B^*)$. In a similar way with above, we can show that there exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $0 \in \sigma_c(M_C^*)$, which implies that $0 \in \sigma_c(M_C)$.

Case III. Suppose that A is injective, $\overline{\mathcal{R}(B)} = \mathcal{K}$ and $n(B) = d(A) = \infty$. Since $n(B) = d(A) = \infty$, there is an operator $C_3 : \mathcal{N}(B) \longrightarrow \mathcal{R}(A)^\perp$ such that $\mathcal{N}(C_3) = \{0\}$ and $\mathcal{R}(C_3) \neq \overline{\mathcal{R}(C_3)} = \mathcal{R}(A)^\perp$. Define $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ by

$$C = \begin{pmatrix} 0 & 0 \\ C_3 & 0 \end{pmatrix} : \mathcal{N}(B) \oplus \mathcal{N}(B)^\perp \longrightarrow \overline{\mathcal{R}(A)} \oplus \mathcal{R}(A)^\perp.$$

Then M_C has an operator matrix representation

$$M_C = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & C_3 & 0 \\ 0 & 0 & B_1 \end{pmatrix} : \mathcal{H} \oplus \mathcal{N}(B) \oplus \mathcal{N}(B)^\perp \longrightarrow \overline{\mathcal{R}(A)} \oplus \mathcal{R}(A)^\perp \oplus \mathcal{K}.$$

Clearly, M_C is injective and $\mathcal{R}(M_C) \neq \overline{\mathcal{R}(M_C)} = \mathcal{H} \oplus \mathcal{K}$, which implies that $0 \in \sigma_c(M_C)$. \square

Finally, an example, to illustrate our results, is given.

Example 2.6. Let $\mathcal{H} = \mathcal{K} = \ell_2$. Define operators $A \in \mathcal{B}(\ell_2)$ and $B \in \mathcal{B}(\ell_2)$ by

$$\begin{aligned} Ax &= (x_1, 0, x_2, 0, x_3, 0, \dots), \\ Bx &= (x_1, x_3, x_5, x_7, x_9, \dots), \end{aligned}$$

where $x = (x_1, x_2, \dots) \in \ell_2$. Clearly, A is left invertible and B is right invertible. Moreover, it is not hard to show that $d(A) = n(B) = \infty$. By Theorems 2.4 and 2.5, there exist $C_1 \in \mathcal{B}(\ell_2)$ and $C_2 \in \mathcal{B}(\ell_2)$ such that $0 \in \sigma_r(M_{C_1})$ and $0 \in \sigma_c(M_{C_2})$. In fact, define $C_1 \in \mathcal{B}(\ell_2)$ and $C_2 \in \mathcal{B}(\ell_2)$ by

$$\begin{aligned} C_1 x &= (0, 0, 0, x_2, 0, x_4, 0, x_6, 0, \dots), \\ C_2 x &= (0, x_2, 0, \frac{1}{2}x_4, 0, \frac{1}{3}x_6, 0, \dots), \end{aligned}$$

where $x = (x_1, x_2, \dots) \in \ell_2$. Then a straightforward calculation shows that $0 \in \sigma_r(M_{C_1})$ and $0 \in \sigma_c(M_{C_2})$.

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