

Fixed Point Theorems Using Cyclic Weaker Meir-Keeler Functions in Partial Metric Spaces

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Abstract. In this paper, we will present some fixed point results for mappings which satisfy cyclic weaker $(\psi \circ \varphi)$ -contractions and cyclic weaker (ψ, φ) -contractions in 0-complete partial metric spaces. Our results generalize or improve many recent fixed point theorems in the literature. Examples are given to support the usability of our results.

1. Introduction

The Banach Contraction Principle is a very popular tool in solving existence problems in many branches of Mathematical Analysis and its applications. It is no surprise that there is a great number of generalizations of this fundamental theorem. They go in several directions—modifying the basic contractive condition or changing the ambient space.

Concerning the first direction we mention so called weakly contractive conditions of Alber and Guerre-Delabriere [4] and Rhoades [21], altering distance functions used by Khan et al. [15] and Boyd and Wong [6], as well as Meir and Keeler [18] generalization of contractive condition.

Cyclic representations and cyclic contractions were introduced by Kirk et al. [16] and further used by several authors to obtain various fixed point results (see, e.g., [8, 9, 13, 20]).

On the other hand, Matthews [17] introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks. In partial metric spaces, self-distance of an arbitrary point need not be equal to zero. Several authors obtained many useful fixed point results in these spaces—we mention just [5, 7, 12, 19, 22, 23].

Some results for cyclic contractions in partial metric spaces were very recently obtained in [1, 3, 10].

In this paper, we will present some new fixed point results for mappings which satisfy cyclic weaker $(\psi \circ \varphi)$ -contractions and cyclic weaker (ψ, φ) -contractions in 0-complete partial metric spaces. Our results are extensions or refinements of recent fixed point theorems of Abbas et al. [1], Agarwal et al. [3], Di Bari and Vetro [10], Ming [8] and some other papers. Examples are given to support the usability of the results and to show that some of these extensions are proper.

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2. Preliminaries

In 2003, Kirk et al. introduced the following notion of cyclic representation.

Definition 2.1. [16] Let X be a nonempty set, $m \in \mathbb{N}$ and let $f : X \rightarrow X$ be a self-mapping. Then $X = \bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to f if

- (a) $A_i, i = 1, \dots, m$ are non-empty subsets of X ;
- (b) $f(A_1) \subset A_2, f(A_2) \subset A_3, \dots, f(A_{m-1}) \subset A_m, f(A_m) \subset A_1$.

They proved the following fixed point result.

Theorem 2.2. [16] Let (X, d) be a complete metric space, $f : X \rightarrow X$ and let $X = \bigcup_{i=1}^m A_i$ be a cyclic representation of X with respect to f . Suppose that f satisfies the following condition

$$d(fx, fy) \leq \psi(d(x, y)), \text{ for all } x \in A_i, y \in A_{i+1}, i \in \{1, 2, \dots, m\},$$

where $A_{m+1} = A_1$ and $\psi : [0, 1) \rightarrow [0, 1)$ is a function, upper semi-continuous from the right and $0 \leq \psi(t) < t$ for $t > 0$. Then, f has a fixed point $z \in \bigcap_{i=1}^m A_i$.

In 2010, Păcurar and Rus introduced the following notion of cyclic weaker φ -contraction.

Definition 2.3. [20] Let (X, d) be a metric space, $m \in \mathbb{N}$, A_1, A_2, \dots, A_m be closed nonempty subsets of X and $X = \bigcup_{i=1}^m A_i$. An operator $f : X \rightarrow X$ is called a cyclic weaker φ -contraction if

- (1) $X = \bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to f ;
- (2) there exists a continuous, non-decreasing function $\varphi : [0, 1) \rightarrow [0, 1)$ with $\varphi(t) > 0$ for $t \in (0, 1)$ and $\varphi(0) = 0$ such that

$$d(fx, fy) \leq d(x, y) - \varphi(d(x, y)),$$

for any $x \in A_i, y \in A_{i+1}, i = 1, 2, \dots, m$, where $A_{m+1} = A_1$.

They proved the following result.

Theorem 2.4. [20] Suppose that f is a cyclic weaker φ -contraction on a complete metric space (X, d) . Then, f has a fixed point $z \in \bigcap_{i=1}^m A_i$.

Recently, Ming [8] introduced two new versions of cyclic weaker contractions and proved fixed point theorems in complete metric spaces.

The following definitions and details can be seen, e.g., in [5, 7, 12, 17, 19, 22, 23].

Definition 2.5. A partial metric on a nonempty set X is a function $p : X \times X \rightarrow \mathbb{R}^+$ such that for all $x, y, z \in X$:

$$(p_1) \quad x = y \iff p(x, x) = p(x, y) = p(y, y),$$

$$(p_2) \quad p(x, x) \leq p(x, y),$$

$$(p_3) \quad p(x, y) = p(y, x),$$

$$(p_4) \quad p(x, y) \leq p(x, z) + p(z, y) - p(z, z).$$

The pair (X, p) is called a partial metric space.

It is clear that, if $p(x, y) = 0$, then from (p_1) and (p_2) $x = y$. But if $x = y$, $p(x, y)$ may not be 0.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p -balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A sequence $\{x_n\}$ in (X, p) converges to a point $x \in X$ (in the sense of τ_p) if $\lim_{n \rightarrow \infty} p(x, x_n) = p(x, x)$. This will be denoted as $x_n \rightarrow x$ ($n \rightarrow \infty$) or $\lim_{n \rightarrow \infty} x_n = x$. Clearly, a limit of a sequence in a partial metric space need not be unique. Moreover, the function $p(\cdot, \cdot)$ need not be continuous in the sense that $x_n \rightarrow x$ and $y_n \rightarrow y$ imply $p(x_n, y_n) \rightarrow p(x, y)$.

If p is a partial metric on X , then the function $p^s : X \times X \rightarrow \mathbb{R}^+$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{1}$$

is a metric on X . It is called the associated metric with the partial metric p .

Example 2.6. (1) A paradigmatic example of a partial metric space is the pair (\mathbb{R}^+, p) , where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. The associated metric is

$$p^s(x, y) = 2 \max\{x, y\} - x - y = |x - y|.$$

(2) If (X, d) is a metric space and $c \geq 0$ is arbitrary, then $p(x, y) = d(x, y) + c$ defines a partial metric on X and the corresponding metric is $p^s(x, y) = 2d(x, y)$.

Other examples of partial metric spaces which are interesting from the computational point of view may be found in [11, 17].

Definition 2.7. Let (X, p) be a partial metric space. Then:

1. A sequence $\{x_n\}$ in (X, p) is called a Cauchy sequence if $\lim_{n, m \rightarrow \infty} p(x_n, x_m)$ exists (and is finite). The space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p , to a point $x \in X$ such that $p(x, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m)$.
2. [22] A sequence $\{x_n\}$ in (X, p) is called 0-Cauchy if $\lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0$. The space (X, p) is said to be 0-complete if every 0-Cauchy sequence in X converges (in τ_p) to a point $x \in X$ such that $p(x, x) = 0$.

Lemma 2.8. Let (X, p) be a partial metric space.

- (a) [2, 14] If $p(x_n, z) \rightarrow p(z, z) = 0$ as $n \rightarrow \infty$, then $p(x_n, y) \rightarrow p(z, y)$ as $n \rightarrow \infty$ for each $y \in X$.
- (b) [22] If (X, p) is complete, then it is 0-complete.

The converse assertion of (b) does not hold as the following easy example shows.

Example 2.9. [22] The space $X = [0, +\infty) \cap \mathbb{Q}$ with the partial metric $p(x, y) = \max\{x, y\}$ is 0-complete, but is not complete. Moreover, the sequence $\{x_n\}$ with $x_n = 1$ for each $n \in \mathbb{N}$ is a Cauchy sequence in (X, p) , but it is not a 0-Cauchy sequence.

It is easy to see that every closed subset of a 0-complete partial metric space is 0-complete.

3. Fixed point theory for cyclic weaker $(\psi \circ \varphi)$ -contractions in partial metric spaces

We will prove some fixed point theorems for self-mappings defined on a 0-complete partial metric space and satisfying certain cyclic weaker Meir-Keeler conditions. To achieve our goal, we recall the notion of a Meir-Keeler function (see [18]).

A function $\psi : [0, +\infty) \rightarrow [0, +\infty)$ is said to be a Meir-Keeler function if for each $\eta > 0$, there exists $\delta > 0$ such that for $t \in [0, +\infty)$ with $\eta \leq t < \eta + \delta$, we have $\psi(t) < \eta$. We now introduce the notion of weaker Meir-Keeler function as follows:

Definition 3.1. [8, 9] We call $\psi : [0, +\infty) \rightarrow [0, +\infty)$ a weaker Meir-Keeler function if for each $\eta > 0$, there exists $\delta > 0$ such that for $t \in [0, +\infty)$ with $\eta \leq t < \eta + \delta$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(t) < \eta$.

As in [8], we assume in this section the following conditions for a weaker Meir-Keeler function $\psi : [0, +\infty) \rightarrow [0, +\infty)$:

- (ψ_1) $\psi(t) > 0$ for $t > 0$ and $\psi(0) = 0$;
- (ψ_2) for all $t \in [0, \infty)$, $\{\psi^n(t)\}_{n \in \mathbb{N}}$ is decreasing;
- (ψ_3) for $t_n \in [0, \infty)$, we have that
 - (a) if $\lim_{n \rightarrow \infty} t_n = \gamma > 0$, then $\lim_{n \rightarrow \infty} \psi(t_n) < \gamma$, and
 - (b) if $\lim_{n \rightarrow \infty} t_n = 0$, then $\lim_{n \rightarrow \infty} \psi(t_n) = 0$.

Also suppose that $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ is a non-decreasing and continuous function satisfying:

- (φ_1) $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$;
- (φ_2) φ is subadditive, that is, for every $\mu_1, \mu_2 \in [0, +\infty)$, $\varphi(\mu_1 + \mu_2) \leq \varphi(\mu_1) + \varphi(\mu_2)$;
- (φ_3) for all $t \in (0, \infty)$, $\lim_{n \rightarrow \infty} \varphi(t_n) = 0$ if and only if $\lim_{n \rightarrow \infty} t_n = 0$.

Similarly as in [8] (in the case of a metric space), we need the notion of a cyclic weaker $(\psi \circ \varphi)$ -contraction in a partial metric space.

Definition 3.2. Let (X, p) be a partial metric space, $m \in \mathbb{N}$, A_1, A_2, \dots, A_m be nonempty subsets of X and $X = \bigcup_{i=1}^m A_i$. An operator $f : X \rightarrow X$ is called a cyclic weaker $(\psi \circ \varphi)$ -contraction if

- (1) $X = \bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to f ;
- (2) for any $x \in A_i, y \in A_{i+1}, i = 1, 2, \dots, m$

$$\varphi(p(fx, fy)) \leq \psi(\varphi(p(x, y))), \tag{2}$$

where $A_{m+1} = A_1$.

The main result of this section is the following:

Theorem 3.3. Let (X, p) be a 0-complete partial metric space, $m \in \mathbb{N}$, A_1, A_2, \dots, A_m be nonempty closed subsets of (X, p) and $Y = \bigcup_{i=1}^m A_i$. Suppose that $f : Y \rightarrow Y$ is a cyclic weaker $(\psi \circ \varphi)$ -contraction. Then, f has a unique fixed point $z \in Y$. Moreover, $z \in \bigcap_{i=1}^m A_i$.

Proof. Let x_0 be an arbitrary point of Y . Then there exists some i_0 such that $x_0 \in A_{i_0}$. Now $x_1 = fx_0 \in A_{i_0+1}$. Similarly, $x_n := fx_{n-1} \in A_{i_0+n}$ for $n \in \mathbb{N}$, where $A_{m+k} = A_k$. In the case $x_{n_0} = x_{n_0+1}$ for some $n_0 = 0, 1, 2, \dots$, it is clear that x_{n_0} is a fixed point of f . Now assume that $x_n \neq x_{n+1}$ for all n . Since $f : Y \rightarrow Y$ is a cyclic weaker $(\psi \circ \varphi)$ -contraction, we have that for all $n \in \mathbb{N}$

$$\varphi(p(x_n, x_{n+1})) = \varphi(p(fx_{n-1}, fx_n)) \leq \psi(\varphi(p(x_{n-1}, x_n))),$$

and so

$$\begin{aligned} \varphi(p(x_n, x_{n+1})) &\leq \psi(\varphi(p(x_{n-1}, x_n))) \\ &\leq \psi(\psi(\varphi(p(x_{n-2}, x_{n-1})))) \\ &= \psi^2(\varphi(p(x_{n-2}, x_{n-1}))) \\ &\leq \\ &\vdots \\ &\leq \psi^n(\varphi(p(x_0, x_1))). \end{aligned}$$

Since $\{\psi^n(\varphi(p(x_0, x_1)))\}_{n \in \mathbb{N}}$ is decreasing, it must converge to some $\eta \geq 0$. We claim that $\eta = 0$. Assume, to the contrary, that $\eta > 0$. Then by the definition of weaker Meir-Keeler function ψ , there exists $\delta > 0$ such that for $x_0, x_1 \in Y$ with $\eta \leq \varphi(p(x_0, x_1)) < \delta + \eta$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(\varphi(p(x_0, x_1))) < \eta$. Since $\lim_{n \rightarrow \infty} \psi^n(\varphi(p(x_0, x_1))) = \eta$, there exists $r_0 \in \mathbb{N}$ such that $\eta \neq \psi^r(\varphi(p(x_0, x_1))) < \delta + \eta$, for all $r \geq r_0$. Thus, we

conclude that $\psi^{r_0+n_0}(\varphi(p(x_0, x_1))) < \eta$. So we get a contradiction. Therefore $\lim_{n \rightarrow \infty} \psi^n(\varphi(p(x_0, x_1))) = 0$, and hence also

$$\lim_{n \rightarrow \infty} \psi^n(\varphi(p(x_n, x_{n+1}))) = 0.$$

Next, we claim that $\{x_n\}$ is a Cauchy sequence. We first prove the following

Claim: for each $\epsilon > 0$, there is $n_0(\epsilon) \in \mathbb{N}$ such that for all $r, q \geq n_0(\epsilon)$,

$$\varphi(p(x_r, x_q)) < \epsilon \tag{3}$$

holds.

We shall prove (3) by contradiction. Suppose that (3) is false. Then there exists some $\epsilon > 0$ such that for all $n \in \mathbb{N}$, there are $r_n, q_n \in \mathbb{N}$ with $r_n > q_n \geq n$ satisfying:

- (i) $\varphi(p(x_{r_n}, x_{q_n})) \geq \epsilon$, and
- (ii) r_n is the smallest number greater than q_n such that condition (i) holds.

Since by (p₄)

$$\begin{aligned} \epsilon &\leq \varphi(p(x_{r_n}, x_{q_n})) \\ &\leq \varphi(p(x_{r_n}, x_{r_n-1}) + p(x_{r_n-1}, x_{q_n}) - p(x_{r_n-1}, x_{r_n-1})) \\ &\leq \varphi(p(x_{r_n}, x_{r_n-1}) + p(x_{r_n-1}, x_{q_n})) \\ &\leq \varphi(p(x_{r_n}, x_{r_n-1})) + \varphi(p(x_{r_n-1}, x_{q_n})) \\ &\leq \varphi(p(x_{r_n}, x_{r_n-1})) + \epsilon, \end{aligned}$$

we conclude that $\lim_{p \rightarrow \infty} \varphi(p(x_{r_n}, x_{q_n})) = \epsilon$. Since φ is subadditive and nondecreasing, we conclude by (p₄)

$$\begin{aligned} \varphi(p(x_{r_n}, x_{q_n})) &\leq \varphi(p(x_{r_n}, x_{r_n+1}) + p(x_{r_n+1}, x_{q_n}) - p(x_{r_n+1}, x_{r_n+1})) \\ &\leq \varphi(p(x_{r_n}, x_{r_n+1}) + p(x_{r_n+1}, x_{q_n})) \\ &\leq \varphi(p(x_{r_n}, x_{r_n+1})) + \varphi(p(x_{r_n+1}, x_{q_n})), \end{aligned}$$

and so again by (p₄)

$$\begin{aligned} \varphi(p(x_{r_n}, x_{q_n})) - \varphi(p(x_{r_n}, x_{r_n+1})) &\leq \varphi(p(x_{r_n+1}, x_{q_n})) \\ &\leq \varphi(p(x_{r_n}, x_{r_n+1}) + p(x_{r_n}, x_{q_n}) - p(x_{r_n}, x_{r_n})) \\ &\leq \varphi(p(x_{r_n}, x_{r_n+1}) + p(x_{r_n}, x_{q_n})) \\ &\leq \varphi(p(x_{r_n}, x_{r_n+1})) + \varphi(p(x_{r_n}, x_{q_n})). \end{aligned}$$

Passing to the limit as $n \rightarrow \infty$, we also have $\lim_{n \rightarrow \infty} \varphi(p(x_{r_n+1}, x_{q_n})) = \epsilon$. Thus, there exists $i, 0 \leq i \leq m - 1$ such that $r_n - q_n + i = 1 \pmod{m}$ for infinitely many n . If $i = 0$, then we have that for such n ,

$$\begin{aligned} \epsilon &\leq \varphi(p(x_{r_n}, x_{q_n})) \\ &\leq \varphi(p(x_{r_n}, x_{r_n+1}) + p(x_{r_n+1}, x_{q_n+1}) + p(x_{q_n+1}, x_{q_n}) - p(x_{r_n+1}, x_{r_n+1}) - p(x_{r_n+1}, x_{q_n+1})) \\ &\leq \varphi(p(x_{r_n}, x_{r_n+1}) + p(x_{r_n+1}, x_{q_n+1}) + p(x_{q_n+1}, x_{q_n})) \\ &\leq \varphi(p(x_{r_n}, x_{r_n+1})) + \varphi(p(x_{r_n+1}, x_{q_n+1})) + \varphi(p(x_{q_n+1}, x_{q_n})) \\ &= \varphi(p(x_{r_n}, x_{r_n+1})) + \varphi(p(fx_{r_n}, fx_{q_n})) + \varphi(p(x_{q_n+1}, x_{q_n})) \\ &\leq \varphi(p(x_{r_n}, x_{r_n+1})) + \psi(\varphi(p(x_{r_n}, x_{q_n}))) + \varphi(p(x_{q_n+1}, x_{q_n})). \end{aligned}$$

Passing to the limit as $n \rightarrow \infty$, we have

$$\epsilon \leq 0 + \lim_{n \rightarrow \infty} \psi(\varphi(p(x_{r_n}, x_{q_n}))) + 0 < \epsilon,$$

a contradiction. Therefore $\lim_{n \rightarrow \infty} \varphi(p(x_{r_n}, x_{q_n})) = 0$. By the condition (φ_3) , we also have $\lim_{n \rightarrow \infty} p(x_{r_n}, x_{q_n}) = 0$. The case $i \neq 0$ is similar. Thus, $\{x_n\}$ is a 0-Cauchy sequence in (Y, p) .

Since Y is closed in (X, p) , then (Y, p) is also 0-complete and there exists $z \in Y = \bigcup_{i=1}^m A_i$ such that $\lim_{n \rightarrow \infty} x_n = z$ in (Y, p) ; equivalently

$$p(z, z) = \lim_{n \rightarrow \infty} p(z, x_n) = \lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0. \tag{4}$$

Notice that the iterative sequence $\{x_n\}$ has an infinite number of terms in A_i for each $i = 1, \dots, m$. Hence, in each $A_i, i = 1, \dots, m$, we can construct a subsequence of $\{x_n\}$ that converges to z . Using that each $A_i, i = 1, \dots, m$, is closed, we conclude that $z \in \bigcap_{i=1}^m A_i$ and thus $\bigcap_{i=1}^m A_i \neq \emptyset$.

Since

$$\begin{aligned} \varphi(p(z, fz)) &= \lim_{n \rightarrow \infty} \varphi(p(fx_{m_n}, fz)) \\ &\leq \lim_{n \rightarrow \infty} \psi(\varphi(p(fx_{m_n-1}, z))) = 0, \end{aligned}$$

we have that $\varphi(p(z, fz)) = 0$, that is, $p(z, fz) = 0$, and z is a fixed point of f .

Finally, to prove the uniqueness of the fixed point, let u be another fixed point of f . By the cyclic character of f , we have $u, z \in \bigcap_{i=1}^m A_i$. Since f is a cyclic weaker $(\psi \circ \varphi)$ -contraction, we have

$$\begin{aligned} \varphi(p(z, u)) &= \varphi(p(z, fu)) = \lim_{n \rightarrow \infty} \varphi(p(fx_{m_n}, fu)) \\ &\leq \lim_{n \rightarrow \infty} \psi(\varphi(p(fx_{m_n-1}, u))) \\ &< \varphi(p(z, u)), \end{aligned}$$

and this is a contradiction unless $\varphi(p(z, u)) = 0$, that is, $z = u$. Thus z is a unique fixed point of f . \square

We illustrate Theorem 3.3 by an example which is obtained by modifying the one from [1].

Example 3.4. Let $X = [0, 1]$ and a partial metric $p : X \times X \rightarrow \mathbb{R}^+$ be given by

$$p(x, y) = \begin{cases} |x - y|, & \text{if } x, y \in [0, 1), \\ 1, & \text{if } x = 1 \text{ or } y = 1. \end{cases}$$

If a mapping $f : X \rightarrow X$ is given by

$$fx = \begin{cases} 1/2, & \text{if } x \in [0, 1), \\ 0, & \text{if } x = 1, \end{cases}$$

and $A_1 = [0, \frac{1}{2}]$, $A_2 = [\frac{1}{2}, 1]$, then $A_1 \cup A_2 = X$ is a cyclic representation of X with respect to f . Moreover, mapping f is a cyclic weaker $(\psi \circ \varphi)$ -contraction, where $\varphi(t) = t$ and $\psi(t) = \frac{3}{4}t$. Indeed, consider the following cases:

1° $x \in [0, \frac{1}{2}]$, $y \in [\frac{1}{2}, 1)$ or $y \in [0, \frac{1}{2}]$, $x \in [\frac{1}{2}, 1)$. Then $p(fx, fy) = p(\frac{1}{2}, \frac{1}{2}) = 0$ and relation (2) is trivially satisfied.

2° $x \in [0, \frac{1}{2}]$, $y = 1$ or $y \in [0, \frac{1}{2}]$, $x = 1$. Then $p(fx, fy) = p(\frac{1}{2}, 0) = \frac{1}{2}$ and $p(x, y) = 1$. Relation (2) holds as it reduces to $\frac{1}{2} < \frac{3}{4}$.

We conclude that f has a unique fixed point (which is $z = \frac{1}{2}$).

Note that, if instead of the given partial metric p its associated metric

$$p^s(x, y) = \begin{cases} 2|x - y|, & \text{if } x, y \in [0, 1), \\ 1, & \text{if } x \in [0, 1), y = 1 \text{ or } x = 1, y \in [0, 1), \\ 0, & \text{if } x = y = 1, \end{cases}$$

is used, then for $x = \frac{1}{2}$ and $y = 1$ the respective condition (i.e., condition (ii) of Definition 4 from [8]) is not satisfied since it reduces to

$$\varphi\left(p^s\left(\frac{1}{2}, 0\right)\right) = 1 < \frac{3}{4} = \psi\left(\varphi\left(p^s\left(\frac{1}{2}, 1\right)\right)\right).$$

Similar conclusion is obtained if the standard Euclidean metric is used.

Hence, this example shows that Theorem 3.3 is a proper extension of [8, Theorem 3].

4. Fixed point theory for cyclic weaker (ψ, φ) -contractions in partial metric spaces

In this section we derive a generalized version of results from [8], [10] and [20].

We assume $\psi : [0, +\infty) \rightarrow [0, +\infty)$ to be a weaker Meir-Keeler function satisfying conditions (ψ_1) , (ψ_2) and (ψ_3) from Section 3. Also consider $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ to be a non-decreasing and continuous function satisfying (φ_1) ((φ_2) and (φ_3) are not needed). To complete the results, we need the following notion of a cyclic weaker (ψ, φ) -contraction, which is the counterpart of the respective notion from [8]:

Definition 4.1. Let (X, p) be a partial metric space, $m \in \mathbb{N}$, and let A_1, A_2, \dots, A_m be nonempty subsets of X such that $X = \bigcup_{i=1}^m A_i$. An operator $f : X \rightarrow X$ is called a cyclic weaker (ψ, φ) -contraction if

- (1) $X = \bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to f ;
- (2) for any $x \in A_i, y \in A_{i+1}, i = 1, 2, \dots, m$

$$p(fx, fy) \leq \psi(p(x, y)) - \varphi(p(x, y)), \tag{5}$$

where $A_{m+1} = A_1$.

Theorem 4.2. Let (X, p) be a 0-complete partial metric space, $m \in \mathbb{N}$, let A_1, A_2, \dots, A_m be nonempty closed subsets of (X, p) and $Y = \bigcup_{i=1}^m A_i$. Suppose that $f : Y \rightarrow Y$ is a cyclic weaker (ψ, φ) -contraction. Then, f has a unique fixed point $z \in Y$. Moreover, $z \in \bigcap_{i=1}^m A_i$.

Proof. Let x_0 be an arbitrary point of Y . Then there exists some i_0 such that $x_0 \in A_{i_0}$. Now $x_1 = fx_0 \in A_{i_0+1}$. Similarly, $x_n := fx_{n-1} \in A_{i_0+n}$ for $n \in \mathbb{N}$, where $A_{m+k} = A_k$. In the case $x_{n_0} = x_{n_0+1}$ for some $n_0 = 0, 1, 2, \dots$, it is clear that x_{n_0} is a fixed point of f . Now assume that $x_n \neq x_{n+1}$ for all n . Since $f : Y \rightarrow Y$ is a cyclic weaker (ψ, φ) -contraction, we have that for all $n \in \mathbb{N}$

$$\begin{aligned} p(x_n, x_{n+1}) &= p(fx_{n-1}, fx_n) \\ &\leq \psi(p(x_{n-1}, x_n)) - \varphi(p(x_{n-1}, x_n)) \\ &\leq \psi(p(x_{n-1}, x_n)) \end{aligned}$$

and so

$$\begin{aligned} p(x_n, x_{n+1}) &\leq \psi(p(x_{n-1}, x_n)) \\ &\leq \psi(\psi(p(x_{n-2}, x_{n-1}))) \\ &= \psi^2(p(x_{n-2}, x_{n-1})) \\ &\vdots \\ &\leq \psi^n(p(x_0, x_1)). \end{aligned}$$

Since $\{\psi^n(p(x_0, x_1))\}_{n \in \mathbb{N}}$ is decreasing, it must converge to some $\eta \geq 0$. We claim that $\eta = 0$. On the contrary, assume that $\eta > 0$. Then by the definition of weaker Meir-Keeler function ψ , there exists $\delta > 0$ such that for $x_0, x_1 \in Y$ with $\eta \leq \varphi(p(x_0, x_1)) < \delta + \eta$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(p(x_0, x_1)) < \eta$. Since $\lim_{n \rightarrow \infty} \psi^n(\varphi(p(x_0, x_1))) = \eta$, there exists $r_0 \in \mathbb{N}$ such that $\eta \neq \psi^r(p(x_0, x_1)) < \delta + \eta$, for all $r \geq r_0$. Thus, we conclude that $\psi^{r_0+n_0}(p(x_0, x_1)) < \eta$. So we get a contradiction. Therefore $\lim_{n \rightarrow \infty} \psi^n(p(x_0, x_1)) = 0$, that is,

$$\lim_{n \rightarrow \infty} p(x_n, x_{n+1}) = 0.$$

Next, we claim that $\{x_n\}$ is a Cauchy sequence. We first prove the following:

Claim: For each $\epsilon > 0$, there is $n \in \mathbb{N}$ such that for all $r, q \geq n$ with $r - q = 1 \pmod{m}$,

$$p(x_p, x_q) < \epsilon, \tag{6}$$

holds.

We shall prove the claim by contradiction. Suppose that (6) is false. Then there exists some $\epsilon > 0$ such that for all $n \in \mathbb{N}$, there are $r_n, q_n \in \mathbb{N}$ with $r_n > q_n \geq n$ with $r_n - q_n = 1 \pmod{m}$, satisfying:

$$p(x_{r_n}, x_{q_n}) \geq \epsilon.$$

Now, we let $n > 2m$. Then corresponding to $q_n \geq n$, we can choose r_n in such a way, that it is the smallest integer with $r_n > q_n \geq n$ satisfying $r_n - q_n = 1 \pmod{m}$ and $p(x_{q_n}, x_{r_n}) \geq \epsilon$. Then, from $p(x_{q_n}, x_{r_n-m}) \leq \epsilon$ and (p_4) , we have

$$\begin{aligned} \epsilon &\leq p(x_{q_n}, x_{r_n}) \\ &\leq p(x_{q_n}, x_{r_n-m}) + \sum_{i=1}^m p(x_{r_{n-i}}, x_{r_{n-i+1}}) - \sum_{i=1}^m p(x_{r_{n-i}}, x_{r_{n-i}}) \\ &\leq p(x_{q_n}, x_{r_n-m}) + \sum_{i=1}^m p(x_{r_{n-i}}, x_{r_{n-i+1}}) \\ &< \epsilon + \sum_{i=1}^m p(x_{r_{n-i}}, x_{r_{n-i+1}}). \end{aligned}$$

Passing to the limit as $n \rightarrow \infty$, we obtain that

$$\lim_{n \rightarrow \infty} p(x_{q_n}, x_{r_n}) = \epsilon.$$

On the other hand, we can conclude that

$$\begin{aligned} \epsilon &\leq p(x_{q_n}, x_{r_n}) \\ &\leq p(x_{q_n}, x_{q_{n+1}}) + p(x_{q_{n+1}}, x_{r_{n+1}}) + p(x_{r_{n+1}}, x_{r_n}) - [p(x_{q_{n+1}}, x_{q_{n+1}}) + p(x_{r_{n+1}}, x_{r_{n+1}})] \\ &\leq p(x_{q_n}, x_{q_{n+1}}) + p(x_{q_{n+1}}, x_{q_n}) + p(x_{q_n}, x_{r_n}) + p(x_{r_n}, x_{r_{n+1}}) + p(x_{r_{n+1}}, x_{r_n}) \\ &\quad - [p(x_{q_{n+1}}, x_{q_{n+1}}) + p(x_{q_n}, x_{q_n}) + p(x_{r_n}, x_{r_n}) + p(x_{r_{n+1}}, x_{r_{n+1}})] \\ &\leq p(x_{q_n}, x_{q_{n+1}}) + p(x_{q_{n+1}}, x_{q_n}) + p(x_{q_n}, x_{r_n}) + p(x_{r_n}, x_{r_{n+1}}) + p(x_{r_{n+1}}, x_{r_n}). \end{aligned}$$

Passing to the limit as $n \rightarrow \infty$, we obtain that

$$\lim_{n \rightarrow \infty} p(x_{q_{n+1}}, x_{r_{n+1}}) = \epsilon.$$

Since x_{q_n} and x_{r_n} lie in different adjacently labelled sets A_i and A_{i+1} for certain $1 \leq i < m$, by using the fact that f is a cyclic weaker (ψ, φ) -contraction, we have

$$p(x_{q_{n+1}}, x_{r_{n+1}}) = p(fx_{q_n}, fx_{r_n}) \leq \psi(p(x_{q_n}, x_{r_n})) - \varphi(p(x_{q_n}, x_{r_n})).$$

Passing to the limit as $n \rightarrow \infty$, by using the condition (ψ_3) of function ψ , we obtain that

$$\epsilon \leq \epsilon - \varphi(\epsilon),$$

and consequently, $\varphi(\epsilon) = 0$. By the definition of the function φ , we get $\epsilon = 0$ which is a contradiction. Therefore, our claim is proved.

Next, we shall show that $\{x_n\}$ is a Cauchy sequence. Let $\epsilon > 0$ be given. By our claim, there exists $n_1 \in \mathbb{N}$ such that if $r, q \geq n_1$ with $r - q = 1 \pmod{m}$, then

$$p(x_r, x_q) \leq \frac{\epsilon}{2}.$$

Since $\lim_{n \rightarrow \infty} p(x_n, x_{n+1}) = 0$, there exists $n_2 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq \frac{\epsilon}{2m},$$

for any $n \geq n_2$.

Let $p, q \geq \max\{n_1, n_2\}$ and $p > q$. Then there exists $k \in \{1, 2, \dots, m\}$ such that $p - q = k \pmod{m}$. Therefore, $p - q + j = 1 \pmod{m}$ for $j = m - k + 1$, and so we have

$$\begin{aligned} p(x_q, x_r) &\leq p(x_q, x_{r+j}) + p(x_{r+j}, x_{r+j-1}) + \dots + p(x_{r+1}, x_r) - [p(x_{r+j}, x_{r+j}) + \dots + p(x_{r+1}, x_{r+1})] \\ &\leq \frac{\epsilon}{2} + j \cdot \frac{\epsilon}{2m} \leq \frac{\epsilon}{2} + m \cdot \frac{\epsilon}{2m} = \epsilon. \end{aligned}$$

Thus, $\{x_n\}$ is a 0-Cauchy sequence in (Y, p) . Since Y is closed in (X, p) , then (Y, p) is also 0-complete. Thus, there exists $z \in Y$ such that

$$p(z, z) = \lim_{n \rightarrow \infty} p(z, x_n) = \lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0. \tag{7}$$

Notice that the iterative sequence $\{x_n\}$ has an infinite number of terms in A_i for each $i = 1, \dots, m$. Hence, in each A_i , $i = 1, \dots, m$, we can construct a subsequence of $\{x_n\}$ that converges to z . Regarding that each A_i , $i = 1, \dots, m$, is closed, we conclude that $z \in \bigcap_{i=1}^m A_i$ and thus $\bigcap_{i=1}^m A_i \neq \emptyset$.

Now for all $i = 1, 2, \dots, m$, we may take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \in A_{i-1}$ and they also all converge to z . Since

$$\begin{aligned} p(x_{n_{k+1}}, fz) &= p(fx_{n_k}, fz) \\ &\leq \psi(p(x_{n_k}, z)) - \varphi(p(x_{n_k}, z)) \\ &\leq \psi(p(x_{n_k}, z)), \end{aligned}$$

passing to the limit as $k \rightarrow \infty$, we have $p(z, fz) \leq 0$, and so $z = fz$.

Finally, to prove the uniqueness of the fixed point, let u be the another fixed point of f . By the cyclic character of f , we have $u, z \in \bigcap_{i=1}^m A_i$. Since f is a cyclic weaker (ψ, φ) -contraction, we have

$$\begin{aligned} p(z, u) &= p(z, fu) \\ &= \lim_{n \rightarrow \infty} p(x_{n_{k+1}}, fu) \\ &= \lim_{n \rightarrow \infty} p(fx_{n_k}, fu) \\ &\leq \lim_{n \rightarrow \infty} [\psi(p(x_{n_k}, u)) - \varphi(p(x_{n_k}, u))] \\ &\leq p(z, u) - \varphi(p(z, u)), \end{aligned}$$

and we can conclude that $\varphi(p(z, u)) = 0$. So we have $u = z$. This completes the proof. \square

We illustrate the use of Theorem 4.2 by the following

Example 4.3. Let $X = [0, 2] = [0, 1] \cup [1, 2] = A_1 \cup A_2$ be equipped by the usual partial metric $p(x, y) = \max\{x, y\}$. Let $f : X \rightarrow X$ be given by $fx = 2 - x$. Then $A_1 \cup A_2 = X$ is a cyclic representation of X with respect to f . If $\psi(t) = 4t$ and $\varphi(t) = 2t$, we will prove that f is a cyclic weaker (ψ, φ) -contraction. Indeed let, e.g., $x \in [0, 1]$ and $y \in [1, 2]$, Then

$$p(fx, fy) = \max\{2 - x, 2 - y\} = 2 - x \leq 2 \leq 2y = 4y - 2y = \psi(p(x, y)) - \varphi(p(x, y)),$$

and condition (5) is fulfilled. All other conditions of Theorem 4.2 are also satisfied and f has a unique fixed point ($z = 1$).

The following example, which is similar to Example 3.4, shows that again the obtained results are stronger than those from [8].

Example 4.4. Let $X = [0, 1]$ and a partial metric $p : X \times X \rightarrow \mathbb{R}^+$ be given by

$$p(x, y) = \begin{cases} |x - y|, & \text{if } x, y \in [0, 1), \\ 1, & \text{if } x = 1 \text{ or } y = 1. \end{cases}$$

If a mapping $f : X \rightarrow X$ is given by

$$fx = \begin{cases} 1/8, & \text{if } x \in [0, 1), \\ 0, & \text{if } x = 1, \end{cases}$$

and $A_1 = [0, \frac{1}{8}]$, $A_2 = [\frac{1}{8}, 1]$, then $A_1 \cup A_2 = X$ is a cyclic representation of X with respect to f . Moreover, mapping f is a cyclic weaker (ψ, φ) -contraction, where

$$\psi(t) = \frac{t^2}{1+t} \quad \text{and} \quad \varphi(t) = \frac{t^2}{2+t}.$$

Indeed, consider the following cases:

1° $x \in [0, \frac{1}{8}]$, $y \in [\frac{1}{8}, 1)$ or $y \in [0, \frac{1}{8}]$, $x \in [\frac{1}{8}, 1)$. Then $p(fx, fy) = p(\frac{1}{8}, \frac{1}{8}) = 0$ and relation (5) is trivially satisfied.

2° $x \in [0, \frac{1}{8}]$, $y = 1$ or $y \in [0, \frac{1}{8}]$, $x = 1$. Then $p(fx, fy) = p(\frac{1}{8}, 0) = \frac{1}{8}$ and $p(x, y) = 1$. Relation (5) holds as it reduces to $\frac{1}{8} < \frac{1}{6} = \frac{1}{2} - \frac{1}{3}$.

We conclude that f has a unique fixed point (which is $z = \frac{1}{8}$).

Note again that, if instead of the given partial metric p its associated metric p^s is used, then for $x = \frac{1}{2}$ and $y = 1$ the respective condition (i.e., condition (ii) of Definition 5 from [8]) is not satisfied since it reduces to

$$\varphi\left(p^s\left(\frac{1}{8}, 0\right)\right) = \frac{1}{4} < \frac{1}{6} = \frac{1}{2} - \frac{1}{3} = \psi\left(p^s\left(\frac{1}{2}, 1\right)\right) - \varphi\left(p^s\left(\frac{1}{2}, 1\right)\right).$$

Similar conclusion is obtained if the standard Euclidean metric is used.

Hence, this example shows that Theorem 4.2 is a proper extension of [8, Theorem 4].

Remark 4.5. The results of this paper are obtained under the assumption that the given partial metric space is 0-complete. Taking into account Lemma 2.8 and Example 2.9, it follows that they also hold if the space is complete, but that our assumption is weaker.

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