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Partial Group (Co)Actions of Hopf Group Coalgebras

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Abstract. We will develop partial group (co)actions of a Hopf group coalgebra on a family of algebras by introducing partial group entwining structure. Then we give necessary and sufficient conditions for a family of functors from the category of partial group entwining modules to the category of modules over a suitable algebra to be separable. Also, the applications of our results are considered.

1. Introduction

As the generalization of Hopf algebra, the notion of a Hopf π-coalgebra was introduced by Turaev [23]. Hopf π-coalgebras are used by the author in [24] to construct Hennings-like and Kuperberg-like invariants of principal π-bundles over link complements and over 3-manifolds. A systematic algebraic study of these new structures has been carried out in recent papers ([7], [25], [26], [27] and [28]).

Partial group actions were considered first by Exel [18] in the context of operator algebras and they turned out to be a powerful tool in the study of C^*-algebras generated by partial isometries on a Hilbert space [19]. A treatment from a purely algebraic point of view was given recently in [1], [14], [16] and [17]. Partial Hopf actions were motivated by an attempt to generalize the notion of partial Galois extensions of commutative rings in [16] to a broader context. The definition of partial Hopf actions and co-actions was introduced by Caenepeel and Janssen in [5] by using the notions of partial entwining structures.

The notion of separable functor was introduced by Năstăsescu, Van den Bergh and Van Oystaeyen in [21], where some applications for group-graded rings were done. Every separable functor between abelian categories encodes a Maschke's Theorem, which explains the interest concentrated in this notion within the module-theoretical developments in recent years. Separable functors have been investigated in the framework of coalgebras ([8]), graded homomorphisms of rings ([9], [12]), Doi-Koppinen modules ([6]), entwined modules ([3]) or coring ([2]).

The idea underlying this article is to consider a more general setting, that is: can we develop a theory of partial (co)actions of Hopf group coalgebras? The aim of this paper is to give a positive answer to these questions.

The paper is organized as follows.

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In Section 2, we recall some definitions of group coalgebras, Hopf group coalgebra and separable functors.

In Section 3, partial π-entwined structures(modules) are introduced, and for all $\alpha \in \pi$, we show that the functor $F^{(\alpha)} : \mathcal{U}_A^{\pi-C}(\psi) \to \mathcal{M}_{A_\alpha}$ which forgets the partial π-C-coaction has an adjoint.

In Section 4, we will develop a theory of partial (co)actions of Hopf group coalgebras and introduce the concepts of partial group co(module) (co)algebra, partial Doi-Hopf group structures(modules).

In Section 5, we will characterize the separability of the forgetful functor $F^{(\alpha)}$ from the category of so-called partial π-entwined modules $\mathcal{U}_A^{\pi-C}(\psi)$ to the category of all A_α-modules (fixed $\alpha \in \pi$) which leads to a generalized notion of integral of a partial π-entwined structure. Finally, the main applications of our results are considered in Section 6.

2. Group Coalgebras, Hopf Group coalgebras and Separable Functors

Throughout this paper, we always let π be a discrete group with a neutral element e and k a field. All (co)algebras and comodules are all over k. Let M and N be vectors, $\tau_{M,N} : M \otimes N \to N \otimes M : m \otimes n \mapsto n \otimes m$ denotes the flip map.

2.1. *Group Coalgebras*

Recall from [23] that π-coalgebra is a family of k-spaces $C = \{C_\alpha\}_{\alpha\in\pi}$ together with a family of k-linear maps $\Delta = \{\Delta_{\alpha,\beta} : C_{\alpha\beta} \to C_\alpha \otimes C_\beta\}_{\alpha,\beta\in\pi}$ (called a *comultiplication*) and a k-linear map $\varepsilon : C_e \to k$ (called a *counit*) such that Δ is coassociative in the sense that

$$(\Delta_{\alpha,\beta} \otimes id_{C_\gamma}) \circ \Delta_{\alpha\beta,\gamma} = (id_{C_\alpha} \otimes \Delta_{\beta,\gamma}) \circ \Delta_{\alpha,\beta\gamma},$$

for any $\alpha, \beta, \gamma \in \pi$ and

$$(id_{C_\alpha} \otimes \varepsilon) \circ \Delta_{\alpha,e} = id_{C_\alpha} = (\varepsilon \otimes id_{C_\alpha}) \circ \Delta_{e,\alpha},$$

for all $\alpha \in \pi$.

Remark 2.1. $(C_e, \Delta_{e,e}, \varepsilon)$ *is an ordinary coalgebra in the sense of Sweedler.*

Following the Sweedler's notation for π-coalgebras, for any $\alpha, \beta \in \pi$ and $c \in C_{\alpha\beta}$, one writes

$$\Delta_{\alpha,\beta}(c) = c_{(1,\alpha)} \otimes c_{(2,\beta)}.$$

2.2. *Hopf Group Coalgebras*

Recall from [7] that a *semi-Hopf π-coalgebra* is a π-coalgebra $H = (\{H_\alpha\}_{\alpha\in\pi}, \Delta = \{\Delta_{\alpha,\beta}\}, \varepsilon)$ such that the following conditions hold:

(SH1) Each H_α is an algebra with multiplication m_α and unit $1_\alpha \in H_\alpha$,
(SH2) For all $\alpha, \beta \in \pi$, $\Delta_{\alpha,\beta}$ and $\varepsilon : H_e \to k$ are algebra maps.

A semi-Hopf π-coalgebra $H = (\{H_\alpha, m_\alpha, 1_\alpha\}_{\alpha\in\pi}, \Delta = \{\Delta_{\alpha,\beta}\}, \varepsilon)$ is called a *Hopf π-coalgebra*, if there exists a family of k-linear maps $S = \{S_\alpha : H_\alpha \to H_{\alpha^{-1}}\}_{\alpha\in\pi}$ (called an *antipode*) such that

$$m_\alpha \circ (id_{H_\alpha} \otimes S_{\alpha^{-1}}) \circ \Delta_{\alpha,\alpha^{-1}} = \varepsilon 1_\alpha = m_\alpha \circ (S_{\alpha^{-1}} \otimes id_{H_\alpha}) \circ \Delta_{\alpha^{-1},\alpha}.$$

2.3. Separable Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and $F : \mathcal{C} \to \mathcal{D}$ a covariant functor. Observe that we have two covariant functors

$$\mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet) : \mathcal{C}^{op} \times \mathcal{C} \to \underline{\mathrm{Sets}} \quad \text{and} \quad \mathrm{Hom}_{\mathcal{D}}(F(\bullet), F(\bullet)) : \mathcal{C}^{op} \times \mathcal{C} \to \underline{\mathrm{Sets}}$$

and a natural transformation

$$\mathcal{F} : \mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet) \to \mathrm{Hom}_{\mathcal{D}}(F(\bullet), F(\bullet)).$$

Recall from [21] that F is called *separable* , if $\mathcal{F}$ splits, this means that we have a natural transformation $\mathcal{P}$ such that $\mathcal{P} \circ \mathcal{F}$ is the identity natural transformation of Hom $_{\mathcal{C}}(\bullet, \bullet)$.

Now suppose that $F : \mathcal{C} \to \mathcal{D}$ has a right adjoint $G : \mathcal{D} \to \mathcal{C}$, and write $\eta : 1_{\mathcal{C}} \to GF$ and $\delta : FG \to 1\mathcal{D}$ for the unit and counit of this adjuction. Then we have the following results [22]:

Rafael's Theorem Let $G : \mathcal{D} \to \mathcal{C}$ be a right adjoint of $F : \mathcal{C} \to \mathcal{D}$. F is separable if and only if η splits, this means that there is a natural transformation $\nu : GF \to 1_{\mathcal{C}}$ such that $\nu \circ \eta$ is the identity natural transformation of $\mathcal{C}$.

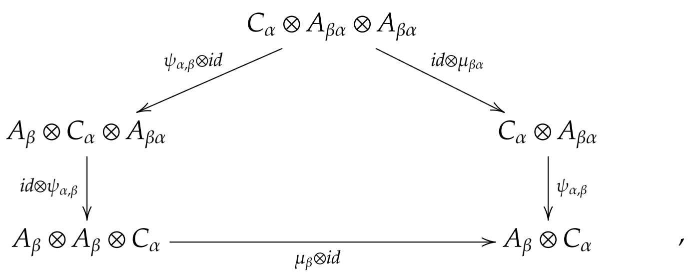
3. Partial Entwining Structures, Partial Entwining Modules and Adjoint Functors

Let $C = (\{C_\alpha\}_{\alpha\in\pi}, \Delta, \varepsilon)$ be a π-coalgebra, and let A be a family of algebras $A = \{A_\alpha, m_\alpha, 1_{A_\alpha}\}_{\alpha\in\pi}$ over k. Let ψ be a family of k-linear maps $\psi = \{\psi_{\alpha,\beta} : C_\alpha \otimes A_{\beta\alpha} \to A_\beta \otimes C_\alpha\}_{\alpha,\beta\in\pi}$ such that the following conditions are satisfied:

(1) For all $\alpha, \beta \in \pi$, $a, b \in A_{\beta\alpha}$ and $c \in C_\alpha$,

$$(ab)_{\psi_{\alpha,\beta}} \otimes c^{\psi_{\alpha,\beta}} = a_{\psi_{\alpha,\beta}} b_{\psi'_{\alpha,\beta}} \otimes c^{\psi_{\alpha,\beta}\psi'_{\alpha,\beta}}, \tag{E3.1}$$

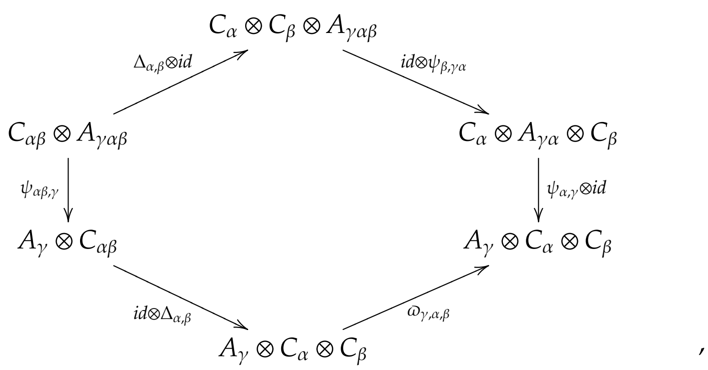
Eq. (E3.1) is equivalent to the following commutative diagram



(2) For all $\alpha, \beta, \gamma \in \pi$, $a \in A_{\gamma\alpha\beta}$ and $d \in C_{\alpha\beta}$,

$$a_{\psi_{\alpha\beta,\gamma}} 1_{A_{\gamma\alpha}\psi_{\alpha,\gamma}} \otimes d^{\psi_{\alpha\beta,\gamma}}{}_{(1,\alpha)}{}^{\psi_{\alpha,\gamma}} \otimes d^{\psi_{\alpha\beta,\gamma}}{}_{(2,\beta)} = a_{\psi_{\beta,\gamma\alpha}\psi_{\alpha,\gamma}} \otimes d_{(1,\alpha)}{}^{\psi_{\alpha,\gamma}} \otimes d_{(2,\beta)}{}^{\psi_{\beta,\gamma\alpha}}, \tag{E3.2}$$

Eq. (E3.2) is equivalent to the following commutative diagram



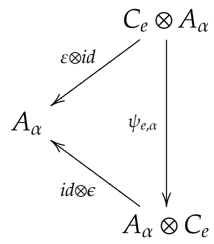
where $\varpi = \{\varpi_{\gamma,\alpha,\beta}\}$ and $\varpi_{\gamma,\alpha,\beta} : A_\gamma \otimes C_\alpha \otimes C_\beta \longrightarrow A_\gamma \otimes C_\alpha \otimes C_\beta$ is defined by

$$\varpi_{\gamma,\alpha,\beta}(a \otimes c \otimes d) = a1_{A_{\gamma\alpha}}{}_{\psi_{\alpha,\gamma}} \otimes c^{\psi_{\alpha,\gamma}} \otimes d.$$

(3) For all $\alpha \in \pi$ and $c \in C_e, a \in A_\alpha$,

$$\varepsilon(c^{\psi_{e,\alpha}})a_{\psi_{e,\alpha}} = \varepsilon(c)a, \tag{E3.3}$$

Eq. (E3.3) is equivalent to the following commutative diagram



The triple (A, C, ψ) is called a (right-right) partial π-*entwining structure* and is denoted by $(A, C)_{\pi-\psi}$. The map ψ is called a partial π-entwining map. For $c \in C_\alpha$ and $a \in A_{\beta\alpha}$, we adopt the notation

$$\psi_{\alpha,\beta}(c \otimes a) = a_{\psi_{\alpha,\beta}} \otimes c^{\psi_{\alpha,\beta}} = a_{\psi'_{\alpha,\beta}} \otimes c^{\psi'_{\alpha,\beta}} = a_{\psi''_{\alpha,\beta}} \otimes c^{\psi''_{\alpha,\beta}} = \cdots .$$

Example 3.1. *Let H be a Hopf π-coalgebra. Assume that there exsits a family of idempotents $p = \{p_\alpha\}_{\alpha\in\pi}$ in H such that $\Delta_{\alpha,\beta}(p_{\alpha\beta})(p_\alpha \otimes 1_\beta) = p_\alpha \otimes p_\beta$ and $\varepsilon(p_e) = 1$. Let $A = \{A_\alpha = k\}_{\alpha\in\pi}$. Then we have a partial π-entwined structure $(A, H)_{\pi-\psi}$, where $\psi = \{\psi_{\alpha,\beta} : H_\alpha \otimes k \to k \otimes H_\alpha\}_{\alpha\in\pi}$, $\psi_{\alpha,\beta}(h \otimes x) = x \otimes hp_\alpha$.*

Example 3.2. *Let H be a Hopf π-coalgebra. Assume that there exsits a family of idempotents $p = \{p_\alpha\}_{\alpha\in\pi}$ in H such that $\Delta_{\alpha,\beta}(p_{\alpha\beta})(p_\alpha \otimes 1_\beta) = p_\alpha \otimes p_\beta$ and $\varepsilon(p_e) = 1$. Let $A = H$. Then we have a partial π-entwined structure $(A, H)_{\pi-\psi}$, where $\psi = \{\psi_{\alpha,\beta} : H_\alpha \otimes H_{\beta\alpha} \to H_\beta \otimes H_\alpha\}_{\alpha\in\pi}$, $\psi_{\alpha,\beta}(h \otimes g) = g_{(1,\beta)} \otimes hg_{(2,\alpha)}e_\alpha$.*

Given a (right-right) partial π-entwining structure $(A, C)_{\pi-\psi}$. Then one can form the category $\mathcal{U}_A^{\pi-C}(\psi)$ of right $(A, C)_{\pi-\psi}$-modules. The objects of $\mathcal{U}_A^{\pi-C}(\psi)$ are a family of vector spaces $M = \{M_\alpha\}_{\alpha\in\pi}$ together with a family of k-linear maps

$$\phi = \{\phi_\alpha : M_\alpha \otimes A_\alpha \to M_\alpha, \phi_\alpha(m \otimes a) = m \cdot a\}_{\alpha\in\pi}$$

and a family of k-linear maps $\rho^M = \{\rho^M_{\alpha,\beta} : M_{\alpha\beta} \to M_\alpha \otimes C_\beta\}_{\alpha,\beta\in\pi}$(called *a partial π-C-coaction* on M) such that

(PM1) For any $\alpha \in \pi$, (M_α, ϕ_α) is a right A_α-module,

(PM2) For any $\alpha, \beta \in \pi$, $m \in M_{\alpha\beta}$ and $a \in A_{\alpha\beta}$,

$$\rho^M_{\alpha,\beta}(m \cdot a) = m_{[0,\alpha]} \cdot a_{\psi_{\beta,\alpha}} \otimes m_{[1,\beta]}{}^{\psi_{\beta,\alpha}},$$

(PM3) For all $\alpha, \beta, \gamma \in \pi$, $m \in M_{\alpha\beta\gamma}$,

$$m_{[0,\alpha\beta][0,\alpha]} \otimes m_{[0,\alpha\beta][1,\beta]} \otimes m_{[1,\gamma]} = m_{[0,\alpha]} \cdot 1_{A_{\alpha\beta\gamma}\psi_{\gamma,\alpha\beta}\psi_{\beta,\alpha}} \otimes m_{[1,\beta\gamma](1,\beta)}{}^{\psi_{\beta,\alpha}} \otimes m_{[1,\beta\gamma](2,\gamma)}{}^{\psi_{\gamma,\alpha\beta}},$$

(PM4) For all $m \in M_\alpha$, $\varepsilon(m_{[1,e]})m_{[0,\alpha]} = m$.

Here we use the following notation for the map $\rho^M_{\alpha,\beta}(m) = m_{[0,\alpha]} \otimes m_{[1,\beta]}$, for all $\alpha, \beta \in \pi$ and $m \in M_{\alpha\beta}$.

Proposition 3.3. *For any $\alpha \in \pi$, the forgetful functor $F^{(\alpha)} : \mathcal{U}_A^{\pi-C}(\psi) \to \mathcal{M}_{A_\alpha}$ has a right adjoint $G^{(\alpha)} : \mathcal{M}_{A_\alpha} \to \mathcal{U}_A^{\pi-C}(\psi)$.*

Proof. For $M \in \mathcal{M}_{A_\alpha}$, we define $G^{(\alpha)}(M) = \{\underline{M \otimes C_{\alpha^{-1}\beta}}\}_{\beta\in\pi}$, where $\underline{M \otimes C_{\alpha^{-1}\beta}}$ is defined as the image of the idempotent map

$$(\phi_\alpha \otimes C_{\alpha^{-1}\beta}) \circ (M \otimes \psi_{\alpha^{-1}\beta,\alpha}) \circ (M \otimes C_{\alpha^{-1}\beta} \otimes \eta_\beta) : M \otimes C_{\alpha^{-1}\beta} \longrightarrow M \otimes C_{\alpha^{-1}\beta}.$$

Explicitly,

$$\underline{M \otimes C_{\alpha^{-1}\beta}} =< m \cdot 1_{A_\beta \psi_{\alpha^{-1}\beta,\alpha}} \otimes c^{\psi_{\alpha^{-1}\beta,\alpha}} | m \in M, c \in C_{\alpha^{-1}\beta} > .$$

Now, we define the A-action and partial π-C-coaction as follows: for all $\beta, \gamma \in \pi$, $a \in A_\beta$, $c \in C_{\alpha^{-1}\beta}$, $d \in C_{\alpha^{-1}\beta\gamma}$ and $m \in M$,

$$(m \cdot 1_{A_\beta \psi_{\alpha^{-1}\beta,\alpha}} \otimes c^{\psi_{\alpha^{-1}\beta,\alpha}}) \cdot a = m \cdot a_{\psi_{\alpha^{-1}\beta,\alpha}} \otimes c^{\psi_{\alpha^{-1}\beta,\alpha}},$$

$${}^r\rho^{G^{(\alpha)}(M)}_{\beta,\gamma}(m \cdot 1_{A_{\beta\gamma}\psi_{\alpha^{-1}\beta\gamma,\alpha}} \otimes d^{\psi_{\alpha^{-1}\beta\gamma,\alpha}}) = m \cdot 1_{A_{\beta\gamma}\psi_{\gamma,\beta}\psi_{\alpha^{-1}\beta,\alpha}} \otimes d_{(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}} \otimes d_{(2,\gamma)}{}^{\psi_{\gamma,\beta}}.$$

With A-action and partial π-C-coaction defined as above, we shall check that (PM2)and (PM3) hold for a partial π-entwined module. First, we shall check Condition (PM2). For all $\alpha, \beta, \gamma \in \pi$, $m \in M$ and $c \in C_{\alpha^{-1}\beta\gamma}$, $a \in A_{\beta\gamma}$, we make a direct calculation as follows:

$$\begin{aligned}
& {}^r\rho^{G^{(\alpha)}(M)}_{\beta,\gamma}((m \cdot 1_{A_{\beta\gamma}\psi_{\alpha^{-1}\beta\gamma,\alpha}} \otimes c^{\psi_{\alpha^{-1}\beta\gamma,\alpha}}) \cdot a) \\
= \ & \rho^r_{\beta,\gamma}(m \cdot a_{\psi_{\alpha^{-1}\beta\gamma,\alpha}} \otimes c^{\psi_{\alpha^{-1}\beta\gamma,\alpha}}) \\
\overset{(E3.1)}{=} \ & \rho^r_{\beta,\gamma}(m \cdot a_{\psi_{\alpha^{-1}\beta\gamma,\alpha}} 1_{A_{\beta\gamma}\psi'_{\alpha^{-1}\beta\gamma,\alpha}} \otimes c^{\psi_{\alpha^{-1}\beta\gamma,\alpha}\psi'_{\alpha^{-1}\beta\gamma,\alpha}}) \\
= \ & m \cdot a_{\psi_{\alpha^{-1}\beta\gamma,\alpha}} 1_{A_{\beta\gamma}\psi_{\gamma,\beta}\psi'_{\alpha^{-1}\beta,\alpha}} \otimes c^{\psi_{\alpha^{-1}\beta\gamma,\alpha}}{}_{(1,\alpha^{-1}\beta)}{}^{\psi'_{\alpha^{-1}\beta,\alpha}} \otimes c^{\psi_{\alpha^{-1}\beta\gamma,\alpha}}{}_{(2,\gamma)}{}^{\psi_{\gamma,\beta}} \\
\overset{(E3.1)}{=} \ & m \cdot a_{\psi_{\alpha^{-1}\beta\gamma,\alpha}} 1_{A_\beta\psi''_{\alpha^{-1}\beta,\alpha}} 1_{A_{\beta\gamma}\psi_{\gamma,\beta}\psi'_{\alpha^{-1}\beta,\alpha}} \otimes c^{\psi_{\alpha^{-1}\beta\gamma,\alpha}}{}_{(1,\alpha^{-1}\beta)}{}^{\psi''_{\alpha^{-1}\beta,\alpha}\psi'_{\alpha^{-1}\beta,\alpha}} \otimes c^{\psi_{\alpha^{-1}\beta\gamma,\alpha}}{}_{(2,\gamma)}{}^{\psi_{\gamma,\beta}} \\
\overset{(E3.2)}{=} \ & m \cdot a_{\psi'_{\gamma,\beta}\psi_{\alpha^{-1}\beta,\alpha}} 1_{A_{\beta\gamma}\psi_{\gamma,\beta}\psi'_{\alpha^{-1}\beta,\alpha}} \otimes c_{(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}\psi'_{\alpha^{-1}\beta,\alpha}} \otimes c_{(2,\gamma)}{}^{\psi'_{\gamma,\beta}\psi_{\gamma,\beta}} \\
\overset{(E3.1)}{=} \ & m \cdot a_{\psi_{\gamma,\beta}\psi_{\alpha^{-1}\beta,\alpha}} \otimes c_{(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}} \otimes c_{(2,\gamma)}{}^{\psi_{\gamma,\beta}}
\end{aligned}$$

and

$$\begin{aligned}
& (m \cdot 1_{A_{\beta\gamma}\psi_{\alpha^{-1}\beta\gamma,\alpha}} \otimes c^{\psi_{\alpha^{-1}\beta\gamma,\alpha}})_{[0,\alpha^{-1}\beta]} \cdot a_{\psi_{\gamma,\beta}} \otimes ((m \cdot 1_{\beta\gamma\psi_{\alpha^{-1}\beta\gamma,\alpha}} \otimes c^{\psi_{\alpha^{-1}\beta\gamma,\alpha}})_{[1,\gamma]})^{\psi_{\gamma,\beta}} \\
= \ & (m \cdot 1_{A_{\beta\gamma}\psi'_{\gamma,\beta}\psi_{\alpha^{-1}\beta,\alpha}} \otimes c_{(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}}) \cdot a_{\psi_{\gamma,\beta}} \otimes c_{(2,\gamma)}{}^{\psi'_{\gamma,\beta}\psi_{\gamma,\beta}} \\
\overset{(E3.1)}{=} \ & (m \cdot 1_{A_{\beta\gamma}\psi'_{\gamma,\beta}\psi_{\alpha^{-1}\beta,\alpha}} 1_{A_\beta\psi'_{\alpha^{-1}\beta,\alpha}} \otimes c_{(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}\psi'_{\alpha^{-1}\beta,\alpha}}) \cdot a_{\psi_{\gamma,\beta}} \otimes c_{(2,\gamma)}{}^{\psi'_{\gamma,\beta}\psi_{\gamma,\beta}} \\
= \ & m \cdot 1_{A_{\beta\gamma}\psi'_{\gamma,\beta}\psi_{\alpha^{-1}\beta,\alpha}} a_{\psi_{\gamma,\beta}\psi'_{\alpha^{-1}\beta,\alpha}} \otimes c_{(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}\psi'_{\alpha^{-1}\beta,\alpha}} \otimes c_{(2,\gamma)}{}^{\psi'_{\gamma,\beta}\psi_{\gamma,\beta}} \\
\overset{(E3.1)}{=} \ & m \cdot a_{\psi_{\gamma,\beta}\psi_{\alpha^{-1}\beta,\alpha}} \otimes c_{(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}} \otimes c_{(2,\gamma)}{}^{\psi_{\gamma,\beta}}.
\end{aligned}$$

Hence Condition(PM2) is proved. For all $\beta, \gamma, \zeta \in \pi$, $m \in M_\alpha$ and $c \in C_{\alpha^{-1}\beta\gamma\zeta}$, one has

$$\begin{aligned}
& (m \cdot 1_{A_{\beta\gamma\zeta}\psi_{\alpha^{-1}\beta\gamma\zeta,\alpha}} \otimes c^{\psi_{\alpha^{-1}\beta\gamma\zeta,\alpha}})_{[0,\alpha^{-1}\beta\gamma][0,\alpha^{-1}\beta]} \\
& \qquad \otimes (m \cdot 1_{A_{\beta\gamma\zeta}\psi_{\alpha^{-1}\beta\gamma\zeta,\alpha}} \otimes c^{\psi_{\alpha^{-1}\beta\gamma\zeta,\alpha}})_{[0,\alpha^{-1}\beta\gamma][1,\gamma]} \otimes (m \cdot 1_{A_{\beta\gamma\zeta}\psi_{\alpha^{-1}\beta\gamma\zeta,\alpha}} \otimes c^{\psi_{\alpha^{-1}\beta\gamma\zeta,\alpha}})_{[1,\zeta]} \\
= \ & (m \cdot 1_{A_{\beta\gamma\zeta}\psi_{\zeta,\beta\gamma}\psi_{\alpha^{-1}\beta\gamma,\alpha}} \otimes c_{(1,\alpha^{-1}\beta\gamma)}{}^{\psi_{\alpha^{-1}\beta\gamma,\alpha}})_{[0,\alpha^{-1}\beta]} \otimes (m \cdot 1_{A_{\beta\gamma\zeta}\psi_{\zeta,\beta\gamma}\psi_{\alpha^{-1}\beta\gamma,\alpha}} \otimes c_{(1,\alpha^{-1}\beta\gamma)}{}^{\psi_{\alpha^{-1}\beta\gamma,\alpha}})_{[1,\gamma]} \otimes c_{(2,\zeta)}{}^{\psi_{\zeta,\beta\gamma}} \\
\overset{(PM2)}{=} \ & ((m \cdot 1_{A_{\beta\gamma}\psi_{\alpha^{-1}\beta\gamma,\alpha}} \otimes c_{(1,\alpha^{-1}\beta\gamma)}{}^{\psi_{\alpha^{-1}\beta\gamma,\alpha}})_{[0,\alpha^{-1}\beta]}) \cdot 1_{A_{\beta\gamma\zeta}\psi_{\zeta,\beta\gamma}\psi'_{\gamma,\beta}} \otimes ((m \cdot 1_{A_{\beta\gamma}\psi_{\alpha^{-1}\beta\gamma,\alpha}} \otimes c_{(1,\alpha^{-1}\beta\gamma)}{}^{\psi_{\alpha^{-1}\beta\gamma,\alpha}})_{[1,\gamma]})^{\psi'_{\gamma,\beta}} \otimes c_{(2,\zeta)}{}^{\psi_{\zeta,\beta\gamma}} \\
= \ & (m \cdot 1_{A_{\beta\gamma}\psi_{\gamma,\beta}\psi_{\alpha^{-1}\beta,\alpha}} \otimes c_{(1,\alpha^{-1}\beta\gamma)(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}}) \cdot 1_{A_{\beta\gamma\zeta}\psi_{\zeta,\beta\gamma}\psi'_{\gamma,\beta}} \otimes c_{(1,\alpha^{-1}\beta\gamma)(2,\gamma)}{}^{\psi_{\gamma,\beta}\psi'_{\gamma,\beta}} \otimes c_{(2,\zeta)}{}^{\psi_{\zeta,\beta\gamma}} \\
= \ & m \cdot 1_{A_{\beta\gamma\zeta}\psi_{\zeta,\beta\gamma}\psi_{\gamma,\beta}\psi_{\alpha^{-1}\beta,\alpha}} \otimes c_{(1,\alpha^{-1}\beta\gamma)(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}} \otimes c_{(1,\alpha^{-1}\beta\gamma)(2,\gamma)}{}^{\psi_{\gamma,\beta}} \otimes c_{(2,\zeta)}{}^{\psi_{\zeta,\beta\gamma}}
\end{aligned}$$

and

$$\begin{aligned}
&(m\cdot 1_{A_{\beta\gamma\zeta}\psi_{\alpha^{-1}\beta\gamma\zeta,\alpha}}\otimes c^{\psi_{\alpha^{-1}\beta\gamma\zeta,\alpha}})_{[0,\alpha^{-1}\beta]}1_{A_{\beta\gamma\zeta}\psi_{\zeta,\beta\gamma}\psi_{\gamma,\beta}}\\
&\qquad\otimes(m\cdot 1_{A_{\beta\gamma\zeta}\psi_{\alpha^{-1}\beta\gamma\zeta,\alpha}}\otimes c^{\psi_{\alpha^{-1}\beta\gamma\zeta,\alpha}})_{[1,\gamma\zeta](1,\gamma)}{}^{\psi_{\gamma,\beta}}\otimes(m\cdot 1_{A_{\beta\gamma\zeta}\psi_{\alpha^{-1}\beta\gamma\zeta,\alpha}}\otimes c^{\psi_{\alpha^{-1}\beta\gamma\zeta,\alpha}})_{[1,\gamma\zeta](2,\zeta)}{}^{\psi_{\zeta,\beta\gamma}}\\
=\;&(m\cdot 1_{A_\beta\psi_{\alpha^{-1}\beta,\alpha}}\otimes c_{(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}})\cdot 1_{A_{\beta\gamma\zeta}\psi_{\gamma\zeta,\beta}}1_{A_{\beta\gamma\zeta}\psi_{\zeta,\beta\gamma}\psi_{\gamma,\beta}}\otimes c_{(2,\gamma\zeta)}{}^{\psi_{\gamma\zeta,\beta}}{}_{(1,\gamma)}{}^{\psi_{\gamma,\beta}}\otimes c_{(2,\gamma\zeta)}{}^{\psi_{\gamma\zeta,\beta}}{}_{(2,\zeta)}{}^{\psi_{\zeta,\beta\gamma}}\\
\overset{(E3.1)}{=}\;&(m\cdot 1_{A_\beta\psi_{\alpha^{-1}\beta,\alpha}}\otimes c_{(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}})\cdot 1_{A_{\beta\gamma\zeta}\psi_{\gamma\zeta,\beta}}1_{A_{\beta\gamma}\psi'_{\gamma,\beta}}1_{A_{\beta\gamma\zeta}\psi_{\zeta,\beta\gamma}\psi_{\gamma,\beta}}\otimes c_{(2,\gamma\zeta)}{}^{\psi_{\gamma\zeta,\beta}}{}_{(1,\gamma)}{}^{\psi'_{\gamma,\beta}\psi_{\gamma,\beta}}\otimes c_{(2,\gamma\zeta)}{}^{\psi_{\gamma\zeta,\beta}}{}_{(2,\zeta)}{}^{\psi_{\zeta,\beta\gamma}}\\
\overset{(E3.2)}{=}\;&(m\cdot 1_{A_\beta\psi_{\alpha^{-1}\beta,\alpha}}\otimes c_{(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}})\cdot 1_{A_{\beta\gamma\zeta}\psi'_{\zeta,\beta\gamma}\psi_{\gamma,\beta}}1_{A_{\beta\gamma\zeta}\psi_{\zeta,\beta\gamma}\psi'_{\gamma,\beta}}\otimes c_{(2,\gamma\zeta)(1,\gamma)}{}^{\psi_{\gamma,\beta}\,\psi'_{\gamma,\beta}}\otimes c_{(2,\gamma\zeta)(2,\zeta)}{}^{\psi'_{\zeta,\beta\gamma}\psi_{\zeta,\beta\gamma}}\\
\overset{(E3.1)}{=}\;&(m\cdot 1_{A_\beta\psi_{\alpha^{-1}\beta,\alpha}}\otimes c_{(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}})\cdot 1_{A_{\beta\gamma\zeta}\psi_{\zeta,\beta\gamma}\psi_{\gamma,\beta}}\otimes c_{(2,\gamma\zeta)(1,\gamma)}{}^{\psi_{\gamma,\beta}}\otimes c_{(2,\gamma\zeta)(2,\zeta)}{}^{\psi_{\zeta,\beta\gamma}}\\
=\;&(m\cdot 1_{A_{\beta\gamma\zeta}\psi_{\zeta,\beta\gamma}\psi_{\gamma,\beta}\psi_{\alpha^{-1}\beta,\alpha}}\otimes c_{(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}})\otimes c_{(2,\gamma\zeta)(1,\gamma)}{}^{\psi_{\gamma,\beta}}\otimes c_{(2,\gamma\zeta)(2,\zeta)}{}^{\psi_{\zeta,\beta\gamma}}\\
=\;&m\cdot 1_{A_{\beta\gamma\zeta}\psi_{\zeta,\beta\gamma}\psi_{\gamma,\beta}\psi_{\alpha^{-1}\beta,\alpha}}\otimes c_{(1,\alpha^{-1}\beta\gamma)(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}}c_{(1,\alpha^{-1}\beta\gamma)(2,\gamma)}{}^{\psi_{\gamma,\beta}}\otimes c_{(2,\zeta)}{}^{\psi_{\zeta,\beta\gamma}}.
\end{aligned}$$

So Condition (PM4) is proved. For A_α-linear map $\mu : M \to M'$, we put

$$G^{(\alpha)}(\mu)=\{G^{(\alpha)}(\mu)_\beta=\mu\otimes id_{C_{\alpha^{-1}\beta}}:\underline{M\otimes C_{\alpha^{-1}\beta}}\to\underline{M'\otimes C_{\alpha^{-1}\beta}}\}_{\beta\in\pi},$$

Standard computations show that $G^{(\alpha)}(\mu)$ is right A_β-linear and partial π-C-colinear. Let us describe the unit η and the counit δ of the adjunction. The unit is described by the partial coaction: for $M\in\mathcal{U}_A^{\pi-C}(\psi)$, we define $\eta^M=\{\eta^M_\beta\}_{\beta\in\pi}:M\to\{\underline{M_\alpha\otimes C_{\alpha^{-1}\beta}}\}_{\beta\in\pi}$, where $\eta^M_\beta:M_\beta\to\underline{M_\alpha\otimes C_{\alpha^{-1}\beta}}$ is defined as the composition of the maps

$$\begin{aligned}
M_\beta\overset{\rho_{\alpha,\alpha^{-1}\beta}}{\longrightarrow}M_\alpha\otimes C_{\alpha^{-1}\beta}\quad&\overset{id_{M_\alpha}\otimes id_{C_{\alpha^{-1}\beta}}\otimes\eta_\beta}{\longrightarrow}\quad M_\alpha\otimes C_{\alpha^{-1}\beta}\otimes A_\beta\\
&\overset{id_{M_\alpha}\otimes\psi_{\alpha^{-1}\beta,\alpha}}{\longrightarrow}\quad M_\alpha\otimes A_\alpha\otimes C_{\alpha^{-1}\beta}\\
&\overset{\phi_\alpha\otimes id_{C_{\alpha^{-1}\beta}}}{\longrightarrow}\quad M_\alpha\otimes C_{\alpha^{-1}\beta}
\end{aligned}$$

i.e., for all $m\in M_\beta$,

$$\eta^M_\beta(m)=m_{[0,\alpha]}\cdot 1_{A_\beta\psi_{\alpha^{-1}\beta,\alpha}}\otimes(m_{[1,\alpha^{-1}\beta]})^{\psi_{\alpha^{-1}\beta,\alpha}}.$$

We can check that $\eta_M\in\mathcal{U}_A^{\pi-C}(\psi)$. For any $N\in\mathcal{M}_{A_\alpha}$, we define $\delta_N:\underline{N\otimes C_e}\to N$, for all $n\in N$ and $c\in C_e$,

$$\delta_N(n\cdot 1_{A_\alpha\psi_{e,\alpha}}\otimes c^{\psi_{e,\alpha}})=\varepsilon(c)n,$$

δ_N is A_α-linear. We can check that η and δ defined above are all natural transformations and they satisfy

$$G^{(\alpha)}(\delta_N)\circ\eta_{G^{(\alpha)}(N)}=I_{G^{(\alpha)}(N)},$$

$$\delta_{F^{(\alpha)}(M)}\circ F^{(\alpha)}(\eta^M)=I_{F^{(\alpha)}(M)},$$

for all $M\in\mathcal{U}_A^{\pi-C}(\psi)$ and $N\in\mathcal{M}_{A_\alpha}$. $\square$

Theorem 3.4. *Fix an $\alpha\in\pi$. Consider a partial π-entwining structure (A,C,ψ) and a partial entwining structure (A',C',ψ') and suppose that we have two linear maps $\mu^\alpha:A_\alpha\to A'$ and $\nu:C_e\to C'$ which are respectively algebra and coalgebra map satisfying, for all $a\in A_\alpha$ and $c\in C_e$,*

$$\mu^\alpha(a_{\psi_{e,\alpha}})1_{A'\psi'}\otimes\nu(c^{\psi_{e,\alpha}})^{\psi'}=\mu^\alpha(a)_{\psi'}\otimes\nu(c)^{\psi'},$$

Then we have a functor

$$F^{(\alpha)}:\mathcal{U}_A^{\pi-C}(\psi)\to\mathcal{U}_{A'}^{C'}(\psi')$$

defined as follows: For $M = \{M_\alpha\}_{\alpha\in\pi} \in \mathcal{U}_A^{\pi-C}(\psi)$,

$$F^{(\alpha)}(M) = M_\alpha \otimes_{A_\alpha} A' = M',$$

where A' *is a left* A*-module via* μ^α *and with structure maps defined by*

$$(m \otimes_{A_\alpha} a')b' = m \otimes_{A_\alpha} a'b',$$

$$\rho^\alpha_{M'}(m \otimes_{A_\alpha} a') = m_{[0,\alpha]} \otimes_{A_\alpha} a'_{\psi'} \otimes \nu(m_{[1,e]})^{\psi'},$$

for all $m \in M_\alpha$, $a', b' \in A$. $F^{(\alpha)}$ *is called the induction functor associated to* α *from* $\mathcal{U}_A^{\pi-C}(\psi)$ *to* $\mathcal{U}_{A'}^{C'}(\psi')$.

Proof. Let us show that $M_\alpha \otimes_{A_\alpha} A'$ is an object of $\mathcal{U}_{A'}^{C'}(\psi')$. Here, we only check (PM3). In fact, for all $m \in M_\alpha$ and $a' \in A'$, we have

$$\begin{aligned}
& m_{[0,\alpha][0,\alpha]} \otimes_{A_\alpha} a'_{\psi'\Psi'} \otimes \nu(m_{[0,\alpha][1,e]})^{\Psi'} \otimes \nu(m_{[1,e]})^{\psi'} \\
&= m_{[0,\alpha]} \cdot 1_{A_\alpha \psi_{e,\alpha}\psi'_{e,\alpha}} \otimes_{A_\alpha} a'_{\psi'\Psi'} \otimes \nu(m_{[0,e](1,e)}{}^{\psi'_{e,\alpha}})^{\Psi'} \otimes \nu(m_{[1,e](2,e)}{}^{\psi_{e,\alpha}})^{\psi'} \\
&= m_{[0,\alpha]} \otimes_{A_\alpha} \mu^\alpha(1_{A_\alpha \psi_{e,\alpha}\psi'_{e,\alpha}})a'_{\psi'\Psi'} \otimes \nu(m_{[0,e](1,e)}{}^{\psi'_{e,\alpha}})^{\Psi'} \otimes \nu(m_{[1,e](2,e)}{}^{\psi_{e,\alpha}})^{\psi'} \\
&= m_{[0,\alpha]} \otimes_{A_\alpha} (\mu^\alpha(1_{A_\alpha \psi_{e,\alpha}})a'_{\psi'})_{\Psi'} \otimes \nu(m_{[0,e](1,e)})^{\Psi'} \otimes \nu(m_{[1,e](2,e)}{}^{\psi_{e,\alpha}})^{\psi'} \\
&= m_{[0,\alpha]} \otimes_{A_\alpha} a'_{\psi'\Psi'} \otimes \nu(m_{[0,e](1,e)})^{\Psi'} \otimes \nu(m_{[1,e](2,e)})^{\psi'} \\
&= m_{[0,\alpha]} \otimes_{A_\alpha} a'_{\psi'\Psi'} \otimes \nu(m_{[0,e]})_{(1)}{}^{\Psi'} \otimes \nu(m_{[1,e]})_{(2)}{}^{\psi'}
\end{aligned}$$

and

$$\begin{aligned}
& (m_{[0,\alpha]} \otimes_{A_\alpha} a'_{\psi'}) \cdot 1_{A'\psi''\Psi'} \otimes (\nu(m_{[1,e]})^{\psi'})_1{}^{\Psi'} \otimes (\nu(m_{[1,e]})^{\psi'})_{(2)}{}^{\psi''} \\
&= m_{[0,\alpha]} \otimes_{A_\alpha} a'_{\psi'} 1_{A'\psi''\Psi'} \otimes (\nu(m_{[1,e]})^{\psi'})_1{}^{\Psi'} \otimes (\nu(m_{[1,e]})^{\psi'})_{(2)}{}^{\psi''} \\
&= m_{[0,\alpha]} \otimes_{A_\alpha} a'_{\psi'} 1_{A'\Psi''} 1_{A'\psi''\Psi'} \otimes (\nu(m_{[1,e]})^{\psi'})_1{}^{\Psi''\Psi'} \otimes (\nu(m_{[1,e]})^{\psi'})_{(2)}{}^{\psi''} \\
&= m_{[0,\alpha]} \otimes_{A_\alpha} a'_{\psi'\Psi''} 1_{A'\psi''\Psi'} \otimes \nu(m_{[1,e]})_1{}^{\Psi''\Psi'} \otimes \nu(m_{[1,e]})_{(2)}{}^{\psi'\psi''} \\
&= m_{[0,\alpha]} \otimes_{A_\alpha} a'_{\psi'\Psi''} \otimes \nu(m_{[1,e]})_1{}^{\Psi''} \otimes \nu(m_{[1,e]})_{(2)}{}^{\psi'}.
\end{aligned}$$

By comparing the equations above, we can get the desired (PM3). □

Theorem 3.5. *Fix an* $\alpha \in \pi$. *Under the assumptions of Theorem 3.4, we have a functor* $G^{(\alpha)} : \mathcal{U}_{A'}^{C'}(\psi') \to \mathcal{U}_A^{\pi-C}(\psi)$ *which is right adjoint to* $F^{(\alpha)}$. $G^{(\alpha)}$ *is defined by* $G^{(\alpha)}(M') = \{(\overline{M'\square_{C'}C})_\beta\}_{\beta\in\pi}$, *its component is given by*

$$(\overline{M'\square_{C'}C})_\beta = < m' \cdot \mu^\alpha(1_{A_\beta \psi_{\alpha^{-1}\beta,\alpha}}) \otimes c^{\psi_{\alpha^{-1}\beta,\alpha}} >,$$

where $m' \otimes c \in M' \otimes C_{\alpha^{-1}\beta}$ *satisfies the following condition:*

$$m'_{[0]} \cdot \mu^\alpha(1_{A_\beta \psi_{\alpha^{-1}\beta,\alpha}})_{\psi'} \otimes m'_{[1]}{}^{\psi'} \otimes c^{\psi_{\alpha^{-1}\beta,\alpha}} = m' \cdot \mu^\alpha(1_{A_\beta \psi_{\alpha^{-1}\beta,\alpha}\psi_{e,\alpha}}) \otimes \nu(c_{(1,e)}{}^{\psi_{e,\alpha}}) \otimes c_{(2,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}}, \tag{1}$$

for all $M' \in \mathcal{U}_{A'}^{C'}(\psi')$, *and with structure maps*

$$\rho^{G^{(\alpha)}}_{\beta,\gamma}(m' \cdot \mu^\alpha(1_{A_\beta \psi_{\alpha^{-1}\beta\gamma,\alpha}}) \otimes c^{\psi_{\alpha^{-1}\beta\gamma,\alpha}}) = m' \cdot \mu^\alpha(1_{A_{\beta\gamma} \psi_{\gamma,\beta}\psi_{\alpha^{-1}\beta,\alpha}}) \otimes c_{(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}} \otimes c_{(2,\gamma)}{}^{\psi_{\gamma,\beta}}, \tag{2}$$

$$(m' \cdot \mu^\alpha(1_{A_\beta \psi_{\alpha^{-1}\beta,\alpha}}) \otimes c^{\psi_{\alpha^{-1}\beta,\alpha}}) \cdot b = m' \cdot \mu^\alpha(b_{\psi_{\alpha^{-1}\beta,\alpha}}) \otimes c^{\psi_{\alpha^{-1}\beta,\alpha}}$$

Proof. In order to prove that $G(M')$ is a right A-module, i.e., each $(\overline{M'\square_{C'}C})_\beta$ is a right A_β-module, we need to show that

$$m'\cdot\mu^\alpha(b_{\psi_{\alpha^{-1}\beta,\alpha}})\otimes c^{\psi_{\alpha^{-1}\beta,\alpha}}\in(\overline{M'\square_{C'}C})_\beta,$$

for all $m'\in M'$, $b\in A_\beta$ and $c\in C_{\alpha^{-1}\beta}$. Indeed,

$$\begin{aligned}
&(m'\cdot\mu^\alpha(b_{\Psi_{\alpha^{-1}\beta,\alpha}}))_{[0]}\cdot\mu^\alpha(1_{A_\beta\psi_{\alpha^{-1}\beta,\alpha}})_{\psi'}\otimes(m'\cdot\mu^\alpha(b_{\psi_{\alpha^{-1}\beta,\alpha}}))_{[1]}{}^{\psi'}\otimes c^{\Psi_{\alpha^{-1}\beta,\alpha}\psi_{\alpha^{-1}\beta,\alpha}}\\
&=m'_{[0]}\cdot\mu^\alpha(b_{\Psi_{\alpha^{-1}\beta,\alpha}})_{\Psi'}\mu^\alpha(1_{A_\beta\psi_{\alpha^{-1}\beta,\alpha}})_{\psi'}\otimes m'_{[1]}{}^{\Psi'\psi'}\otimes c^{\Psi_{\alpha^{-1}\beta,\alpha}\psi_{\alpha^{-1}\beta,\alpha}}\\
&=m'_{[0]}\cdot\mu^\alpha(b_{\Psi_{\alpha^{-1}\beta,\alpha}}1_{A_\beta\psi_{\alpha^{-1}\beta,\alpha}})_{\Psi'}\otimes m'_{[1]}{}^{\Psi'}\otimes c^{\Psi_{\alpha^{-1}\beta,\alpha}\psi_{\alpha^{-1}\beta,\alpha}}\\
&=m'_{[0]}\cdot\mu^\alpha(b_{\Psi_{\alpha^{-1}\beta,\alpha}})_{\Psi'}\otimes m'_{[1]}{}^{\Psi'}\otimes c^{\Psi_{\alpha^{-1}\beta,\alpha}}\\
&=m'_{[0]}\cdot\mu^\alpha(1_{A_\beta\psi_{\alpha^{-1}\beta,\alpha}})_{\psi'}\mu^\alpha(b_{\Psi_{\alpha^{-1}\beta,\alpha}})_{\Psi'}\otimes m'_{[1]}{}^{\psi'\Psi'}\otimes c^{\psi_{\alpha^{-1}\beta,\alpha}\Psi_{\alpha^{-1}\beta,\alpha}}\\
&=m'\cdot\mu^\alpha(1_{A_\beta\psi_{\alpha^{-1}\beta,\alpha}\psi_{e,\alpha}})\mu^\alpha(b_{\Psi_{\alpha^{-1}\beta,\alpha}})_{\Psi'}\otimes\nu(c_{(1,e)}{}^{\psi_{e,\alpha}})^{\Psi'}\otimes c_{(2,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}\Psi_{\alpha^{-1}\beta,\alpha}}\\
&=m'\cdot\mu^\alpha(1_{A_\beta\psi_{\alpha^{-1}\beta,\alpha}\psi_{e,\alpha}})\mu^\alpha(b_{\Psi_{\alpha^{-1}\beta,\alpha}\Psi'_{e,\alpha}})\otimes\nu(c_{(1,e)}{}^{\psi_{e,\alpha}\Psi'_{e,\alpha}})\otimes c_{(2,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}\Psi_{\alpha^{-1}\beta,\alpha}}\\
&=m'\cdot\mu^\alpha(b_{\Psi_{\alpha^{-1}\beta,\alpha}\Psi'_{e,\alpha}})\otimes\nu(c_{(1,e)}{}^{\Psi'_{e,\alpha}})\otimes c_{(2,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}}
\end{aligned}$$

This is exactly what we have to show. Let us finally show that $G^{(\alpha)}$ is a right adjoint to $F^{(\alpha)}$. Take $M\in\mathcal{U}_A^{\pi-C}(\psi)$, we define a family of linear maps

$$\eta^M=\{\eta^M_\beta\}_{\beta\in\pi}:M\to G^{(\alpha)}F^{(\alpha)}(M)=\overline{(M_\alpha\otimes_{A_\alpha}A')\square_C C'},$$

where

$$\eta^M_\beta:M_\beta\to(\overline{(M_\alpha\otimes_{A_\alpha}A')\square_{C'}C})_\beta$$

$$\eta^M_\beta(m)=(m_{[0,\alpha]}\otimes_{A_\alpha}\mu^\alpha(1_{A_\beta\psi_{\alpha^{-1}\beta,\alpha}}))\otimes m_{[1,\alpha^{-1}\beta]}{}^{\psi_{\alpha^{-1}\beta,\alpha}}$$

We can check that η^M is a homomorphism in $\mathcal{U}_A^{\pi-C}(\psi)$ in a straightforward way. For any $M'\in\mathcal{U}_{A'}^{C'}(\psi')$, we define $\delta:F^{(\alpha)}G^{(\alpha)}\to\iota$ (where ι is the identity functor) as follows:

$$\delta^{M'}:(\overline{M'\square_{C'}C_e})\otimes_{A_\alpha}A'\to M',\ \delta^{M'}((m'\otimes c)\otimes_{A_\alpha}a')=m'\cdot a'\varepsilon(c).$$

This ends the proof. □

4. Partial Group Coactions

Let H be a Hopf π-coalgebra and $A=\{A_\alpha,m_\alpha,1_{A_\alpha}\}_{\alpha\in\pi}$ a family of algebras. Consider a family of maps $\rho^A=\{\rho^A_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$, where

$$\rho^A_{\alpha,\beta}:A_{\alpha\beta}\to A_\alpha\otimes H_\beta,\rho^A_{\alpha,\beta}(a)=a_{[0,\alpha]}\otimes a_{[1,\beta]}.$$

To ρ^A, we associate a family of maps $\psi=\{\psi_{\alpha,\beta}:H_\alpha\otimes A_{\beta\alpha}\to A_\beta\otimes H_\alpha\}_{\alpha,\beta\in\pi}$, where

$$\psi_{\alpha,\beta}(h\otimes a)=a_{[0,\beta]}\otimes ha_{[1,\alpha]}=a_{\psi_{\alpha,\beta}}\otimes h^{\psi_{\alpha,\beta}}.$$

Proposition 4.1. *(A,H,ψ) is a partial entwining structure if and only if*

$$a_{[0,\gamma\alpha][0,\gamma]}\otimes a_{[0,\gamma\alpha][1,\alpha]}\otimes a_{[1,\beta]}=a_{[0,\gamma]}1_{A_{\gamma\alpha}[0,\gamma]}\otimes a_{[1,\alpha\beta](1,\alpha)}1_{A_{\gamma\alpha}[1,\alpha]}\otimes a_{[1,\alpha\beta](2,\beta)},\tag{E4.1}$$

$$(a'b)_{[0,\beta]}\otimes(a'b)_{[1,\alpha]}=a'_{[0,\beta]}b_{[0,\beta]}\otimes a'_{[1,\alpha]}b_{[1,\alpha]},\tag{E4.2}$$

$$\varepsilon(b'_{[1,e]})b'_{[0,\alpha]}=b',\tag{E4.3}$$

for any $\alpha,\beta,\gamma\in\pi$, $a\in A_{\gamma\alpha\beta}$, $a',b\in A_{\beta\alpha}$ and $b'\in A_\alpha$.

Proof. Straightforward. □

Definition 4.2. *Let H be a Hopf π-coalgebra and $A = \{A_\alpha, m_\alpha, 1_{A_\alpha}\}_{\alpha\in\pi}$ a family of algebras. A is called a right partial π-H-comodule algebra, if there exists a family of k-linear maps $\rho^A = \{\rho^A_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$ such that (E4.1)-(E4.3) hold.*

Let $C = (\{C_\alpha\}_{\alpha\in\pi}, \Delta = \{\Delta_{\alpha,\beta}\}, \varepsilon_C)$ be a π-coalgebra. Consider a family of k-linear maps

$$\kappa = \{\kappa_\alpha : C_\alpha \otimes H_\alpha \to C_\alpha, \kappa_\alpha(c \otimes h) = c \cdot h\}_{\alpha\in\pi},$$

and define $\psi = \{\psi_{\alpha,\beta} : C_\alpha \otimes H_{\beta\alpha} \to H_\beta \otimes C_\alpha\}_{\alpha,\beta\in\pi}$ by the formula

$$\psi_{\alpha,\beta}(c \otimes h) = h_{(1,\beta)} \otimes c \cdot h_{(2,\alpha)}.$$

Proposition 4.3. (H, C, ψ) *is a partial entwining structure if and only if*

(1) For all $\alpha, \beta \in \pi$, $h, h' \in H_\alpha$ and $c \in C_\alpha$,

$$c \cdot (hh') = (c \cdot h) \cdot h', \tag{F4.1}$$

(2) For all $\alpha, \beta \in \pi$, $h \in H_{\alpha\beta}$ and $d \in C_{\alpha\beta}$,

$$(d \cdot h)_{(1,\alpha)} \cdot 1_\alpha \otimes (d \cdot h)_{(2,\beta)} = d_{(1,\alpha)} \cdot h_{(2,\alpha)} \otimes d_{(2,\beta)} \cdot h_{(2,\beta)} \tag{F4.2}$$

(3) For all $\alpha \in \pi$ and $c \in C_e, h \in H_e$,

$$\varepsilon_C(d \cdot h) = \varepsilon_C(d)\varepsilon(h). \tag{F4.3}$$

Definition 4.4. *Let H be a Hopf π-coalgebra and $C = (\{C_\alpha\}_{\alpha\in\pi}, \Delta = \{\Delta_{\alpha,\beta}\}, \varepsilon_C)$ a π-coalgebra. We call C a right partial π-H-module coalgebra, if there exists a family of k-linear maps*

$$\kappa = \{\kappa_\alpha : C_\alpha \otimes H_\alpha \to C_\alpha, \kappa_\alpha(c \otimes h) = c \cdot h\}_{\alpha\in\pi},$$

such that (F4.1)-(F4.3) hold.

We are now able to define partial Doi-Hopf π-data.

Proposition 4.5. *Let H be a semi-Hopf π-coalgebra, A a right partial π-H-comodule algebra, and C a right partial π-H-module coalgebra. Consider the family of k-linear maps $\psi = \{\psi_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$, where*

$$\psi_{\alpha,\beta} : C_\alpha \otimes A_{\beta\alpha} \to A_\beta \otimes C_\alpha, \psi_{\alpha,\beta}(c \otimes a) = a_{[0,\beta]} \otimes c \cdot a_{[1,\alpha]}.$$

Then (A, C, ψ) is a partial entwining structure. We will say the (H, A, C) is a (right-right) partial Doi-Hopf π-structure.

Proof. We have to show that the $\psi = \{\psi_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$ satisfies the conditions (E3.1)-(E3.3). Notice that (E3.3) is easily implied from (F4.3). For all $\alpha, \beta \in \pi$, $a, b \in A_{\beta\alpha}$ and $c \in C_\alpha$, we compute that

$$\begin{aligned} (ab)_{\psi_{\alpha,\beta}} \otimes c^{\psi_{\alpha,\beta}} &= (ab)_{[0,\beta]} \otimes c \cdot (ab)_{[1,\alpha]} \\ &\overset{(E4.2),(F4.1)}{=} a_{[0,\beta]}b_{[0,\beta]} \otimes (c \cdot a_{[1,\alpha]}) \cdot b_{[1,\alpha]} \\ &= a_{\psi_{\alpha,\beta}} b_{\psi'_{\alpha,\beta}} \otimes c^{\psi_{\alpha,\beta}\psi'_{\alpha,\beta}}. \end{aligned}$$

So (E3.1) is checked. it is left to show that (E3.2) holds. For all $\alpha, \beta, \gamma \in \pi$, $a \in A_{\gamma\alpha\beta}$ and $d \in C_{\alpha\beta}$, we have

$$\begin{aligned} & a_{\psi_{\alpha\beta,\gamma}} 1_{A_{\gamma\alpha}\psi_{\alpha,\gamma}} \otimes {d^{\psi_{\alpha\beta,\gamma}}}_{(1,\alpha)}{}^{\psi_{\alpha,\gamma}} \otimes {d^{\psi_{\alpha\beta,\gamma}}}_{(2,\beta)} \\ =\; & a_{[0,\gamma]}1_{A_{\gamma\alpha}[0,\gamma]} \otimes (d \cdot a_{[1,\alpha\beta]})_{(1,\alpha)} \cdot 1_{A_{\gamma\alpha}[1,\alpha]} \otimes (d \cdot a_{[1,\alpha\beta]})_{(2,\beta)} \\ \overset{(F4.1),(F4.2)}{=}\; & a_{[0,\gamma]}1_{A_{\gamma\alpha}[0,\gamma]} \otimes d_{(1,\alpha)} \cdot a_{[1,\alpha\beta](1,\alpha)}1_{A_{\gamma\alpha}[1,\alpha]} \otimes d_{(2,\beta)} \cdot a_{[1,\alpha\beta](2,\beta)} \\ \overset{(E4.1)}{=}\; & a_{[0,\gamma\alpha][0,\gamma]} \otimes d_{(1,\alpha)} \cdot a_{[0,\gamma\alpha][1,\alpha]} \otimes d_{(2,\beta)} \cdot a_{[1,\beta]} \\ =\; & a_{\psi_{\beta,\gamma\alpha}\psi_{\alpha,\gamma}} \otimes {d_{(1,\alpha)}}^{\psi_{\alpha,\gamma}} \otimes {d_{(2,\beta)}}^{\psi_{\beta,\gamma\alpha}}. \end{aligned}$$

This ends the proof. □

Definition 4.6. *Let (H, A, C) be a (right-right) partial Doi-Hopf π-structure. A partial Doi-Hopf π-module M is a family of k-vector spaces$\{M_\alpha\}_{\alpha\in\pi}$ together with a family of k-linear maps*

$$\phi = \{\phi_\alpha : M_\alpha \otimes A_\alpha \to M_\alpha, \phi_\alpha(m \otimes a) = m \cdot a\}_{\alpha\in\pi}$$

and a family of k-linear maps $\rho^M = \{\rho^M_{\alpha,\beta} : M_{\alpha\beta} \to M_\alpha \otimes C_\beta\}_{\alpha,\beta\in\pi}$ such that

(DM1) For any $\alpha \in \pi$, (M_α, ϕ_α) is a right A_α-module,

(DM2) For any $\alpha, \beta \in \pi$, $m \in M_{\alpha\beta}$ and $a \in A_{\alpha\beta}$,

$$\rho^M_{\alpha,\beta}(m \cdot a) = m_{[0,a]} \cdot a_{[0,\alpha]} \otimes m_{[1,\beta]} \cdot a_{[1,\beta]},$$

(DM3) For all $\alpha, \beta, \gamma \in \pi$, $m \in M_{\alpha\beta\gamma}$,

$$m_{[0,\alpha\beta][0,\alpha]} \otimes m_{[0,\alpha\beta][1,\beta]} \otimes m_{[1,\gamma]} = m_{[0,\alpha]} \cdot 1_{A_{\alpha\beta\gamma}[0,\alpha\beta][0,\alpha]} \otimes m_{[1,\beta\gamma](1,\beta)} \cdot 1_{A_{\alpha\beta\gamma}[0,\alpha\beta][1,\beta]} \otimes m_{[1,\beta\gamma](2,\gamma)} \cdot 1_{A_{\alpha\beta\gamma}[1,\gamma]},$$

(DM4) For all $m \in M_\alpha$, $\varepsilon(m_{[1,e]})m_{[0,\alpha]} = m$.

$\mathcal{U}^{\pi-C}_A$ will denote the category of all partial Doi-Hopf π-modules.

5. Separable Functors For The Category of the Partial π-Entwining Modules

In the section, for a fixed $\alpha \in \pi$, we shall give necessary and sufficient conditions for the functor $F^{(\alpha)}$ to be separable.

Definition 5.1. *Let $(A, C)_{\pi-\psi}$ be a partial π-entwining structure. For any $\alpha \in \pi$, a k-linear map*

$$\theta^{(\alpha)} = \{\theta^{(\alpha)}_\beta : C_{(\alpha^{-1}\beta)^{-1}} \otimes C_{\alpha^{-1}\beta} \to A_\beta\}_{\beta\in\pi}$$

is called a partial normalised integral, if θ satisfies the following conditions:

(1)For all $\beta \in \pi$, $b \in C_e$,

$$\theta^{(\alpha)}_\beta(b_{(1,(\alpha^{-1}\beta)^{-1})} \otimes b_{(2,\alpha^{-1}\beta)}) = 1_{A_\beta}\varepsilon(b), \tag{E5.1}$$

Eq.(E5.1) is equivalent to the following commutative diagram,

$$\begin{array}{ccc} C_e & \xrightarrow{\Delta_{(\alpha^{-1}\beta)^{-1},\alpha^{-1}\beta}} & C_{(\alpha^{-1}\beta)^{-1}} \otimes C_{\alpha^{-1}\beta} \\ \downarrow{\scriptstyle id_{C_e}\otimes\eta_\beta} & & \downarrow{\scriptstyle \theta^{(\alpha)}_\beta} \\ C_e \otimes A_\beta & \xrightarrow[\varepsilon\otimes id_{A_\beta}]{} & A_\beta \end{array}$$

(2) For all $\beta, \gamma \in \pi$, $c \in C_{\alpha^{-1}\beta\gamma}$ and $d \in C_{(\alpha^{-1}\beta)^{-1}}$,

$$d_{(1,\gamma)}{}^{\psi_{\gamma,\beta}\psi'_{\gamma,\beta}} \otimes 1_{A_\alpha\psi_{(\alpha^{-1}\beta\gamma)^{-1},\beta\gamma}\psi_{\gamma,\beta}}\theta^{(\alpha)}_{\beta\gamma}(d_{(2,(\alpha^{-1}\beta\gamma)^{-1})}{}^{\psi_{(\alpha^{-1}\beta\gamma)^{-1},\beta\gamma}} \otimes c)_{\psi'_{\gamma,\beta}} \tag{E5.2}$$

$$= c_{(2,\gamma)}{}^{\psi_{\gamma,\beta}} \otimes 1_{A_{\beta\gamma}\psi_{\gamma,\beta}\psi_{\alpha^{-1}\beta,\alpha}\psi_{(\alpha^{-1}\beta)^{-1},\beta}}\theta^{(\alpha)}_\beta(d_{(\alpha^{-1}\beta)^{-1}}{}^{\psi_{(\alpha^{-1}\beta)^{-1},\beta}} \otimes c_{(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}}),$$

Eq.(E5.2) means that the following diagram is commutative,

$$\begin{array}{ccc}
C_{(\alpha^{-1}\beta)^{-1}} \otimes C_{\alpha^{-1}\beta\gamma} & \xrightarrow{id\otimes\Delta_{\alpha^{-1}\beta,\gamma}} & C_{(\alpha^{-1}\beta)^{-1}}\otimes C_{\alpha^{-1}\beta}\otimes C_\gamma \\
\downarrow \Delta_{\gamma,\gamma^{-1}(\alpha^{-1}\beta)^{-1}}\otimes id & & \downarrow id\otimes id\otimes id\otimes\eta_{\beta\gamma} \\
C_\gamma\otimes C_{\gamma^{-1}(\alpha^{-1}\beta)^{-1}}\otimes C_{\alpha^{-1}\beta\gamma} & & C_{(\alpha^{-1}\beta)^{-1}}\otimes C_{\alpha^{-1}\beta}\otimes C_\gamma\otimes A_{\beta\gamma}\\
\downarrow id\otimes id\otimes\eta_\alpha\otimes id & & \downarrow id\otimes id\otimes\psi_{\gamma,\beta}\\
C_\gamma\otimes C_{\gamma^{-1}(\alpha^{-1}\beta)^{-1}}\otimes A_\alpha\otimes C_{\alpha^{-1}\beta\gamma} & & C_{(\alpha^{-1}\beta)^{-1}}\otimes C_{\alpha^{-1}\beta}\otimes A_\beta\otimes C_\gamma\\
\downarrow id\otimes\psi_{\gamma^{-1}(\alpha^{-1}\beta)^{-1},\beta\gamma}\otimes id & & \downarrow id\otimes\psi_{\alpha^{-1}\beta,\alpha}\otimes id\\
C_\gamma\otimes A_{\beta\gamma}\otimes C_{\gamma^{-1}(\alpha^{-1}\beta)^{-1}}\otimes C_{\alpha^{-1}\beta\gamma} & & C_{(\alpha^{-1}\beta)^{-1}}\otimes A_\alpha\otimes C_{\alpha^{-1}\beta}\otimes C_\gamma\\
\downarrow \psi_{\gamma,\beta}\otimes id\otimes id & & \downarrow \psi_{\beta^{-1}\alpha,\beta}\otimes id\otimes id\\
A_\beta\otimes C_\gamma\otimes C_{\gamma^{-1}(\alpha^{-1}\beta)^{-1}}\otimes C_{\alpha^{-1}\beta\gamma} & & A_\beta\otimes C_{(\alpha^{-1}\beta)^{-1}}\otimes C_{\alpha^{-1}\beta}\otimes C_\gamma\\
\downarrow id\otimes\theta^{(\alpha)}_{\beta\gamma} & & \\
A_\beta\otimes C_\gamma\otimes A_{\beta\gamma} & & \downarrow (m_\beta\otimes id)\circ(id\otimes\theta^{(\alpha)}_\beta\otimes id)\\
\downarrow id\otimes\psi_{\gamma,\beta} & & \\
A_\beta\otimes A_\beta\otimes C_\gamma & \xrightarrow{m_\beta\otimes id} & A_\beta\otimes C_\gamma
\end{array}$$

(3) For all $a \in A_\beta, b \in C_{\alpha^{-1}\beta}$ and $d \in C_{(\alpha^{-1}\beta)^{-1}}$.

$$(a_{\psi_{\alpha^{-1}\beta,\alpha}})_{\psi_{(\alpha^{-1}\beta)^{-1},\beta}}\theta^{(\alpha)}_\beta(d^{\psi_{(\alpha^{-1}\beta)^{-1},\beta}} \otimes b^{\psi_{\alpha^{-1}\beta,\alpha}}) = \theta^{(\alpha)}_\beta(d \otimes b)a, \tag{E5.3}$$

Eq.(E5.3) is equivalent to the following commutative diagram,

$$\begin{array}{ccc}
C_{(\alpha^{-1}\beta)^{-1}}\otimes C_{\alpha^{-1}\beta}\otimes A_\beta & \xrightarrow{\theta^{(\alpha)}_\beta\otimes id} & A_\beta\otimes A_\beta\\
\downarrow (\psi_{(\alpha^{-1}\beta)^{-1},\beta}\otimes id)\circ(id\otimes\psi_{\alpha^{-1}\beta,\alpha}) & & \downarrow m_\beta\\
A_\beta\otimes C_{(\alpha^{-1}\beta)^{-1}}\otimes C_{\alpha^{-1}\beta} & \xrightarrow[m_\beta\circ(id\otimes\theta^{(\alpha)}_\beta)]{} & A_\beta
\end{array}$$

Theorem 5.2. *For a partial π-entwining structure $\psi = \{\psi_{\beta,\gamma} : C_\beta \otimes A_{\gamma\beta} \to A_\gamma \otimes C_\beta\}_{\beta,\gamma\in\pi}$, for any $\alpha \in \pi$, the following assertions are equivalent.*

(1) The unit η of the adjunction in Prop. 3.3 is a split natural monomorphism.

(2) The left adjoint $F^{(\alpha)}$ in Prop. 3.3 is separable.

(3) There exists a partial normalised integral

$$\theta^{(\alpha)} = \{\theta^{(\alpha)}_\beta : C_{(\alpha^{-1}\beta)^{-1}} \otimes C_{\alpha^{-1}\beta} \to A_\beta\}_{\beta\in\pi}.$$

Proof. In view of Prop. 3.3, (1) $\Longleftrightarrow$ (2) follows by Rafael's Theorem.

(3) $\Longrightarrow$ (1). We construct a natural transformation $\nu : G^{(\alpha)}F^{(\alpha)} \to 1_{\mathcal{U}^{\pi-C}_A(\psi)}$ For any partial π-entwined module $M = \{M_\beta\}_{\beta\in\pi}$, we consider a family of maps

$$\nu^M = \{\nu^M_\beta : \underline{M_\alpha \otimes C_{\alpha^{-1}\beta}} \to M_\beta\}_{\beta\in\pi},$$

where ν_β^M is the composition of the following maps,

$$\underline{M_\alpha \otimes C_{\alpha^{-1}\beta}} \hookrightarrow M_\alpha \otimes C_{\alpha^{-1}\beta} \overset{\rho_{\beta,\beta^{-1}\alpha}\otimes id}{\longrightarrow} M_\beta \otimes C_{\beta^{-1}\alpha} \otimes C_{\alpha^{-1}\beta}$$
$$\overset{\phi_\beta\circ(id\otimes\theta_\beta^{(\alpha)})}{\longrightarrow} M_\beta$$

That is, for all $m \in M_\alpha$ and $c \in C_{\alpha^{-1}\beta}$,

$$\begin{aligned}
& \nu_\beta^M(m \cdot 1_{A_\beta \psi_{\alpha^{-1}\beta,\alpha}} \otimes c^{\psi_{\alpha^{-1}\beta,\alpha}}) \\
= \; & m_{[0,\beta]} \cdot 1_{A_\beta \psi_{\alpha^{-1}\beta,\alpha}\psi_{\beta^{-1}\alpha,\beta}} \theta_\beta^{(\alpha)}(m_{[1,\beta^{-1}\alpha]}{}^{\psi_{\beta^{-1}\alpha,\beta}} \otimes c^{\psi_{\alpha^{-1}\beta,\alpha}}) \\
\overset{(E5.3)}{=} \; & m_{[0,\beta)} \cdot \theta_\beta^{(\alpha)}(m_{[1,\beta^{-1}\alpha]} \otimes c).
\end{aligned}$$

For all $\beta \in \pi$, it is a morphism of A_β-modules, in fact, for all $m \in M_\alpha$, $c \in C_{\alpha^{-1}\beta}$ and $a \in A_\beta$, we have

$$\begin{aligned}
& \nu_\beta^M((m \cdot 1_{A_\beta \psi_{\alpha^{-1}\beta,\alpha}} \otimes c^{\psi_{\alpha^{-1}\beta,\alpha}}) \cdot a) \\
= \; & \nu_\beta^M(m \cdot a_{\psi_{\alpha^{-1}\beta,\alpha}} \otimes c^{\psi_{\alpha^{-1}\beta,\alpha}}) \\
= \; & (m \cdot a_{\psi_{\alpha^{-1}\beta,\alpha}})_{[0,\beta]} \theta_\beta^{(\alpha)}((m \cdot a_{\psi_{\alpha^{-1}\beta,\alpha}})_{[1,\beta^{-1}\alpha]} \otimes c^{\psi_{\alpha^{-1}\beta,\alpha}}) \\
= \; & m_{[0,\beta]} \cdot a_{\psi_{\alpha^{-1}\beta,\alpha}\psi_{\beta^{-1}\alpha,\beta}} \theta_\beta^{(\alpha)}(m_{[1,\beta^{-1}\alpha]}{}^{\psi_{\beta^{-1}\alpha,\beta}} \otimes c^{\psi_{\alpha^{-1}\beta,\alpha}}) \\
\overset{(E5.3)}{=} \; & m_{[0,\beta]} \cdot \theta_\beta^{(\alpha)}(m_{[1,\beta^{-1}\alpha]} \otimes c)a \\
= \; & \nu_\beta^M(m \cdot 1_{A_\beta \psi_{\alpha^{-1}\beta,\alpha}} \otimes c^{\psi_{\alpha^{-1}\beta,\alpha}}) \cdot a.
\end{aligned}$$

So ν^M is A-linear. Also, $\nu^M = \{\nu_\beta^M\}_{\beta\in\pi}$ constitutes a morphism of partial π-comodules over $\{C_\beta\}_{\beta\in\pi}$. For this, we shall check that the following diagram commutes: for all $\beta, \gamma \in \pi$,

$$\begin{array}{ccc}
\underline{M_\alpha \otimes C_{\alpha^{-1}\beta\gamma}} & \xrightarrow{\nu_{\beta\gamma}^M} & M_{\beta\gamma} \\
\Big\downarrow {\scriptstyle {}^r\rho_{\beta,\gamma}^{G^{(\alpha)}(M)}} & & \Big\downarrow {\scriptstyle \rho_{\beta,\gamma}^M} \\
\underline{M_\alpha \otimes C_{\alpha^{-1}\beta}} \otimes C_\gamma & \xrightarrow[\nu_\beta^M\otimes id_{C_\gamma}]{} & M_\beta \otimes C_\gamma
\end{array}$$

For all $m \in M_\alpha$, $c \in C_{\alpha^{-1}\beta\gamma}$, we have

$$\begin{aligned}
& \rho_{\beta,\gamma}^M \circ \nu_{\beta\gamma}^M(m \cdot 1_{A_{\beta\gamma}\psi_{\alpha^{-1}\beta\gamma,\alpha}} \otimes c^{\psi_{\alpha^{-1}\beta\gamma,\alpha}}) \\
= \; & \rho_{\beta,\gamma}^M(m_{[0,\beta\gamma]} \cdot \theta_{\beta\gamma}^{(\alpha)}(m_{[1,\gamma^{-1}\beta^{-1}\alpha]} \otimes c)) \\
= \; & (m_{[0,\beta\gamma]} \cdot \theta_{\beta\gamma}^{(\alpha)}(m_{[1,\gamma^{-1}\beta^{-1}\alpha]} \otimes c))_{[0.\beta]} \otimes (m_{[0,\beta\gamma]} \cdot \theta_{\beta\gamma}^{(\alpha)}(m_{[1,\gamma^{-1}\beta^{-1}\alpha]} \otimes c))_{[1,\gamma]} \\
= \; & m_{[0,\beta\gamma][0.\beta]} \cdot \theta_{\beta\gamma}^{(\alpha)}(m_{[1,\gamma^{-1}\beta^{-1}\alpha]} \otimes c)_{\psi_{\gamma,\beta}} \otimes m_{[0,\beta\gamma][1,\gamma]}{}^{\psi_{\gamma,\beta}} \\
= \; & m_{[0.\beta]} \cdot 1_{A_\alpha \psi_{\gamma^{-1}\beta^{-1}\alpha,\beta\gamma}\psi_{\gamma,\beta}} \theta_{\beta\gamma}^{(\alpha)}(m_{[1,\beta^{-1}\alpha](2,\gamma^{-1}\beta^{-1}\alpha)}{}^{\psi_{\gamma^{-1}\beta^{-1}\alpha,\beta\gamma}} \otimes c)_{\psi'_{\gamma,\beta}} \otimes m_{[1,\beta^{-1}\alpha](1,\gamma)}{}^{\psi_{\gamma,\beta}\psi'_{\gamma,\beta}} \\
= \; & m_{[0.\beta]} \cdot 1_{A_{\beta\gamma}\psi_{\gamma,\beta}\psi_{\alpha^{-1}\beta,\alpha}\psi_{(\alpha^{-1}\beta)^{-1},\beta}} \theta_\beta^{(\alpha)}(m_{[1,\beta^{-1}\alpha]}{}^{\psi_{(\alpha^{-1}\beta)^{-1},\beta}} \otimes c_{(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}}) \otimes c_{(2,\gamma)}{}^{\psi_{\gamma,\beta}}
\end{aligned}$$

and

$$
\begin{aligned}
& (\nu_\beta^M \otimes id_{C_\gamma})({}^r\rho_{\beta,\gamma}^{G^{(\alpha)}(M)})(m \cdot 1_{A_{\beta\gamma}\psi_{\alpha^{-1}\beta\gamma,\alpha}} \otimes c^{\psi_{\alpha^{-1}\beta\gamma,\alpha}}) \\
= \quad & (\nu_\beta^M \otimes id_{C_\gamma})(m \cdot 1_{A_{\beta\gamma}\psi_{\gamma,\beta}\psi_{\alpha^{-1}\beta,\alpha}} \otimes c_{(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}} \otimes c_{(2,\gamma)}{}^{\psi_{\gamma,\beta}}) \\
\overset{(E3.1)}{=} \quad & (\nu_\beta^M \otimes id_{C_\gamma})(m \cdot 1_{A_{\beta\gamma}\psi_{\gamma,\beta}\psi_{\alpha^{-1}\beta,\alpha}} 1_{A_\beta\psi'_{\alpha^{-1}\beta,\alpha}} \otimes c_{(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}\psi'_{\alpha^{-1}\beta,\alpha}} \otimes c_{(2,\gamma)}{}^{\psi_{\gamma,\beta}}) \\
= \quad & (m \cdot 1_{A_{\beta\gamma}\psi_{\gamma,\beta}\psi_{\alpha^{-1}\beta,\alpha}})_{[0,\beta]}\theta_\beta^{(\alpha)}((m \cdot 1_{A_{\beta\gamma}\psi_{\gamma,\beta}\psi_{\alpha^{-1}\beta,\alpha}})_{[1,\beta^{-1}\alpha]} \otimes c_{(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}}) \otimes c_{(2,\gamma)}{}^{\psi_{\gamma,\beta}} \\
= \quad & m_{[0,\beta]} \cdot 1_{A_{\beta\gamma}\psi_{\gamma,\beta}\psi_{\alpha^{-1}\beta,\alpha}\psi_{\beta^{-1}\alpha,\beta}}\theta_\beta^{(\alpha)}(m_{[1,\beta^{-1}\alpha]}{}^{\psi_{\beta^{-1}\alpha,\beta}} \otimes c_{(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}}) \otimes c_{(2,\gamma)}{}^{\psi_{\gamma,\beta}}.
\end{aligned}
$$

Hence the diagram above is commutative.

Now, we shall check that ν splits the unit of the adjunction in Prop.3.3, i.e., $\nu \circ \eta = I$. For any partial π-entwined module $M = \{M_\beta\}_{\beta\in\pi}$, for all $\beta \in \pi$, $m \in M_\beta$, we make a direct computation as follows:

$$
\begin{aligned}
\nu_\beta^M \circ \eta_\beta^M(m) \quad &= \quad \nu_\beta^M(m_{[0,\alpha]} \cdot 1_{A_\beta\psi_{\alpha^{-1}\beta,\alpha}} \otimes m_{[1,\alpha^{-1}\beta]}{}^{\psi_{\alpha^{-1}\beta,\alpha}}) \\
&= \quad m_{[0,\alpha][0,\beta]} \cdot \theta_\beta^{(\alpha)}(m_{[0,\alpha][1,\beta^{-1}\alpha]} \otimes m_{[1,\alpha^{-1}\beta]}) \\
&= \quad m_{[0,\beta]} \cdot 1_{A_\beta\psi_{\alpha^{-1}\beta,\alpha}\psi_{\beta^{-1}\alpha,\beta}}\theta_\beta^{(\alpha)}(m_{[1,e](1,\beta^{-1}\alpha)}{}^{\psi_{\beta^{-1}\alpha,\beta}} \otimes m_{[1,e](2,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}}) \\
&= \quad m_{[0,\beta]} \cdot \theta_\beta^{(\alpha)}(m_{[1,e](1,\beta^{-1}\alpha)} \otimes m_{[1,e](2,\alpha^{-1}\beta)}) \\
&\overset{(E5.1)}{=} \quad m_{[0,\beta]}\varepsilon(m_{[1,e]}) = m.
\end{aligned}
$$

It is evidently natural in $\mathcal{U}_A^{\pi-C}(\psi)$.

(1) $\Longrightarrow$ (3). For any $\beta \in \pi$, we consider the following partial π-entwined module $R^{(\beta)} = \{R_\gamma^{(\beta)}\}_{\gamma\in\pi}$, where $R_\gamma^{(\beta)} := \underline{A_\beta \otimes C_{\beta^{-1}\gamma}}$. Evaluating at this object, the retraction ν of the unit of the adjunction in Prop.3.3. yields a morphism $\nu^{R^{(\beta)}} = \{\nu_\gamma^{R^{(\beta)}}\}_{\gamma\in\pi}$, for all $\beta, \gamma \in \pi$,

$$\nu_\gamma^{R^{(\beta)}} : \underline{A_\beta \otimes C_{\beta^{-1}\alpha} \otimes C_{\alpha^{-1}\gamma}} \to \underline{A_\beta \otimes C_{\beta^{-1}\gamma}},$$

where

$$
\begin{aligned}
& \underline{A_\beta \otimes C_{\beta^{-1}\alpha} \otimes C_{\alpha^{-1}\gamma}} \\
& =< (a1_{A_\alpha\psi_{\beta^{-1}\alpha,\beta}} \otimes c^{\psi_{\beta^{-1}\alpha,\beta}}) \cdot 1_{A_\gamma\psi_{\alpha^{-1}\gamma,\alpha}} \otimes d^{\psi_{\alpha^{-1}\gamma,\alpha}} | a \in A_\beta, c \in C_{\beta^{-1}\alpha}, d \in C_{\alpha^{-1}\gamma} > \\
& =< a1_{A_\gamma\psi_{\alpha^{-1}\gamma,\alpha}\psi_{\beta^{-1}\alpha,\beta}} \otimes c^{\psi_{\beta^{-1}\alpha,\beta}} \otimes d^{\psi_{\alpha^{-1}\gamma,\alpha}} | a \in A_\beta, c \in C_{\beta^{-1}\alpha}, d \in C_{\alpha^{-1}\gamma} > .
\end{aligned}
$$

In particular, we also have

$$\nu_\gamma^{R^{(\beta)}}(a1_{A_\gamma\psi_{\alpha^{-1}\gamma,\alpha}\psi_{\beta^{-1}\alpha,\beta}} \otimes c_{(1,\beta^{-1}\alpha)}{}^{\psi_{\beta^{-1}\alpha,\beta}} \otimes c_{(2,\alpha^{-1}\gamma)}{}^{\psi_{\alpha^{-1}\gamma,\alpha}}) = a1_{A_\gamma\psi_{\beta^{-1}\gamma,\beta}} \otimes c^{\psi_{\beta^{-1}\gamma,\beta}}, \tag{E5.4}$$

for all $a \in A_\beta, c \in C_{\beta^{-1}\gamma}$.

It can be used to construct $\theta_\beta^{(\alpha)}$ as

$$
\begin{aligned}
C_{(\alpha^{-1}\beta)^{-1}} \otimes C_{\alpha^{-1}\beta} \quad & \overset{id\otimes\eta_\alpha\otimes id}{\longrightarrow} \quad C_{(\alpha^{-1}\beta)^{-1}} \otimes A_\alpha \otimes C_{\alpha^{-1}\beta} \\
& \overset{\psi_{(\alpha^{-1}\beta)^{-1},\beta}\otimes id}{\longrightarrow} \quad A_\beta \otimes C_{(\alpha^{-1}\beta)^{-1}} \otimes C_{\alpha^{-1}\beta} \\
& \overset{\nu_\beta^{R^{(\beta)}}\circ P}{\longrightarrow} \quad \underline{A_\beta \otimes C_e} \\
& \overset{id\otimes\varepsilon}{\longrightarrow} \quad A_\beta
\end{aligned}
$$

where $P : A_\beta \otimes C_{(\alpha^{-1}\beta)^{-1}} \otimes C_{\alpha^{-1}\beta} \to \underline{A_\beta \otimes C_{(\alpha^{-1}\beta)^{-1}} \otimes C_{\alpha^{-1}\beta}}$ is the natural projection. Explicitly, for all $c \in C_{(\alpha^{-1}\beta)^{-1}}, d \in C_{\alpha^{-1}\beta}$,

$$\theta^{(\alpha)}_\beta(c \otimes d) = (id_{A_\beta} \otimes \varepsilon)\nu^{R^{(\beta)}}_\beta(1_{A_\beta\psi_{\alpha^{-1}\beta,\alpha}\psi_{\beta^{-1}\alpha,\beta}} \otimes c^{\psi_{\beta^{-1}\alpha,\beta}} \otimes d^{\psi_{\alpha^{-1}\beta,\alpha}}).$$

Now, we shall check $\theta^{(\alpha)}$ is a partial normalize integral, that is, $\theta^{(\alpha)}$ satisfies the conditions (E5.1)-(E5.3). As to Condition (E5.1), we make the following calculation: for all $\beta \in \pi$, $b \in C_e$,

$$\begin{aligned}
& \theta^{(\alpha)}_\beta(b_{(1,(\alpha^{-1}\beta)^{-1})} \otimes b_{(2,\alpha^{-1}\beta)}) \\
= \; & (id_{A_\beta} \otimes \varepsilon)\nu^{R^{(\beta)}}_\beta(1_{A_\beta\psi_{\alpha^{-1}\beta,\alpha}\psi_{\beta^{-1}\alpha,\beta}} \otimes (b_{(1,(\alpha^{-1}\beta)^{-1})})^{\psi_{\beta^{-1}\alpha,\beta}} \otimes (b_{(2,\alpha^{-1}\beta)})^{\psi_{\alpha^{-1}\beta,\alpha}}) \\
\overset{(E5.4)}{=} \; & (id_{A_\beta} \otimes \varepsilon)(1_{A_\beta\psi_{e,\beta}} \otimes b^{\psi_{e,\beta}}) = 1_{A_\beta\psi_{e,\beta}}\varepsilon(b^{\psi_{e,\beta}}) = 1_{A_\beta}\varepsilon(b).
\end{aligned}$$

So Condition (E5.1) holds.

For $a \in A_\beta$, let f^a be the map

$$f^a = \{f^a_\gamma : \underline{A_\beta \otimes C_{\beta^{-1}\gamma}} \to \underline{A_\beta \otimes C_{\beta^{-1}\gamma}}\}_{\gamma\in\pi},$$

where

$$f^a_\gamma(b1_{A_\gamma\psi_{\beta^{-1}\gamma,\beta}} \otimes c^{\psi_{\beta^{-1}\gamma,\beta}}) = ab1_{A_\gamma\psi_{\beta^{-1}\gamma,\beta}} \otimes c^{\psi_{\beta^{-1}\gamma,\beta}},$$

for all $\beta, \gamma \in \pi$, $b \in A_\beta$ and $c \in C_{\beta^{-1}\gamma}$. Notice that f^a is a morphism in the category $\mathcal{U}^{\pi-C}_A(\psi)$. By naturality of ν, we have the following commutative diagram

$$\begin{array}{ccc}
\underline{(A_\beta \otimes C_{\beta^{-1}\alpha}) \otimes C_{\alpha^{-1}\gamma}} & \xrightarrow{\nu^{C\otimes_\beta A}_\gamma} & \underline{A_\beta \otimes C_{\beta^{-1}\gamma}} \\
\Big\downarrow {\scriptstyle f^a_\alpha \otimes id} & & \Big\downarrow {\scriptstyle f^a_\gamma} \\
\underline{(A_\beta \otimes C_{\beta^{-1}\alpha}) \otimes C_{\alpha^{-1}\gamma}} & \xrightarrow[\nu^{R^{(\beta)}}_\gamma]{} & \underline{A_\beta \otimes C_{\beta^{-1}\gamma}}
\end{array}$$

The commutative diagram above means that, for all $\beta \in \pi$, then, for all $\gamma \in \pi$, the morphism $\nu^{C\otimes_\beta A}_\gamma$ is A_β-linear. So for all $a \in A_\beta, b \in C_{\alpha^{-1}\beta}$ and $d \in C_{(\alpha^{-1}\beta)^{-1}}$.

$$\begin{aligned}
& (a_{\psi_{\alpha^{-1}\beta,\alpha}})_{\psi_{(\alpha^{-1}\beta)^{-1},\beta}}\theta^{(\alpha)}_\beta(d^{\psi_{(\alpha^{-1}\beta)^{-1},\beta}} \otimes b^{\psi_{\alpha^{-1}\beta,\alpha}}) \\
= \; & (a_{\psi_{\alpha^{-1}\beta,\alpha}})_{\psi_{(\alpha^{-1}\beta)^{-1},\beta}}(id_{A_\beta} \otimes \varepsilon)\nu^{R^{(\beta)}}_\beta(1_{A_\beta\psi'_{\alpha^{-1}\beta,\alpha}\psi'_{\beta^{-1}\alpha,\beta}} \otimes d^{\psi_{(\alpha^{-1}\beta)^{-1},\beta}\psi'_{\beta^{-1}\alpha,\beta}} \otimes b^{\psi_{\alpha^{-1}\beta,\alpha}\psi'_{\alpha^{-1}\beta,\alpha}}) \\
= \; & (id_{A_\beta} \otimes \varepsilon)\nu^{R^{(\beta)}}_\beta((a_{\psi_{\alpha^{-1}\beta,\alpha}})_{\psi_{(\alpha^{-1}\beta)^{-1},\beta}}1_{A_\beta\psi'_{\alpha^{-1}\beta,\alpha}\psi'_{\beta^{-1}\alpha,\beta}} \otimes d^{\psi_{(\alpha^{-1}\beta)^{-1},\beta}\psi'_{\beta^{-1}\alpha,\beta}} \otimes b^{\psi_{\alpha^{-1}\beta,\alpha}\psi'_{\alpha^{-1}\beta,\alpha}}) \\
\overset{(E3.1)}{=} \; & (id_{A_\beta} \otimes \varepsilon)\nu^{R^{(\beta)}}_\beta(a_{\psi_{\alpha^{-1}\beta,\alpha}\psi_{(\alpha^{-1}\beta)^{-1},\beta}} \otimes d^{\psi_{(\alpha^{-1}\beta)^{-1},\beta}} \otimes b^{\psi_{\alpha^{-1}\beta,\alpha}}) \\
= \; & (id_{A_\beta} \otimes \varepsilon)\nu^{R^{(\beta)}}_\beta(1_{A_\beta\psi_{\alpha^{-1}\beta,\alpha}\psi_{(\alpha^{-1}\beta)^{-1},\beta}} \otimes d^{\psi_{(\alpha^{-1}\beta)^{-1},\beta}} \otimes b^{\psi_{\alpha^{-1}\beta,\alpha}})a \\
= \; & \theta^{(\alpha)}_\beta(d \otimes b)a.
\end{aligned}$$

Hence Condition(5.3)holds.

The verification of Condition (5.2) is more difficult. For the sake of completion, we proceed the proof as follows. For any π-comodule $M = \{M_\beta\}_{\beta\in\pi}$, we consider the partial π-entwined module $M \otimes_\beta A = \{M_{\beta^{-1}\gamma} \otimes A_\gamma\}_{\gamma\in\pi}$. The A-action and partial π-C-coaction are induced by the respective maps

$$M_{\beta^{-1}\gamma} \otimes A_\gamma \otimes A_\gamma \overset{id\otimes m_\gamma}{\longrightarrow} M_{\beta^{-1}\gamma} \otimes A_\gamma,$$

$$M_{\beta^{-1}\gamma}\otimes A_\gamma \overset{\rho_{\beta^{-1}\zeta,\zeta^{-1}\gamma}\otimes id}{\longrightarrow} M_{\beta^{-1}\zeta}\otimes C_{\zeta^{-1}\gamma}\otimes A_\gamma \overset{id\otimes\psi_{\zeta^{-1}\gamma,\zeta}}{\longrightarrow} M_{\beta^{-1}\zeta}\otimes A_\zeta\otimes C_{\zeta^{-1}\gamma}$$

That is, for all $m\in M_{\beta^{-1}\gamma}, a, b\in A_\gamma$,

$$\begin{cases} (m\otimes a)\cdot b = m\otimes ab, & \text{for all } m\in M_{\beta^{-1}\gamma}, a, b\in A_\gamma,\\ \rho^{M\otimes_\beta A}_{\gamma,\zeta}(m\otimes a) = m_{[0,\beta^{-1}\gamma]}\otimes a_{\psi_{\zeta,\gamma}}\otimes m_{[1,\zeta]}{}^{\psi_{\zeta,\gamma}}, & \text{for all } m\in M_{\beta^{-1}\gamma\zeta}, a\in A_{\gamma\zeta}.\end{cases}$$

In particular, there is a partial π-entwined module $C\otimes_\beta A$ and

$$\psi^{(\beta)} = \{\psi^{(\beta)}_\gamma : C_{\beta^{-1}\gamma}\otimes A_\gamma \to \underline{A_\beta\otimes C_{\beta^{-1}\gamma}}, \psi^{(\beta)}_\gamma(c\otimes a) = a_{\psi_{\beta^{-1}\gamma,\beta}}\otimes c^{\psi_{\beta^{-1}\gamma,\beta}}\}_{\gamma\in\pi}.$$

Standard computations show that $\psi^{(\beta)}$ is a morphism of partial π-entwined modules $C\otimes_\beta A\to R^{(\beta)}$. Thus by naturality of ν, the following diagram commutes, for all $\beta,\gamma\in\pi$,

$$\begin{array}{ccc} \underline{(C_{\beta^{-1}\alpha}\otimes A_\alpha)\otimes C_{\alpha^{-1}\gamma}} & \xrightarrow{\nu^{C\otimes_\beta A}_\gamma} & C_{\beta^{-1}\gamma}\otimes A_\gamma \\ \Big\downarrow{\scriptstyle \psi^{(\beta)}_\alpha\otimes id} & & \Big\downarrow{\scriptstyle \psi^{(\beta)}_\gamma} \\ \underline{(A_\beta\otimes C_{\beta^{-1}\alpha})\otimes C_{\alpha^{-1}\gamma}} & \xrightarrow[\nu^{R^{(\beta)}}_\gamma]{} & \underline{A_\beta\otimes C_{\beta^{-1}\gamma}} \end{array}$$

From the commutative diagram above, we have the following equation:

$$\psi^{(\beta)}_\gamma\nu^{C\otimes_\beta A}_\gamma(c\otimes a1_{A_\gamma\psi_{\alpha^{-1}\gamma,\alpha}}\otimes d^{\psi_{\alpha^{-1}\gamma,\alpha}}) = \nu^{R^{(\beta)}}_\gamma(a_{\psi_{\beta^{-1}\alpha,\beta}}1_{A_\gamma\psi_{\alpha^{-1}\gamma,\alpha}\psi'_{\beta^{-1}\alpha,\beta}}\otimes c^{\psi_{\beta^{-1}\alpha,\beta}\psi'_{\beta^{-1}\alpha,\beta}}\otimes d^{\psi_{\alpha^{-1}\gamma,\alpha}}), \tag{E5.5}$$

for all $c\in C_{\beta^{-1}\alpha}, a\in A_\alpha, d\in C_{\alpha^{-1}\gamma}$.

We consider next the following partial π-comodule $D^{(\beta)} = \{D^{(\beta)}_\gamma\}_{\gamma\in\pi}$, where $D^{(\beta)}_\gamma = C_\beta\otimes C_{\beta^{-1}\gamma}$ with π-C-coaction given by comultiplication in the second factor. Then

$$\Delta^{(\beta)} = \{\Delta^{(\beta)}_\gamma : C_{\beta^{-1}\gamma}\otimes A_\gamma\to D^{(\beta^{-1}\zeta)}_{\beta^{-1}\gamma}\otimes A_\gamma\}_{\gamma\in\pi},$$

$$\Delta^{(\beta)}_\gamma(c\otimes a) = c_{(1,\beta^{-1}\zeta)}\otimes c_{(2,\zeta^{-1}\gamma)}\otimes a.$$

Standard computations show that $\Delta^{(\beta)}$ is a morphism a morphism of partial π-entwined modules $C\otimes_\beta A\to D^{(\beta^{-1}\zeta)}\otimes_\beta A$. Thus by naturality of ν, the following diagram commutes, for all $\beta,\gamma,\zeta\in\pi$,

$$\begin{array}{ccc} (C_{\beta^{-1}\alpha}\otimes \underline{A_\alpha)\otimes C_{\alpha^{-1}\gamma}} & \xrightarrow{\nu^{C\otimes_\beta A}_\gamma} & C_{\beta^{-1}\gamma}\otimes A_\gamma \\ \Big\downarrow{\scriptstyle \Delta^{(\beta)}_\alpha\otimes id} & & \Big\downarrow{\scriptstyle \Delta^{(\beta)}_\gamma} \\ (D^{(\beta^{-1}\zeta)}_{\beta^{-1}\alpha}\otimes \underline{A_\alpha)\otimes C_{\alpha^{-1}\gamma}} & \xrightarrow[\nu^{D^{(\beta^{-1}\zeta)\otimes_\beta A}}_\gamma]{} & D^{(\beta^{-1}\zeta)}_{\beta^{-1}\gamma}\otimes A_\gamma \end{array}$$

So we have the following equation: for all $c\in C_{\beta^{-1}\alpha}, a\in A_\alpha, d\in C_{\alpha^{-1}\gamma}$,

$$\Delta^{(\beta)}_\gamma\nu^{C\otimes_\beta A}_\gamma(c\otimes a1_{A_\gamma\psi_{\alpha^{-1}\gamma,\alpha}}\otimes d^{\psi_{\alpha^{-1}\gamma,\alpha}}) = \nu^{D^{(\beta^{-1}\zeta)\otimes_\beta A}}_\gamma(c_{(1,\beta^{-1}\zeta)}\otimes c_{(2,\zeta^{-1}\alpha)}\otimes a1_{A_\gamma\psi_{\alpha^{-1}\gamma,\alpha}}\otimes d^{\psi_{\alpha^{-1}\gamma,\alpha}}). \tag{E5.6}$$

Finally, for any $c \in C_\beta$, the map

$$^{c}f^{(\gamma\beta)} = \{^{c}f^{(\gamma\beta)}_{\zeta} : C_{(\gamma\beta)^{-1}\zeta} \otimes A_\zeta \to D^{(\beta)}_{\gamma^{-1}\zeta} \otimes A_\zeta,\ \ d \otimes a \mapsto c \otimes d \otimes a\}_{\zeta\in\pi}$$

is a morphism of partial π-entwined modules $C \otimes_{\gamma\beta} A \to D^{(\beta)} \otimes_\gamma A$. Hence by naturality of ν,

$$\nu^{D^{(\beta)}\otimes_\gamma A}_{\zeta} = C_\beta \otimes \nu^{C\otimes_{\gamma\beta}A}_{\zeta}$$

as maps $C_\beta \otimes (C_{(\gamma\beta)^{-1}\alpha} \otimes \underline{A_\alpha) \otimes C_{\alpha^{-1}\zeta}} \to D^{(\beta)}_{\gamma^{-1}\zeta} \otimes A_\zeta$. So we obtain the following commutative diagram, for all $\beta, \gamma, \zeta \in \pi$,

$$\begin{array}{ccc}
(C_{\beta^{-1}\alpha} \otimes \underline{A_\alpha) \otimes C_{\alpha^{-1}\gamma}} & \xrightarrow{\ \nu^{C\otimes_\beta A}_{\gamma}\ } & C_{\beta^{-1}\gamma} \otimes A_\gamma \\
\Big\downarrow{\scriptstyle \Delta^{(\beta)}_{\alpha}\otimes id} & & \Big\downarrow{\scriptstyle \Delta^{(\beta)}_{\gamma}} \\
(D^{(\beta^{-1}\zeta)}_{\beta^{-1}\alpha} \otimes \underline{A_\alpha) \otimes C_{\alpha^{-1}\gamma}} & \xrightarrow[\ id\otimes\nu^{C\otimes_\zeta A}_{\gamma}\ ]{} & D^{(\beta^{-1}\zeta)}_{\beta^{-1}\gamma} \otimes A_\gamma
\end{array}$$

i.e., for all $c \in C_{\beta^{-1}\alpha}, a \in A_\alpha, d \in C_{\alpha^{-1}\gamma}$,

$$\Delta^{(\beta)}_{\gamma}\nu^{C\otimes_\beta A}_{\gamma}(c \otimes a1_{A_\gamma\psi_{\alpha^{-1}\gamma,\alpha}} \otimes d^{\psi_{\alpha^{-1}\gamma,\alpha}}) = c_{(1,\beta^{-1}\zeta)} \otimes \nu^{C\otimes_\zeta A}_{\gamma}(c_{(2,\zeta^{-1}\alpha)} \otimes a1_{A_\gamma\psi_{\alpha^{-1}\gamma,\alpha}} \otimes d^{\psi_{\alpha^{-1}\gamma,\alpha}}). \quad \text{(E5.7)}$$

From $\nu^{R^{(\beta)}}$ being a partial π-C-colinear, It follows that

$$\nu^{R^{(\beta)}}_{\gamma}(a1_{A_{\gamma\zeta}\psi_{\zeta,\gamma}\psi_{\alpha^{-1}\gamma,\alpha}\psi_{\beta^{-1}\alpha,\beta}} \otimes d^{\psi_{\beta^{-1}\alpha,\beta}} \otimes c_{(1,\alpha^{-1}\gamma)}{}^{\psi_{\alpha^{-1}\gamma,\alpha}}) \otimes c_{(2,\zeta)}{}^{\psi_{\zeta,\gamma}} = \rho^{R^{(\beta)}}_{\gamma,\zeta}\nu^{R^{(\beta)}}_{\gamma,\zeta}(a1_{A_{\gamma\zeta}\psi_{\alpha^{-1}\gamma\zeta,\alpha}\psi_{\beta^{-1}\alpha,\beta}} \otimes d^{\psi_{\beta^{-1}\alpha,\beta}} \otimes c^{\psi_{\alpha^{-1}\gamma\zeta,\alpha}}), \quad \text{(E5.8)}$$

for all $a \in A_\beta$, $d \in C_{\beta^{-1}\alpha}$ and $c \in C_{\alpha^{-1}\gamma\zeta}$.

For all $\beta, \gamma \in \pi$, $c \in C_{\alpha^{-1}\beta\gamma}$ and $d \in C_{(\alpha^{-1}\beta)^{-1}}$,

$$\begin{aligned}
& c_{(2,\gamma)}{}^{\psi_{\gamma,\beta}} \otimes 1_{A_{\beta\gamma}\psi_{\gamma,\beta}\psi_{\alpha^{-1}\beta,\alpha}\psi_{(\alpha^{-1}\beta)^{-1},\beta}}\theta^{(\alpha)}_{\beta}(d^{\,\psi_{(\alpha^{-1}\beta)^{-1},\beta}} \otimes c_{(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}}) \\
= \ & c_{(2,\gamma)}{}^{\psi_{\gamma,\beta}} \otimes 1_{A_{\beta\gamma}\psi_{\gamma,\beta}\psi_{\alpha^{-1}\beta,\alpha}\psi_{(\alpha^{-1}\beta)^{-1},\beta}}(id_{A_\beta} \otimes \varepsilon)\nu^{R^{(\beta)}}_{\beta}(1_{A_\beta\psi'_{\alpha^{-1}\beta,\alpha}\psi'_{\beta^{-1}\alpha,\beta}} \otimes d^{\,\psi_{(\alpha^{-1}\beta)^{-1},\beta}\psi'_{\beta^{-1}\alpha,\beta}} \otimes c_{(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}\psi'_{\alpha^{-1}\beta,\alpha}}) \\
= \ & c_{(2,\gamma)}{}^{\psi_{\gamma,\beta}} \otimes (id_{A_\beta} \otimes \varepsilon)\nu^{R^{(\beta)}}_{\beta}(1_{A_{\beta\gamma}\psi_{\gamma,\beta}\psi_{\alpha^{-1}\beta,\alpha}\psi_{(\alpha^{-1}\beta)^{-1},\beta}}1_{A_\beta\psi'_{\alpha^{-1}\beta,\alpha}\psi'_{\beta^{-1}\alpha,\beta}} \otimes d^{\,\psi_{(\alpha^{-1}\beta)^{-1},\beta}\psi'_{\beta^{-1}\alpha,\beta}} \otimes c_{(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}\psi'_{\alpha^{-1}\beta,\alpha}}) \\
\overset{(E3.1)}{=} \ & c_{(2,\gamma)}{}^{\psi_{\gamma,\beta}} \otimes (id_{A_\beta} \otimes \varepsilon)\nu^{R^{(\beta)}}_{\beta}(1_{A_{\beta\gamma}\psi_{\gamma,\beta}\psi_{\alpha^{-1}\beta,\alpha}\psi_{(\alpha^{-1}\beta)^{-1},\beta}} \otimes d^{\,\psi_{(\alpha^{-1}\beta)^{-1},\beta}} \otimes c_{(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}}) \\
= \ & \tau_{A_\beta,C_\gamma}(id_{A_\beta} \otimes \varepsilon \otimes id_{C_\gamma})\nu^{R^{(\beta)}}_{\beta}(1_{A_{\beta\gamma}\psi_{\gamma,\beta}\psi_{\alpha^{-1}\beta,\alpha}\psi_{(\alpha^{-1}\beta)^{-1},\beta}} \otimes d^{\,\psi_{(\alpha^{-1}\beta)^{-1},\beta}} \otimes c_{(1,\alpha^{-1}\beta)}{}^{\psi_{\alpha^{-1}\beta,\alpha}}) \otimes c_{(2,\gamma)}{}^{\psi_{\gamma,\beta}}) \\
\overset{(E5.8)}{=} \ & \tau_{A_\beta,C_\gamma}(id_{A_\beta} \otimes \varepsilon \otimes id_{C_\gamma})(\rho^{R^{(\beta)}}_{\beta,\gamma}\nu^{R^{(\beta)}}_{\beta,\gamma}(1_{A_{\beta\gamma}\psi_{\alpha^{-1}\beta\gamma,\alpha}\psi_{\beta^{-1}\alpha,\beta}} \otimes d^{\psi_{\beta^{-1}\alpha,\beta}} \otimes c^{\psi_{\alpha^{-1}\beta\gamma,\alpha}})) \\
= \ & \tau_{A_\beta,C_\gamma}(\nu^{R^{(\beta)}}_{\beta,\gamma}(1_{A_{\beta\gamma}\psi_{\alpha^{-1}\beta\gamma,\alpha}\psi_{\beta^{-1}\alpha,\beta}} \otimes d^{\psi_{\beta^{-1}\alpha,\beta}} \otimes c^{\psi_{\alpha^{-1}\beta\gamma,\alpha}})).
\end{aligned}$$

and

$$\begin{aligned}
& d_{(1,\gamma)}{}^{\psi_{\gamma,\beta}\psi'_{\gamma,\beta}} \otimes 1_{A_\alpha\psi_{(\alpha^{-1}\beta\gamma)^{-1},\beta\gamma}\psi_{\gamma,\beta}}\theta^{(\alpha)}_{\beta\gamma}(d_{(2,(\alpha^{-1}\beta\gamma)^{-1})}{}^{\psi_{(\alpha^{-1}\beta\gamma)^{-1},\beta\gamma}} \otimes c)_{\psi'_{\gamma,\beta}} \\
= \ & d_{(1,\gamma)}{}^{\psi_{\gamma,\beta}\psi'_{\gamma,\beta}} \otimes 1_{A_\alpha\psi_{(\alpha^{-1}\beta\gamma)^{-1},\beta\gamma}\psi_{\gamma,\beta}}\theta^{(\alpha)}_{\beta\gamma}(d_{(2,(\alpha^{-1}\beta\gamma)^{-1})}{}^{\psi_{(\alpha^{-1}\beta\gamma)^{-1},\beta\gamma}} \otimes c)_{\psi'_{\gamma,\beta}} \\
= \ & d_{(1,\gamma)}{}^{\psi_{\gamma,\beta}} \otimes (1_{A_\alpha\psi_{(\alpha^{-1}\beta\gamma)^{-1},\beta\gamma}}\theta^{(\alpha)}_{\beta\gamma}(d_{(2,(\alpha^{-1}\beta\gamma)^{-1})}{}^{\psi_{(\alpha^{-1}\beta\gamma)^{-1},\beta\gamma}} \otimes c))_{\psi_{\gamma,\beta}} \\
= \ & d_{(1,\gamma)}{}^{\psi_{\gamma,\beta}} \otimes (1_{A_\alpha\psi_{(\alpha^{-1}\beta\gamma)^{-1},\beta\gamma}}(id_{A_{\beta\gamma}} \otimes \varepsilon)\nu^{R^{(\beta)}}_{\beta\gamma}(1_{A_{\beta\gamma}\psi_{\alpha^{-1}\beta\gamma,\alpha}\psi'_{(\beta\gamma)^{-1}\alpha,\beta\gamma}} \otimes d_{(2,(\alpha^{-1}\beta\gamma)^{-1})}{}^{\psi_{(\alpha^{-1}\beta\gamma)^{-1},\beta\gamma}\psi'_{(\beta\gamma)^{-1}\alpha,\beta\gamma}} \otimes c^{\psi_{\alpha^{-1}\beta\gamma,\alpha}}))_{\psi_{\gamma,\beta}} \\
= \ & d_{(1,\gamma)}{}^{\psi_{\gamma,\beta}} \otimes (id_{A_{\beta\gamma}} \otimes \varepsilon)\nu^{R^{(\beta\gamma)}}_{\beta\gamma}(1_{A_\alpha\psi_{(\alpha^{-1}\beta\gamma)^{-1},\beta\gamma}}1_{A_{\beta\gamma}\psi_{\alpha^{-1}\beta\gamma,\alpha}\psi'_{(\beta\gamma)^{-1}\alpha,\beta\gamma}} \otimes d_{(2,(\alpha^{-1}\beta\gamma)^{-1})}{}^{\psi_{(\alpha^{-1}\beta\gamma)^{-1},\beta\gamma}\psi'_{(\beta\gamma)^{-1}\alpha,\beta\gamma}} \otimes c^{\psi_{\alpha^{-1}\beta\gamma,\alpha}}))_{\psi_{\gamma,\beta}}
\end{aligned}$$

$$\overset{(E3.1)}{=} \quad d_{(1,\gamma)}{}^{\psi_{\gamma,\beta}} \otimes (id_{A_{\beta\gamma}} \otimes \varepsilon)\nu^{R(\beta\gamma)}_{\beta\gamma}(1_{A_{\beta\gamma}\psi_{\alpha^{-1}\beta\gamma,\alpha}\psi_{(\beta\gamma)^{-1}\alpha,\beta\gamma}} \otimes d_{(2,(\alpha^{-1}\beta\gamma)^{-1})}{}^{\psi_{(\alpha^{-1}\beta\gamma)^{-1},\beta\gamma}} \otimes c^{\psi_{\alpha^{-1}\beta\gamma,\alpha}}))_{\psi_{\gamma,\beta}}$$

$$= \quad d_{(1,\gamma)}{}^{\psi_{\gamma,\beta}} \otimes (id_{A_{\beta\gamma}} \otimes \varepsilon)\nu^{R(\beta\gamma)}_{\beta\gamma}(1_{A_{\beta\gamma}\psi_{\alpha^{-1}\beta\gamma,\alpha}\psi_{(\beta\gamma)^{-1}\alpha,\beta\gamma}} \otimes d_{(2,(\alpha^{-1}\beta\gamma)^{-1})}{}^{\psi_{(\alpha^{-1}\beta\gamma)^{-1},\beta\gamma}} \otimes c^{\psi_{\alpha^{-1}\beta\gamma,\alpha}}))_{\psi_{\gamma,\beta}}$$

$$\overset{(E5.5)}{=} \quad (d_{(1,\gamma)})^{\psi_{\gamma,\beta}} \otimes ((id_{A_{\beta\gamma}} \otimes \varepsilon)\psi^{(\beta\gamma)}_{\beta\gamma}\nu^{C\otimes_{\beta\gamma}A}_{\beta\gamma}(d_{(2,\gamma^{-1}(\alpha^{-1}\beta)^{-1})} \otimes 1_{A_{\beta\gamma}\psi_{\alpha^{-1}\beta\gamma,\alpha}} \otimes c^{\psi_{\alpha^{-1}\beta\gamma,\alpha}}))_{\psi_{\gamma,\beta}}$$

(Let $\nu^{C\otimes_\beta A}_{\beta\gamma}(d \otimes 1_{A_{\beta\gamma}\psi_{\alpha^{-1}\beta\gamma,\alpha}} \otimes c^{\psi_{\alpha^{-1}\beta\gamma,\alpha}}) = \sum_i e_i \otimes f_i$, where $e_i \in C_\gamma$, $f_i \in A_{\beta\gamma}$)

$$\overset{(E5.7)}{=} \quad \sum_i (e_{i(1,\gamma)})^{\psi_{\gamma,\beta}} \otimes ((id_{A_{\beta\gamma}} \otimes \varepsilon)\psi^{\beta\gamma}_{\beta\gamma}(e_{i(2,e)} \otimes f_i))_{\psi_{\gamma,\beta}}$$

$$= \quad {}_i(e_{i(1,\gamma)})^{\psi_{\gamma,\beta}} \otimes ((id_{A_{\beta\gamma}} \otimes \varepsilon)(f_{i\psi_{e,\beta\gamma}} \otimes (e_{i(2,e)})^{\psi_{e,\beta\gamma}}))_{\psi_{\gamma,\beta}}$$

$$= \quad \sum_i e_i^{\psi_{\gamma,\beta}} \otimes f_{i\psi_{\gamma,\beta}}$$

$$= \quad \tau_{A_\beta,C_\gamma}\psi^{(\beta)}_{\beta\gamma}(\nu^{C\otimes_\beta A}_{\beta\gamma}(d \otimes 1_{A_{\beta\gamma}\psi_{\alpha^{-1}\beta\gamma,\alpha}} \otimes c^{\psi_{\alpha^{-1}\beta\gamma,\alpha}}))$$

$$\overset{(E5.5)}{=} \quad \tau_{A_\beta,C_\gamma}\nu^{R(\beta)}_{\beta\gamma}(1_{A_{\beta\gamma}\psi_{\alpha^{-1}\beta\gamma,\alpha}\psi_{\beta^{-1}\alpha,\beta}} \otimes d^{\psi_{\beta^{-1}\alpha,\beta}} \otimes c^{\psi_{\alpha^{-1}\beta\gamma,\alpha}}).$$

So we get (E5.2). □

6. Applications

6.1. Maschke-type Theorems for Partial Group Entwined Modules

From Theorem 5.2, we shall prove the Maschke-type theorems for partial π-entwined modules.

Corollary 6.1. *Let $(A, C)_{\pi-\psi}$ be a partial π-entwining structure and $M = \{M_\beta\}_{\beta\in\pi}$, $N = \{N_\beta\}_{\beta\in\pi} \in \mathcal{U}^{\pi-C}_A(\psi)$. For any $\alpha \in \pi$, suppose that there exists a partial normalized integral $\theta^{(\alpha)} = \{\theta^{(\alpha)}_\beta : C_{(\alpha^{-1}\beta)^{-1}} \otimes C_{\alpha^{-1}\beta} \to A_\beta\}_{\beta\in\pi}$. Then a monomorphism (resp. epimorphism) $f = (f_\beta : M_\beta \to N_\beta)$ splits in $\mathcal{U}^{\pi-C}_A(\psi)$ if the monomorphism (resp. epimorphism) f_α splits as an A_α-module morphism.*

If $\pi = \{e\}$ is a trivial group, partial π-entwined structures (modules) are just partial entwined structures (modules) in sense of [5], so we have the following conclusion.

Theorem 6.2. *For a partial entwining structure $(A, C)_\psi$, the following assertions are equivalent:*
(1) The forgetful functor $F : \mathcal{U}^C_A(\psi) \to \mathcal{M}_A$ is separable,
(2) There exists a k-linear map $\theta : C \otimes C \to A$ such that
(i) For all $c, d \in C$,

$$d_{(1)}{}^{\psi\Psi} \otimes 1_{A\psi'\psi}\theta(d_{(2)}{}^{\psi'} \otimes c)_\Psi = c_{(2)}{}^{\psi'} \otimes 1_{A\psi'\psi\Psi}\theta(d^{\Psi} \otimes c_{(1)}{}^{\psi}),$$

(ii)For all $b \in C$,

$$\theta(b_{(1)} \otimes b_{(2)}) = 1_A\varepsilon(b),$$

(iii) For all $a \in A_\beta, b \in C_{\alpha^{-1}\beta}$ and $d \in C_{(\alpha^{-1}\beta)^{-1}}$,

$$(a_\psi)_\Psi\theta(d^\Psi \otimes b^\psi) = \theta(d \otimes b)a.$$

Corollary 6.3. *Let $(A, C)_\psi$ be a partial π-entwining structure and $M, N \in \mathcal{U}^C_A(\psi)$. Suppose that there exists a partial normalized integral $\theta : C \otimes C \to A$. Then a monomorphism (resp. epimorphism) $f : M \to N$ splits in $\mathcal{U}^C_A(\psi)$ if the monomorphism (resp. epimorphism) f splits as an A-module morphism.*

6.2. Partial Doi-Hopf Group Modules

Let (H, A, C) be a (right-right) partial Doi-Hopf π-structure, where H be a Hopf π-coalgebra, A a right partial π-H-comodule algebra and C a right partial π-H-module coalgebra. From Definition 5.1 and Theorem 5.2, we have

Theorem 6.4. *For a partial Doi-Hopf π-structure (H, A, C) and $\alpha \in \pi$, the following assertions are equivalent:*

(1) The forgetful functor $F^{(\alpha)} : \mathcal{U}_A^{\pi-C} \to \mathcal{M}_{A_\alpha}$ is separable.

(2) There exists a family of k-linear maps

$$\theta^{(\alpha)} = \{\theta_\beta^{(\alpha)} : C_{(\alpha^{-1}\beta)^{-1}} \otimes C_{\alpha^{-1}\beta} \to A_\beta\}_{\beta\in\pi}$$

such that

(i) For all $\beta, \gamma \in \pi$, $c \in C_{\alpha^{-1}\beta\gamma}$ and $d \in C_{(\alpha^{-1}\beta)^{-1}}$,

$$d_{(1,\gamma)} \cdot 1_{A_\alpha[0,\beta\gamma][1,\gamma]}\theta_{\beta\gamma}^{(\alpha)}(d_{(2,(\alpha^{-1}\beta\gamma)^{-1})} \cdot 1_{A_\alpha[1,(\alpha^{-1}\beta\gamma)^{-1}]} \otimes c)_{[1,\gamma]} \otimes 1_{A_\alpha[0,\beta\gamma][0,\beta]}\theta_{\beta\gamma}^{(\alpha)}(d_{(2,(\alpha^{-1}\beta\gamma)^{-1})} \cdot 1_{A_\alpha[1,(\alpha^{-1}\beta\gamma)^{-1}]} \otimes c)_{[0,\beta]} \quad \text{(E6.1)}$$
$$= c_{(2,\gamma)} \cdot 1_{A_{\beta\gamma}[1,\gamma]} \otimes 1_{A_{\beta\gamma}[0,\beta][0,\alpha][0,\beta]}\theta_\beta^{(\alpha)}(d_{(\alpha^{-1}\beta)^{-1}} \cdot 1_{A_{\beta\gamma}[0,\beta][0,\alpha][1,(\alpha^{-1}\beta)^{-1}]} \otimes c_{(1,\alpha^{-1}\beta)} \cdot 1_{A_{\beta\gamma}[0,\beta][1,\alpha^{-1}\beta]}),$$

(ii)For all $\beta \in \pi$, $b \in C_e$,

$$\theta_\beta^{(\alpha)}(b_{(1,(\alpha^{-1}\beta)^{-1})} \otimes b_{(2,\alpha^{-1}\beta)}) = 1_{A_\beta}\varepsilon(b), \quad \text{(E6.2)}$$

(iii) For all $a \in A_\beta, b \in C_{\alpha^{-1}\beta}$ and $d \in C_{(\alpha^{-1}\beta)^{-1}}$,

$$a_{[0,\alpha][0,\beta]}\theta_\beta^{(\alpha)}(d \cdot a_{[0,\alpha][1,(\alpha^{-1}\beta)^{-1}]} \otimes b \cdot a_{[1,\alpha^{-1}\beta]}) = \theta_\beta^{(\alpha)}(d \otimes b)a. \quad \text{(E6.3)}$$

We call that a family of k-linear maps

$$\theta^{(\alpha)} = \{\theta_\beta^{(\alpha)} : C_{(\alpha^{-1}\beta)^{-1}} \otimes C_{\alpha^{-1}\beta} \to A_\beta\}_{\beta\in\pi}$$

which satisfies (E6.1)-(E6.3) is the partial nomalised integral for the Doi-Hopf π-structure.

6.3. Partial Relative Group Modules

Let H be a Hopf π-coalgebra and A a partial π-H-comodule algebra. Then the threetupe (H, A, H) is a partial Doi-Hopf π-data. From Theorem 6.4, we have

Theorem 6.5. *Let H be a Hopf π-coalgebra, A a right partial π-H-comodule algebra. For a fixed $\alpha \in \pi$, the following assertions are equivalent:*

(1) The forgetful functor $F^{(\alpha)} : \mathcal{U}_A^{\pi-H} \to \mathcal{M}_{A_\alpha}$ is separable.

(2) There exists a family of k-linear maps

$$\theta^{(\alpha)} = \{\theta_\beta^{(\alpha)} : H_{(\alpha^{-1}\beta)^{-1}} \otimes H_{\alpha^{-1}\beta} \to A_\beta\}_{\beta\in\pi}$$

such that

(i) For all $\beta, \gamma \in \pi$, $h \in H_{\alpha^{-1}\beta\gamma}$ and $d \in H_{(\alpha^{-1}\beta)^{-1}}$,

$$d_{(1,\gamma)}1_{A_\alpha[0,\beta\gamma][1,\gamma]}\theta_{\beta\gamma}^{(\alpha)}(d_{(2,(\alpha^{-1}\beta\gamma)^{-1})}1_{A_\alpha[1,(\alpha^{-1}\beta\gamma)^{-1}]} \otimes h)_{[1,\gamma]} \otimes 1_{A_\alpha[0,\beta\gamma][0,\beta]}\theta_{\beta\gamma}^{(\alpha)}(d_{(2,(\alpha^{-1}\beta\gamma)^{-1})}1_{A_\alpha[1,(\alpha^{-1}\beta\gamma)^{-1}]} \otimes h)_{[0,\beta]} \quad \text{(F6.1)}$$
$$= h_{(2,\gamma)}1_{A_{\beta\gamma}[1,\gamma]} \otimes 1_{A_{\beta\gamma}[0,\beta][0,\alpha][0,\beta]}\theta_\beta^{(\alpha)}(d_{(\alpha^{-1}\beta)^{-1}}1_{A_{\beta\gamma}[0,\beta][0,\alpha][1,(\alpha^{-1}\beta)^{-1}]} \otimes h_{(1,\alpha^{-1}\beta)}1_{A_{\beta\gamma}[0,\beta][1,\alpha^{-1}\beta]}),$$

(ii)For all $\beta \in \pi$, $b \in H_e$,

$$\theta_\beta^{(\alpha)}(b_{(1,(\alpha^{-1}\beta)^{-1})} \otimes b_{(2,\alpha^{-1}\beta)}) = 1_{A_\beta}\varepsilon(b), \quad \text{(F6.2)}$$

(iii) For all $a \in A_\beta, b \in H_{\alpha^{-1}\beta}$ and $d \in H_{(\alpha^{-1}\beta)^{-1}}$,

$$a_{[0,\alpha][0,\beta]}\theta_\beta^{(\alpha)}(da_{[0,\alpha][1,(\alpha^{-1}\beta)^{-1}]} \otimes ba_{[1,\alpha^{-1}\beta]}) = \theta_\beta^{(\alpha)}(d \otimes b)a. \quad \text{(F6.3)}$$

We will now introduce the partial total integral for the partial right π-comodule algebra, and investigate the difference between the partial total integral and the total integral in sense of Doi.

Proposition 6.6. *Let H be a Hopf π-coalgebra and A a right partial π-H-comodule algebra. For a fixed $\alpha \in \pi$. If*

$$\theta^{(\alpha)} = \{\theta^{(\alpha)}_{\beta} : H_{(\alpha^{-1}\beta)^{-1}} \otimes H_{\alpha^{-1}\beta} \to A_{\beta}\}_{\beta\in\pi}$$

is the partial normalised integral for (H, A, H), the family of k-linear map

$$\varphi^{(\alpha)} = \{\varphi^{(\alpha)}_{\beta} : H_{\alpha^{-1}\beta} \to A_{\beta}, \varphi^{(\alpha)}_{\beta}(h) = \theta^{(\alpha)}_{\beta}(1_{\beta^{-1}\alpha} \otimes h)\}_{\beta\in\pi},$$

satisfies the following relations:

$$\varphi^{(\alpha)}_{\beta}(h_{(1,\alpha^{-1}\beta)})1_{A_{\beta\gamma}[0,\beta]} \otimes h_{(2,\gamma)}1_{A_{\beta\gamma}[1,\gamma]} = \varphi^{(\alpha)}_{\beta\gamma}(h)_{[0,\beta]} \otimes \varphi^{(\alpha)}_{\beta\gamma}(h)_{[1,\gamma]} \tag{G6.1}$$

for any $\beta, \gamma \in \pi$ and $h \in H_{\alpha^{-1}\beta\gamma}$ and

$$\varphi^{(\alpha)}_{\beta}(1_{\alpha^{-1}\beta}) = 1_{A_{\beta}}. \tag{G6.2}$$

Proof. Notice first that

$$\varphi^{(\alpha)}_{\beta}(1_{\alpha^{-1}\beta}) = \theta^{(\alpha)}_{\beta}(1_{\beta^{-1}\alpha} \otimes 1_{\alpha^{-1}\beta}) \overset{(F6.2)}{=} 1_{A_{\beta}}.$$

Since

$$\begin{aligned}
& h_{(2,\gamma)}1_{A_{\beta\gamma}[1,\gamma]} \otimes \varphi^{(\alpha)}_{\beta}(h_{(1,\alpha^{-1}\beta)})1_{A_{\beta\gamma}[0,\beta]} \\
= \ & h_{(2,\gamma)}1_{A_{\beta\gamma}[1,\gamma]} \otimes \theta^{(\alpha)}_{\beta}(1_{\beta^{-1}\alpha} \otimes h_{(1,\alpha^{-1}\beta)})1_{A_{\beta\gamma}[0,\beta]} \\
\overset{(F6.2)}{=} \ & h_{(2,\gamma)}1_{A_{\beta\gamma}[1,\gamma]} \otimes 1_{A_{\beta\gamma}[0,\beta][0,\alpha][0,\beta]}\theta^{(\alpha)}_{\beta}(1_{A_{\beta\gamma}[0,\beta][0,\alpha][1,\beta^{-1}\alpha]} \otimes h_{(1,\alpha^{-1}\beta)}1_{A_{\beta\gamma}[0,\beta][1,\alpha^{-1}\beta]}) \\
\overset{(F6.1)}{=} \ & 1_{A_{\alpha}[0,\beta\gamma][1,\gamma]}\theta^{(\alpha)}_{\beta\gamma}(1_{A_{\alpha}[1,(\beta\gamma)^{-1}\alpha]} \otimes h)_{[1,\gamma]} \otimes 1_{A_{\alpha}[0,\beta\gamma][0,\beta]}\theta^{(\alpha)}_{\beta\gamma}(1_{A_{\alpha}[1,(\beta\gamma)^{-1}\alpha]} \otimes h)_{[0,\beta]} \\
= \ & (1_{A_{\alpha}[0,\beta\gamma]}\theta^{(\alpha)}_{\beta\gamma}(1_{A_{\alpha}[1,(\beta\gamma)^{-1}\alpha]} \otimes h))_{[1,\gamma]} \otimes (1_{A_{\alpha}[0,\beta\gamma]}\theta^{(\alpha)}_{\beta\gamma}(1_{A_{\alpha}[1,(\beta\gamma)^{-1}\alpha]} \otimes h))_{[0,\beta]} \\
= \ & \theta^{(\alpha)}_{\beta\gamma}(1_{A_{\alpha}} \otimes h)_{[1,\gamma]} \otimes \theta^{(\alpha)}_{\beta\gamma}(1_{A_{\alpha}} \otimes h)_{[0,\beta]}.
\end{aligned}$$

So we get the desired result. □

Definition 6.7. *Fix an $\alpha \in \pi$. A family of k-linear maps*

$$\varphi^{(\alpha)} = \{\varphi^{(\alpha)}_{\beta} : H_{\alpha^{-1}\beta} \to A_{\beta}\}_{\beta\in\pi}$$

is called a partial total integral integral for partial group comodule algebra associated to α, if $\varphi^{(\alpha)}$ satisfies the conditions (G6.1) and (G6.2).

Remark 6.8. *If π is a trivial group and $1_{A[0]} \otimes 1_{A[1]} = 1_A \otimes 1_H$, the partial total integral reduces to the form in sense of Doi ([13]).*

6.4. Partial π-entwining Structure in Example 3.1

Corollary 6.9. *Under the assumptions of Example 3.1. Then the following statements are equivalent:*

(1) The forgetful functor $F^{(e)} : \mathcal{U}^{\pi-H} \to M_k$(the category of all vector spaces) is separable.

(2) There exist a family of k-linear maps $\theta = \{\theta_{\beta} : H_{\beta^{-1}} \otimes H_{\beta} \to k\}_{\beta\in\pi}$ such that the following conditions are satisfied

(i) For all $\beta, \gamma \in \pi$, $c \in H_{\beta\gamma}$ and $d \in H_{\beta^{-1}}$,

$$\theta_{\beta\gamma}(p_{(\beta\gamma)^{-1}}d_{(2,(\beta\gamma)^{-1})} \otimes c)p_{\gamma}d_{(1,\gamma)} = \theta_{\beta}(d \otimes c_{(1,\beta)})p_{\gamma}c_{(2,\gamma)}, \tag{G6.3}$$

(ii)For all $\beta \in \pi$, $b \in H_e$,

$$\theta_\beta(b_{(1,\beta^{-1})} \otimes b_{(2,\beta)}) = \varepsilon(b), \tag{G6.4}$$

(iii) For all $b \in H_\beta$ and $d \in H_{\beta^{-1}}$.

$$\theta_\beta(p_{\beta^{-1}} d \otimes p_\beta b) = \theta_\beta(d \otimes b). \tag{G6.5}$$

Take $p = \{1_\alpha\}_{\alpha\in\pi}$. Then partial π-entwining structure in Example 3.1 is just the π-entwining structure in sense of Wang in [26]. Recall from [24] that a *left(resp. right) π-integral* for H is a family of k-linear forms $\lambda = \{\lambda\}_{\alpha\in\pi} \in \prod_{\alpha\in\pi} H_\alpha^*$ such that, for all $\alpha, \beta \in \pi$,

$$(id_{H_\alpha} \otimes \lambda_\beta) \circ \Delta_{\alpha,\beta} = \lambda_{\alpha\beta} 1_\alpha, \ (resp.\ (\lambda_\alpha \otimes id_{H_\beta}) \circ \Delta_{\alpha,\beta} = \lambda_{\alpha\beta} 1_\beta).$$

Note that λ_e is a usual left (resp. right) integral for the Hopf algebra H_e^*. Suppose that $\lambda = \{\lambda_\alpha\}_{\alpha\in\pi}$ is a right π-integral, a family of k-linear maps $\theta = \{\theta_\alpha : H_{\alpha^{-1}} \otimes H_\alpha \to k\}_{\alpha\in\pi}$ are defined by

$$\theta_\alpha(h \otimes g) = \lambda_\alpha(g S_\alpha^{-1}(h)),\ h \in H_{\alpha^{-1}}, g \in H_\alpha.$$

Lemma 6.10. *For any $\alpha, \beta \in \pi$, $g \in H_\alpha$, $h \in H_{\alpha\beta}$, we have*

$$(\lambda_\alpha, g h_{(1,\alpha)}) h_{(2,\beta)} = (\lambda_{\alpha\beta}, g_{(1,\alpha\beta)} h) S_\beta^{-1}(g_{(2,\beta^{-1})}).$$

Proof. For all $\alpha, \beta \in \pi$, $g \in H_\alpha$, $h \in H_{\alpha\beta}$, we have

$$\begin{aligned}
(\lambda_\alpha, g h_{(1,\alpha)}) h_{(2,\beta)} &= \varepsilon(g_{(2,e)})(\lambda_\alpha, g_{(1,\alpha)} h_{(1,\alpha)}) h_{(2,\beta)} \\
&= S_\beta^{-1}(g_{(2,e)(2,\beta^{-1})}) g_{(2,e)(1,\beta)} (\lambda_\alpha, g_{(1,\alpha)} h_{(1,\alpha)}) h_{(2,\beta)} \\
&= S_\beta^{-1}(g_{(1,\beta^{-1})}) g_{(1,\alpha\beta)(2,\beta)} (\lambda_\alpha, g_{(1,\alpha\beta)(1,\alpha)} h_{(1,\alpha)}) h_{(2,\beta)} \\
&= S_\beta^{-1}(g_{(1,\beta^{-1})}) (\lambda_{\alpha\beta}, g_{(1,\alpha\beta)} h).
\end{aligned}$$

The proof of the lemma is completed. □

By Lemma 6.10, we can check that θ satisfies (G6.3)-(G6.5). So we have

Corollary 6.11. *Let H be a cosemisimple Hopf π-coalgebra. Then the forgetful functor $F^{(e)} : \mathcal{U}^{\pi-H} \to M_k$ is separable.*

Proof. Since H is cosemisimple, it follows that there exists a right π-integral $\lambda = \{\lambda_\alpha\}_{\alpha\in\pi}$ such that $\lambda_\alpha(1_\alpha) = 1$. The desired partial normalised integral θ can be constructed by using λ as above. □

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