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α-ψ-Geraghty Contraction Type Mappings and Some Related Fixed Point Results

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Abstract. In this paper, we consider a generalization of α-ψ-Geraghty contractions and investigate the existence and uniqueness of fixed point for the mapping satisfying this condition. We illustrate an example and an application to support our results. In particular, we extend, improve and generalize some earlier results in the literature on this topic.

1. Introduction and preliminaries

A fixed-point theory investigate whether a function T, defined on abstract space, have at least one fixed point (a point x such that $Tx = x$), under some conditions on T and on abstract space. Furthermore, the uniqueness of fixed point is examined if existence of fixed point(s) of T is guaranteed. Results of this theory have been used in many fields and directions. In particular, fixed point theory techniques plays a crucial role in the solutions of differential equations and hence improvement of fixed point theory develop the differential equations theory. Banach contraction principle [3] is one of the initial and also fundamental results in the theory of fixed point: Every contraction on a complete metric space has a unique fixed point. Regarding the application potential of theory, several authors have studied to generalize, improve and extend the fixed point theory by defining new contractive conditions and replacing complete metric spaces with some convenient abstract space (see e.g. [1],[2]-[18].) Among them, we mention one of interesting results given by Geraghty [8]. In this remarkable report [8], the author extended the Banach contraction mapping principle in complete metric space. For the sake of completeness, we recall Geraghty's theorem. For this purpose, we first remind the class of $\mathcal{F}$ all functions $\beta : [0,\infty) \to [0,1)$ which satisfies the condition:

$$\lim_{n\to\infty} \beta(t_n) = 1 \text{ implies } \lim_{n\to\infty} t_n = 0.$$

Also, the author proved the following result:

Theorem 1.1. *(Geraghty [8].) Let (X,d) be a complete metric space and $T : X \to X$ be an operator. If T satisfies the following inequality:*

$$d(Tx,Ty) \leq \beta(d(x,y))d(x,y), \text{ for any } x,y \in X, \tag{1}$$

where $\beta \in \mathcal{F}$, then T has a unique fixed point.

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The results of Geraghty have attracted a numbers of authors [2, 4–7, 12].

On the other hand, recently, Samet *et al.* [16] reported interesting fixed point results by introducing the notion of α-ψ-contractive mappings. Let $T : X \to X$ be a map and $\alpha : X \times X \to \mathbb{R}$ be a function. Then, T is said to be *α-admissible* [16] if

$$\alpha(x, y) \geq 1 \text{ implies } \alpha(Tx, Ty) \geq 1.$$

Very recently, the authors of [13] improved this idea by defining the concept of α-ψ-Meir-Keeler contractive mappings. An α-admissible map T is said to be *triangular α-admissible* [13] if

$$\alpha(x, z) \geq 1 \text{ and } \alpha(z, y) \geq 1 \text{ imply } \alpha(x, y) \geq 1. \tag{2}$$

For more details and examples of α-admissible maps, see e.g. [13, 14, 16] and also [1, 18].

In this manuscript, the notion of generalized α-ψ-Geraghty contraction type mappings is introduced and the existence and uniqueness of a fixed point of mappings, under the assumption of α-Geraghty contraction, is researched in the setting of complete metric spaces.

2. Fixed point theorems

We recollect the following auxiliary result which will be used efficiently in the proof of main results.

Lemma 2.1. *[13] Let $T : X \to X$ be a triangular α-admissible map. Assume that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then, we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Now, we define the following class of auxiliary functions which will be used densely in the sequel: Let Ψ denote the class of the functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

(a) ψ is nondecreasing;
(b) ψ is subadditive, that is, $\psi(s + t) \leq \psi(s) + \psi(t)$;
(c) ψ is continuous;
(d) $\psi(t) = 0 \Leftrightarrow t = 0$.

We introduce the following contraction.

Definition 2.2. *Let (X, d) be a metric space, and let $\alpha : X \times X \to \mathbb{R}$ be a function. A mapping $T : X \to X$ is said to be a generalized α-ψ-Geraghty contraction if there exists $\beta \in \mathcal{F}$ such that*

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)) \text{ for any } \quad x, y \in X, \tag{3}$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\},$$

and $\psi \in \Psi$.

Notice that if take $\psi(t) = t$ in Definition 2.2, then T is called generalized α-Geraghty contraction mapping [4].

Remark 2.3. *Notice that since $\beta : [0, \infty) \to [0, 1)$, we have*

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)) < \psi(M(x, y)) \text{ for any } x, y \in X \text{ with } x \neq y. \tag{4}$$

Theorem 2.4. *Let (X, d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function, and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

(i) T is generalized α-ψ-Geraghty contraction type map;

(ii) T is triangular α-admissible;
(iii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;
(iv) T is continuous.

Then, T has a fixed point $x^* \in X$, and $\{T^n x_1\}$ converges to x^*.

Proof. Let $x_1 \in X$ be such that $\alpha(x_1, Tx_1) \geq 1$. Define a sequence $\{x_n\} \subset X$ by $x_{n+1} = Tx_n$ for $n \in \mathbb{N}$.

Suppose that $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$. Then, it is clear that x_{n_0} is a fixed point of T and hence the proof is completed. From now on, we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Due to Lemma 2.1, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \tag{5}$$

for all $n \in \mathbb{N}$. Taking (3) into account, we derive

$$\begin{aligned} \psi(d(x_{n+1}, x_{n+2})) = &\ \psi(d(Tx_n, Tx_{n+1})) \\ \leq &\ \alpha(x_n, x_{n+1})\psi(d(Tx_n, Tx_{n+1})) \\ \leq &\ \beta(\psi(M(x_n, x_{n+1})))\psi(M(x_n, x_{n+1})) \end{aligned} \tag{6}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1})\} \\ &= \max\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}. \end{aligned}$$

Notice that the case $M(x_n, x_{n+1}) = d(x_{n+1}, x_{n+2})$ is impossible due to the definition of β. Indeed,

$$\begin{aligned} \psi(d(x_{n+1}, x_{n+2})) &\leq \beta(\psi(M(x_n, x_{n+1})))\psi(M(x_n, x_{n+1})) \\ &\leq \beta(\psi(d(x_{n+1}, x_{n+2})))\psi(d(x_{n+1}, x_{n+2})) < \psi(d(x_{n+1}, x_{n+2})). \end{aligned}$$

Thus, we conclude that $M(x_n, x_{n+1}) = d(x_n, x_{n+1}))$. Keeping the inequality (6) in the mind, we get $\psi(d(x_{n+1}, x_{n+2})) < \psi(d(x_n, x_{n+1}))$ for all $n \in \mathbb{N}$. Regarding the properties of ψ, we conclude that $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Hence, we deduce that the sequence $\{d(x_n, x_{n+1})\}$ is nonnegative and nonincreasing. Consequently, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. We claim that $r = 0$. Suppose, on the contrary, that $r > 0$. Then, due to (6), we have

$$\frac{\psi(d(x_{n+1}, x_{n+2}))}{\psi(M(x_n, x_{n+1}))} \leq \beta(\psi(M(x_n, x_{n+1}))) < 1.$$

In what follows that $\lim\limits_{n\to\infty} \beta(\psi(M(x_n, x_{n+1}))) = 1$. Owing to the fact that $\beta \in \mathcal{F}$, we have

$$\lim_{n\to\infty} \psi(M(x_n, x_{n+1})) = 0, \tag{7}$$

which yields that

$$r = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{8}$$

We observe that

$$\begin{aligned} M(x_m, x_n) &= \max\{d(x_m, x_n), d(x_m, Tx_m), d(x_n, Tx_n)\} \\ &= \max\{d(x_m, x_n), d(x_m, x_{m+1}), d(x_n, x_{n+1})\} \end{aligned}$$

By using the result $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$, we find that

$$\lim_{m,n\to\infty} M(x_m, x_n) = \lim_{m,n\to\infty} d(x_m, x_n). \tag{9}$$

We assert that $\{x_n\}$ is a Cauchy sequence. Suppose, on the contrary, that we have

$$\varepsilon = \limsup_{m,n\to\infty} d(x_n, x_m) > 0. \tag{10}$$

By using the triangular inequality, we derive

$$d(x_n, x_m) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m). \tag{11}$$

Combining (3), (11) with the properties of ψ, we get

$$\begin{aligned}\psi(d(x_n, x_m)) &\le \psi\left(d(x_n, x_{n+1}) + d(Tx_n, Tx_m) + d(x_{m+1}, x_m)\right)\\ &\le \psi\left(d(x_n, x_{n+1})\right) + \psi\left(d(Tx_n, Tx_m)\right) + \psi\left(d(x_{m+1}, x_m)\right)\\ &\le \psi\left(d(x_n, x_{n+1})\right) + \beta(\psi\left(M(x_n, x_m)\right))\,\psi\left(M(x_n, x_m)\right) + \psi\left(d(x_{m+1}, x_m)\right).\end{aligned} \tag{12}$$

Together with (9), (12) and (8), we deduce that

$$\begin{aligned}\lim_{m,n\to\infty} \psi(d(x_n, x_m)) &\le \lim_{m,n\to\infty} \beta(\psi\left(M(x_n, x_m)\right)) \lim_{m,n\to\infty} \psi\left(M(x_m, x_n)\right)\\ &\le \lim_{m,n\to\infty} \beta(\psi\left(M(x_n, x_m)\right)) \lim_{m,n\to\infty} \psi\left(d(x_m, x_n)\right).\end{aligned}$$

Hence by (10), we get

$$1 \le \lim_{m,n\to\infty} \beta(\psi\left(M(x_n, x_m)\right)),$$

which implies $\lim_{m,n\to\infty} \beta(\psi\left(M(x_n, x_m)\right)) = 1$. Consequently, we get $\lim_{m,n\to\infty} M(x_n, x_m) = 0$ and hence $\lim_{m,n\to\infty} d(x_n, x_m) = 0$. It is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. Recalling the completeness of X, we conclude that there exists

$$x^* = \lim_{n\to\infty} x_n \in X.$$

Since the mapping T is continuous, we find $\lim_{n\to\infty} x_n = Tx^*$, and so $x^* = Tx^*$. □

It is also possible to remove the continuity of the mapping T by replacing a weaker condition:

Definition 2.5. *Let (X, d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function, and let $T : X \to X$ be a map. We say that the sequence $\{x_n\}$ is α-regular the following condition is satisfied:*
If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $x_n \to x \in X$ as $n \to +\infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \ge 1$ for all k.

In the following theorem, we omit the continuity condition of the mapping T in Theorem 2.4.

Theorem 2.6. *Let (X, d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function, and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

(i) T is a generalized α-ψ-Geraghty contraction type map;
(ii) T is triangular α-admissible;
(iii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \ge 1$;
(iv) $\{x_n\}$ is α-regular

Then, T has a fixed point $x^* \in X$, and $\{T^n x_1\}$ converges to x^*.

Proof. Following the proof of Theorem 2.4, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \ge 0$, converges to some $x^* \in X$. From (5) and assumption (iv) of the theorem, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} \alpha(x_{n(k)}, x^*) \ge 1$. Applying (3), for all k, we get that

$$\begin{aligned}\alpha(x_{n(k)}, x^*)\psi(d(x_{n(k)+1}, Tx^*)) &= \alpha(x_{n(k)}, x^*)\psi(d(Tx_{n(k)}, Tu))\\ &\le \beta(\psi(M(x_{n(k)}, x^*)))\psi(M(x_{n(k)}, x^*)).\end{aligned} \tag{13}$$

On the other hand, we have

$$\begin{aligned} M(x_{n(k)}, x^*) &= \max\{d(x_{n(k)}, x^*), d(x_{n(k)}, Tx_{n(k)}), d(x^*, Tx^*)\} \\ &= \max\{d(x_{n(k)}, x^*), d(x_{n(k)}, x_{n(k)+1}), d(x^*, Tx^*)\}, \end{aligned}$$

and hence,

$$\lim_{k\to\infty} \psi(M(x_{n(k)}, x^*)) = \psi(d(x^*, Tx^*)). \tag{14}$$

From (13) we have

$$\alpha(x_{n(k)}, x^*)\frac{\psi(d(x_{n(k)+1}, Tx^*))}{\psi(M(x_{n(k)}, x^*))} \leq \beta(\psi(M(x_{n(k)}, x^*))) < 1.$$

Letting $k \to \infty$ in the above inequality, we obtain $\lim_{n\to\infty} \beta(\psi(M(x_{n(k)}, x^*))) = 1$, and so $\psi(d(x^*, Tx^*)) = \lim_{k\to\infty} \psi(M(x_{n(k)}, x^*)) = 0$. Hence, $x^* = Tx^*$. □

For the uniqueness of a fixed point of a α-Geraghty contractive mapping, we will consider the following condition.

(H1) For all $x, y \in \mathrm{Fix}(T)$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.

Theorem 2.7. *Adding condition (H1) to the hypotheses of Theorem 2.4 (resp. Theorem 2.6), we obtain that x^* is the unique fixed point of T.*

Proof. *Due to Theorem 2.4 (resp. Theorem 2.6), we have a fixed point, say $x^* \in X$. Let $y^* \in X$ be another fixed point of T.*

Then, by assumption, there exists $z \in X$ such that

$$\alpha(x^*, z) \geq 1 \textit{ and } \alpha(y^*, z) \geq 1. \tag{15}$$

Since T is $\alpha-$admissible, from (15)*, we have*

$$\alpha(x^*, T^n z) \geq 1 \textit{ and } \alpha(y^*, T^n z) \geq 1, \textit{ for all } n.$$

Hence we have

$$\begin{aligned} d(x^*, T^n z) &\leq \alpha(x^*, T^{n-1}z)d(Tx^*, TT^{n-1}z) \\ &\leq \beta(d(x^*, T^{n-1}z))d(x^*, T^{n-1}z) \\ &< d(x^*, T^{n-1}z) \end{aligned} \tag{16}$$

for all $n \in \mathbb{N}$.

Thus, the sequence $\{d(x^*, T^n z)\}$ is nonincreasing, and there exists $u \geq 0$ such that

$$\lim_{n\to\infty} d(x^*, T^n z) = u.$$

From (16) *we have*

$$\frac{d(x^*, T^n z)}{d(x^*, T^{n-1}z)} \leq \beta(d(x^*, T^{n-1}z))$$

and thus $\lim_{n\to\infty} \beta(d(x^*, T^n z)) = 1$. Hence $\lim_{n\to\infty} d(x^*, T^n z) = 0$ which yields $\lim_{n\to\infty} T^n z = x^*$.

Similarly, we have $\lim_{n\to\infty} T^n z = y^*$. Thus, we have $x^* = y^*$. □

3. Consequences

We start this section with following definition.

Definition 3.1. *Let (X, d) be a metric space, and let $\alpha : X \times X \to \mathbb{R}$ be a function. A map $T : X \to X$ is called α-ψ-Geraghty contraction type map if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$*

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq \beta(\psi(d(x, y)))\psi(d(x, y)), \tag{17}$$

where $\psi \in \Psi$.

Note that if take $\psi(t) = t$ in Definition 3.1, then T is called generalized α-Geraghty contraction mapping [4].

Theorem 3.2. *Let (X, d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function, and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

(1) T is α-ψ-Geraghty contraction type map;
(2) T is triangular α-admissible;
(3) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;
(4) T is continuous.

Then, T has a fixed point $x^* \in X$, and $\{T^n x_1\}$ converges to x^*.

Proof. Let $x_1 \in X$ be such that $\alpha(x_1, Tx_1) \geq 1$.

Following the lines in the proof of Theorem 2.4, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all n, converges to some $x^* \in X$, and $\alpha(x_n, x_{n+1}) \geq 1$ for all n.

Since T is continuous, then obviously, x^* is a fixed point of T.

□

Theorem 3.3. *Let (X, d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function, and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

(i) T is α-Geraghty contraction type map;
(ii) T is triangular α-admissible;
(iii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;
(iv) $\{x_n\}$ is α-regular

Then, T has a fixed point $x^* \in X$, and $\{T^n x_1\}$ converges to x^*.

Proof. Let $x_1 \in X$ be such that $\alpha(x_1, Tx_1) \geq 1$.

Following the lines in the proof of Theorem 2.4, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all n, converges to some $x^* \in X$, and $\alpha(x_n, x_{n+1}) \geq 1$ for all n.

Suppose that the condition (iv) holds. Consequently, we have $\lim_{n\to\infty} \sup \alpha(x_n, x^*) > 0$.

Thus, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} \alpha(x_{n(k)}, x^*) = p > 0$.

Then we have

$$\begin{aligned}
\psi(d(x_{n(k)+1}, Tx^*)) &= \psi(d(Tx_{n(k)}, Tx^*)) \\
&\leq \frac{1}{\alpha(x_{n(k)}, x^*)} \beta(\psi(d(x_{n(k)}, x^*)))\psi(d(x_{n(k)}, x^*)) \\
&\leq \frac{1}{\alpha(x_{n(k)}, x^*)} \psi(d(x_{n(k)}, x^*))
\end{aligned}$$

for all sufficiently large k.

Hence, we obtain

$$\begin{aligned}\psi(d(x^*,Tx^*)) &= \lim_{k\to\infty}\psi(d(x_{n(k)+1},Tx^*))\\ &\leq \frac{1}{p}\lim_{n\to\infty}\psi(d(x_{n(k)},x^*)),\\ &=0.\end{aligned}$$

Therefore, x^* is a fixed point of T. □

Theorem 3.4. *Adding condition (H1) to the hypotheses of Theorem 3.2 (resp. Theorem 3.3), we obtain that x^* is the unique fixed point of T.*

Proof. Due to Theorem 3.2 (resp. Theorem 3.3), we have a fixed point, say $x^* \in X$. Let $y^* \in X$ be another fixed point of T.

Then, by assumption, there exists $z \in X$ such that

$$\alpha(x^*,z) \geq 1 \text{ and } \alpha(y^*,z) \geq 1. \tag{18}$$

Since T is $\alpha-$admissible, from (18), we have

$$\alpha(x^*,T^nz) \geq 1 \text{ and } \alpha(y^*,T^nz) \geq 1, \text{ for all } n.$$

Hence we have

$$\begin{aligned}\psi(d(x^*,T^nz)) \leq\ & \alpha(x^*,T^{n-1}z)\psi(d(Tx^*,TT^{n-1}z))\\ \leq\ & \beta(\psi(d(x^*,T^{n-1}z)))\psi(d(x^*,T^{n-1}z))\\ <\ & \psi(d(x^*,T^{n-1}z))\end{aligned} \tag{19}$$

for all $n \in \mathbb{N}$. Consequently, the sequence $\{\psi(d(x^*,T^nz))\}$ is nonincreasing, and there is $u \geq 0$ such that $\lim_{n\to\infty}\psi(d(x^*,T^nz)) = u$.

From (19) we have

$$\frac{\psi(d(x^*,T^nz))}{\psi(d(x^*,T^{n-1}z))} \leq \beta(\psi(d(x^*,T^{n-1}z)))$$

and hence $\lim_{n\to\infty}\beta(\psi(d(x^*,T^nz))) = 1$. In what follows that $\lim_{n\to\infty}\psi(d(x^*,T^nz)) = 0$ which implies $\lim_{n\to\infty}T^nz = x^*$.

Similarly, we have $\lim_{n\to\infty}T^nz = y^*$. Thus, we have $x^* = y^*$. □

Corollary 3.5. *Let (X,d) be a complete metric space, $\alpha : X\times X \to [0,\infty)$ be a function, and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

(i) T is generalized α-Geraghty contraction type map;
(ii) T is triangular α-admissible;
(iii) there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq 1$;
(iv) either T is continuous or $\{x_n\}$ is α-regular

Then, T has a fixed point $x^* \in X$, and $\{T^nx_1\}$ converges to x^*. Further if, for all $x,y \in \mathrm{Fix}(T)$, there exists $z \in X$ such that $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$, then T has a unique fixed point.

Proof. Combine Theorem 2.4-Theorem 2.7 by taking $\psi(t) = t$. □

Corollary 3.6. *Let (X,d) be a complete metric space, $\alpha : X\times X \to [0,\infty)$ be a function, and let $T : X \to X$ be a map. Suppose that the following conditions hold:*

(i) T *is* α*-Geraghty contraction type map;*
(ii) T *is triangular* α*-admissible;*
(iii) there exists $x_1 \in X$ *such that* $\alpha(x_1, Tx_1) \geq 1$*;*
(iv) either T *is continuous or* $\{x_n\}$ *is* α*-regular*

Then, T *has a fixed point* $x^* \in X$*, and* $\{T^n x_1\}$ *converges to* x^**. Further if, for all* $x, y \in \mathrm{Fix}(T)$*, there exists* $z \in X$ *such that* $\alpha(x, z) \geq 1$ *and* $\alpha(y, z) \geq 1$*, then* T *has a unique fixed point.*

Proof. Combine Theorem 3.2-Theorem 3.4 by taking $\psi(t) = t$. □

In Corollary 3.6, let $\alpha(x, y) = 1$ for all $x, y \in X$. Then, we have the following corollary.

Corollary 3.7. *[8] Let* (X, d) *be a metric space, and let* $T : X \to X$ *be a map. Suppose that there exists* $\beta \in \mathcal{F}$ *such that for all* $x, y \in X$

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y). \tag{20}$$

Then, T *has a unique fixed point* $x^* \in X$*, and* $\{T^n x\}$ *converges to* x^**, for each* $x \in X$*.*

Corollary 3.8. *[9] Let* $(X, \preceq)$ *be a partially ordered set and suppose that there exists a metric* d *on* X *such that* (X, d) *is a complete metric space. Suppose that* $T : X \to X$ *is a map. Assume that the following conditions are satisfied.*

(i) there exists $\beta \in \mathcal{F}$ *such that*

$$\psi(d(Tx, Ty)) \leq \beta(\psi(d(x, y)))\psi(d(x, y))$$

for all $x, y \in X$ *with* $y \preceq x$*, where* $\psi \in \Psi$*,*

(ii) there exists $x_1 \in X$ *such that* $x_1 \preceq Tx_1$*;*
(iii) T *is increasing;*
(iv) either T *is continuous or, if* $\{x_n\}$ *is increasing sequence with* $\lim_{n\to\infty} x_n = x$*, then* $x_n \preceq x$ *for all* $n \in \mathbb{N}$*.*

Then, T *has a fixed point* $x^* \in X$*, and* $\{T^n x_1\}$ *converges to* x^**. Further if, for any* $x, y \in X$*, there exists* $z \in X$ *such that* z *is comparable to* x *and* y*, then* T *has a unique fixed point in* X*.*

Proof. Define a function $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{otherwise.} \end{cases}$$

Then, from (*i*) we have $\alpha(x, y)\psi(d(Ty, Tx)) \leq \beta(\psi(d(x, y)))\psi(d(x, y))$ for all $x, y \in X$, and hence (17) is satisfied.

Obviously, condition (2) is satisfied. Since T is increasing, $\alpha(x, y) = 1$ implies $\alpha(Tx, Ty) = 1$ for all $x, y \in X$. Thus, the condition (*iii*) of Theorem 3.2 and Theorem 3.3are satisfied.

Condition (*ii*) implies that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) = 1$, and so condition (*iii*) of Theorem 3.2 and Theorem 3.3 are satisfied.

Condition (*iv*) implies that condition (*iv*) of Theorem 3.2 and Theorem 3.3 satisfied.

Thus, all conditions of Theorem 3.2 and Theorem 3.3 are satisfied. By Corollary 3.6, T has a fixed point in X. □

If we take $\psi(t) = t$ in Corollary 3.8, then we get the following result.

Corollary 3.9. *[2] Let* $(X, \preceq)$ *be a partially ordered set and suppose that there exists a metric* d *on* X *such that* (X, d) *is a complete metric space. Suppose that* $T : X \to X$ *is a map. Assume that the following conditions hold:*

(i) there exists $\beta \in \mathcal{F}$ *such that*

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y)$$

for all $x, y \in X$ *with* $y \preceq x$*;*

(ii) there exists $x_1 \in X$ such that $x_1 \preceq Tx_1$;
(iii) T is increasing;
(iv) either T is continuous or if $\{x_n\}$ is increasing sequence with $\lim_{n\to\infty} x_n = x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Then, T has a fixed point $x^* \in X$, and $\{T^n x_1\}$ converges to x^*. Further if, for any $x, y \in X$, there exists $z \in X$ such that z is comparable to x and y, then T has a unique fixed point in X.

We give an example to illustrate Theorem 3.2.

Example 3.10. *Let $X = [0,\infty)$, and let $d(x,y) = | x - y |$ for all $x, y \in X$. Let $\beta(t) = \frac{1}{1+t}$ for all $t \geq 0$. Then, $\beta \in \mathcal{F}$. Let $\psi(t) = \frac{t}{2}$ and a mapping $T : X \to X$ be defined by*

$$Tx = \begin{cases} \dfrac{1}{3}x & (0 \leq x \leq 1), \\ 3x & (x > 1). \end{cases}$$

Also, we define a function $\alpha : X \times X \to [0,\infty)$ in the following way

$$\alpha(x,y) = \begin{cases} 1 & (0 \leq x, y \leq 1), \\ 0 & \text{otherwise}. \end{cases}$$

Condition (iii) of Theorem 3.2 is satisfied with $x_1 = 1$. Condition (iv) of Theorem 3.2 is satisfied with $x_n = T^n x_1 = \frac{1}{3^n}$. Obviously, condition (ii) is satisfied. Let $x, y \in X$ be such that $\alpha(x,y) \geq 1$. Then, $x, y \in [0,1]$, and so $Tx \in [0,1]$, $Ty \in [0,1]$ and $\alpha(Tx,Ty) = 1$. Hence T is α-admissible, and hence (ii) is satisfied. Finally, we shall prove that (i) is satisfied.

If $0 \leq x, y \leq 1$, then $\alpha(x,y) = 1$, and we have

$$\begin{aligned} \beta(\psi(d(x,y)))\psi(d(x,y)) - \alpha(x,y)\psi(d(Tx,Ty)) &= \beta(\psi(d(x,y)))\psi(d(x,y)) - \psi(d(Tx,Ty)) \\ &= \frac{\frac{|x-y|}{2}}{1 + \frac{|x-y|}{2}} - \frac{1}{6} | x - y | \\ &= \frac{| x - y | (6 - 2 | x - y |)}{6(2 + | x - y |)} \\ &\geq 0. \end{aligned}$$

Therefore, we derive that $\alpha(x,y)\psi(d(Tx,Ty)) \leq \beta(\psi(d(x,y)))\psi(d(x,y))$ for $0 \leq x, y \leq 1$. If $0 \leq x \leq 1$ and $y > 1$, then, obviously, we have $\alpha(x,y)\psi(d(Tx,Ty)) \leq \beta(\psi(d(x,y)))\psi(d(x,y))$, since $\alpha(x,y) = 0$. Consequently, all assumptions of Theorem 3.2 are satisfied, and hence T has a fixed point $x^* = 0$.

We also notice that (20) is not satisfied. In fact, for $x = 1, y = 2$, we have

$$d(T1,T2) = \frac{17}{3} > \frac{1}{2} > \beta(d(2,1))d(2,1).$$

4. Application to ordinary differential equations

We consider the following two-point boundary value problem of second order differential equation:

$$\begin{cases} -\frac{d^2x}{dt^2} = f(t,x(t)), & t \in [0,1] \\ x(0) = x(1) = 0, \end{cases} \tag{21}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous function.

The Green function associated to (21) is given by

$$G(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1 \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases}$$

Let $C(I)$ be the space of all continuous functions defined on I, where $I = [0, 1]$, and let

$$d(x,y) = \|x-y\|_\infty = \sup_{t\in I} \mid x(t) - y(t) \mid \text{ for all } x, y \in C(I).$$

Then, $(C(I), d)$ is a complete metric space.

We consider the following conditions:

(a) there exists a function $\xi : \mathbb{R}^2 \to \mathbb{R}$ such that for all $t \in I$, for all $a, b \in \mathbb{R}$ with $\xi(a, b) \geq 0$, we have

$$\mid f(t,a) - f(t,b) \mid \leq \ln(\mid a-b \mid +1);$$

(b) there exists $x_1 \in C(I)$ such that for all $t \in I$

$$\xi(x_1(t), \int_0^1 G(t,s) f(s, x_1(s))ds) \geq 0;$$

(c) for all $t \in I$ and for all $x, y \in C(I)$,

$$\xi(x(t), y(t)) \geq 0 \text{ implies } \xi\left(\int_0^1 G(t,s)f(s,x(s))ds, \int_0^1 G(t,s)f(s,y(s))ds\right) \geq 0;$$

(d) for any cluster point x of a sequence $\{x_n\}$ of points in $C(I)$ with $\xi(x_n, x_{n+1}) \geq 0$, $\lim_{n\to\infty} \inf \xi(x_n, x) \geq 0$.

Theorem 4.1. *Suppose that conditions (a)-(d) are satisfied. Then, (21) has at least one solution $x^* \in C^2(I)$.*

Proof. It is known that $x \in C^2(I)$ is a solution of (21) if and only if $x \in C(I)$ is a solution of the integral equation

$$x(t) = \int_0^1 G(t,s) f(s,x(s))ds \text{ for all } t \in I.$$

We define $T : C(I) \to C(I)$ by

$$Tx(t) = \int_0^1 G(t,s) f(s,x(s))ds \text{ for all } t \in I.$$

Then, the problem (21) is equivalent to finding $x^* \in C(I)$ that is a fixed point of T.
Let $x, y \in C(I)$ such that $\xi(x(t), y(t)) \geq 0$ for all $t \in I$.
From (a) we have

$$\begin{aligned} d(Tx,Ty) = \mid Tx(t) - Ty(t) \mid &= \left| \int_0^1 G(t,s)[f(s,x(s)) - f(s,y(s))]ds \right| \\ &\leq \int_0^1 G(t,s) \mid f(s,x(s)) - f(s,y(s)) \mid ds \\ &\leq \int_0^1 G(t,s) \ln(\mid x(s) - y(s) \mid +1)ds \\ &\leq \sup_{t\in I} \int_0^1 G(t,s)ds \ln(\mid x(s) - y(s) \mid +1) \\ &= \frac{1}{8} \ln(\mid x(s) - y(s) \mid +1) \leq \ln(\mid x(s) - y(s) \mid +1) = \ln(d(x,y)+1) \end{aligned}$$

which yields that

$$\ln(d(Tx,Ty)+1) \le \ln(\ln(d(x,y)+1)+1) = \frac{\ln(\ln(d(x,y)+1)+1)}{\ln(d(x,y)+1)} \ln(d(x,y)+1)$$

Put $\psi(x) = \ln(x+1)$ and $\beta(x) = \frac{\psi(x)}{x}$. Obviously, $\psi : [0,\infty) \to [0,\infty)$ is continuous, subadditive, nondecreasing and ψ is positive in $(0,\infty)$ with $\psi(0) = 0$ and also $\psi(x) < x$ for any $\beta \in \mathcal{F}$.

Thus we have $\psi(d(Tx,Ty)) < \beta(\psi(d(x,y)))\psi(d(x,y))$ for all $x, y \in C(I)$ such that $\xi(x(t), y(t)) \ge 0$ for all $t \in I$.

We define $\alpha : C(I) \times C(I) \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } \xi(x(t),y(t)) \ge 0, t \in I, \\ 0, & \text{otherwise .} \end{cases}$$

Then, for all $x, y \in C(I)$, we have

$$\alpha(x,y)d(Tx,Ty) < \beta(d(x,y))d(x,y).$$

Obviously, $\alpha(x,y) = 1$ and $\alpha(y,z) = 1$ implies $\alpha(x,z) = 1$ for all $x, y, z \in C(I)$.

If $\alpha(x,y) = 1$ for all $x, y \in C(I)$, then $\xi(x(t), y(t)) \ge 0$. From (c) we have $\xi(Tx(t), Ty(t)) \ge 0$, and so $\alpha(Tx,Ty) = 1$. Thus, T is triangular α-admissible.

From (b) there exists $x_1 \in C(I)$ such that $\alpha(x_1, Tx_1) = 1$.

By (d), we have that, for any cluster point x of a sequence $\{x_n\}$ of points in $C(I)$ with $\alpha(x_n, x_{n+1}) = 1$, $\liminf_{n\to\infty} \alpha(x_n, x) = 1$.

By applying Theorem 3.2, T has a fixed point in $C(I)$, i.e. there exists $x^* \in C(I)$ such that $Tx^* = x^*$, and x^* is a solution of (21). □

Remark 4.2. *Notice that from Theorem 3.2 and Theorem 3.3. we deduce the main theorem of [9], Corollary 3.8. Consequently, we can easily derive the their application, the following initial-value problem*

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + F(x,t,u,u_x), & -\infty < x < \infty,\ 0 < t < T \\ u(x,t) = \varphi(x), & -\infty < x < \infty \end{cases} \tag{22}$$

where we assumed that φ is continuously differentiable and that φ and φ' are bounded and $F(x,t,u,u_x)$ is a continuous function. For more detail see [9].

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