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A New Note on Almost Increasing and Quasi Monotone Sequences

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Abstract. In [22], we proved a main theorem dealing an application of almost increasing and quasi monotone sequences. In this paper, we prove that theorem under weaker conditions. We also obtained some new and known results.

1. Introduction

A positive sequence (b_n) is said to be an almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). A sequence (d_n) is said to be δ-quasi monotone, if $d_n \to 0$, $d_n > 0$ ultimately and $\Delta d_n \geq -\delta_n$, where $\Delta d_n = d_n - d_{n+1}$ and δ=(δ_n) is a sequence of positive numbers (see [2]). Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by t_n the nth (C,1) mean of the sequence (na_n). A series $\sum a_n$ is said to be summable $\mid C,1 \mid_k$, $k \geq 1$, if (see [24])

$$\sum_{n=1}^{\infty} \frac{1}{n} \mid t_n \mid^k < \infty. \tag{1}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{2}$$

The sequence-to-sequence transformation

$$R_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{3}$$

defines the sequence (R_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [25]). Let (θ_n) be any sequence of positive constants. The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \mid R_n - R_{n-1} \mid^k < \infty, \tag{4}$$

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and it is said to be summable $\mid \bar{N}, p_n, \theta_n \mid_k, k \geq 1$, if (see [27])

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \mid R_n - R_{n-1} \mid^k < \infty. \tag{5}$$

In the special case $p_n = 1$ for all values of n, $\mid \bar{N}, p_n \mid_k$ summability is the same as $\mid C, 1 \mid_k$ summability. If we take $\theta_n = \frac{P_n}{p_n}$, then $\mid \bar{N}, p_n, \theta_n \mid_k$ summability reduces to $\mid \bar{N}, p_n \mid_k$ summability. Also, if we take $\theta_n = n$ and $p_n = 1$ for all values of n, then we get $\mid C, 1 \mid_k$ summability. Furthermore, if we take $\theta_n = n$, then $\mid \bar{N}, p_n, \theta_n \mid_k$ summability reduces to $\mid R, p_n \mid_k$ summability (see [4]). Finally, if we take $p_n = 1$ for all values of n, then we get $\mid C, 1, \theta_n \mid_k$ summability.

2. Known result

Many works dealing with an application of increasing sequences to the some absolute summability methods of infinite series have been done (see [5-23], [26], [29]). Among them, in [22], the following main theorem has been proved.

Theorem 2.1 Let (X_n) be an almost increasing sequence such that $\mid \Delta X_n \mid = O(X_n/n)$ and let $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (A_n) such that it is δ-quasi-monotone with $\sum n\delta_n X_n < \infty$, $\sum A_n X_n$ is convergent and $\mid \Delta\lambda_n \mid \leq \mid A_n \mid$ for all n. If the conditions

$$\sum_{n=1}^{m} \frac{1}{n} \mid \lambda_n \mid = O(1) \quad as \quad m \to \infty, \tag{6}$$

$$\sum_{n=1}^{m} \frac{1}{n} \mid t_n \mid^k = O(X_m) \quad as \quad m \to \infty, \tag{7}$$

and

$$\sum_{n=1}^{m} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \mid t_n \mid^k = O(X_m) \quad as \quad m \to \infty \tag{8}$$

are satisfied, then the series $\sum a_n\lambda_n$ is summable $\mid \bar{N}, p_n, \theta_n \mid_k$, $k \geq 1$, where (θ_n) is any sequence of positive constants such that $\left(\frac{\theta_n p_n}{P_n}\right)$ is a non-increasing sequence.

3. The main result

The aim of this paper is to prove Theorem 2.1 under weaker conditions. Now we shall prove the following theorem.

Theorem 3.1 Let (X_n) be an almost increasing sequence such that $\mid \Delta X_n \mid = O(X_n/n)$ and let $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (A_n) such that it is δ-quasi-monotone with $\sum n\delta_n X_n < \infty$, $\sum A_n X_n$ is convergent and $\mid \Delta\lambda_n \mid \leq \mid A_n \mid$ for all n. If the condition (6) of Theorem 2.1 is satisfied and if the conditions

$$\sum_{n=1}^{m} \frac{\mid t_n \mid^k}{n X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty \tag{9}$$

$$\sum_{n=1}^{m} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{\mid t_n \mid^k}{X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{10}$$

are satisfied, where (θ_n) is as in Theorem B, then the series $\sum a_n\lambda_n$ is summable $|\bar{N}, p_n, \theta_n|_k$, $k \geq 1$.

Remark 3.2 It should be noted that conditions (9) and (10) are the same as conditions (7) and (8), respectively, when k=1. When $k > 1$, conditions (9) and (10) are weaker than conditions (7) and (8), respectively, but the converses are not true. As in [28] we can show that if (7) is satisfied, then we get that

$$\sum_{n=1}^{m} \frac{|t_n|^k}{nX_n^{k-1}} = O\left(\frac{1}{X_1^{k-1}}\right)\sum_{n=1}^{m} \frac{|t_n|^k}{n} = O(X_m).$$

If (9) is satisfied, then for $k > 1$ we obtain that

$$\sum_{n=1}^{m} \frac{|t_n|^k}{n} = \sum_{n=1}^{m} X_n^{k-1}\frac{|t_n|^k}{nX_n^{k-1}} = O(X_m^{k-1})\sum_{n=1}^{m} \frac{|t_n|^k}{nX_n^{k-1}} = O(X_m^k) \neq O(X_m).$$

The similar argument is also valid for the conditions (8) and (10).
We need following lemmas for the proof of our theorem.
Lemma 3.3 ([5]) Under the conditions of the theorem, we have that

$$|\lambda_n| X_n = O(1) \quad as \quad n \to \infty. \tag{11}$$

Lemma 3.4 ([6]) Let (X_n) be an almost increasing sequence such that $n \mid \Delta X_n \mid = O(X_n)$. If (A_n) is a δ-quasi-monotone with $\sum n\delta_n X_n < \infty$, and $\sum A_n X_n$ is convergent, then

$$nA_nX_n = O(1) \quad as \quad n \to \infty, \tag{12}$$

$$\sum_{n=1}^{\infty} nX_n \mid \Delta A_n \mid < \infty. \tag{13}$$

4. Proof of Theorem 3.1 Let (T_n) be denote the $(\bar{N}, p_n)$ mean of the series $\sum a_n\lambda_n$. Then, by definition and changing the order of summation, we have

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{i=0}^{v} a_i\lambda_i = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})a_v\lambda_v.$$

Then, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} P_{v-1}a_v\lambda_v = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} \frac{P_{v-1}\lambda_v}{v} va_v.$$

By Abel's transformation, we have

$$\begin{aligned} T_n - T_{n-1} &= \frac{n+1}{nP_n}p_nt_n\lambda_n - \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} p_vt_v\lambda_v\frac{v+1}{v} + \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_v\Delta\lambda_vt_v\frac{v+1}{v} \\ &+ \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_vt_v\lambda_{v+1}\frac{1}{v} \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}. \end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality for $k > 1$, it is enough to show that

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \mid T_{n,r} \mid^k < \infty, \quad for \quad r = 1, 2, 3, 4.$$

Firstly, we have that

$$\begin{aligned}
\sum_{n=1}^{m}\theta_n^{k-1}\mid T_{n,1}\mid^k &= \sum_{n=1}^{m}\theta_n^{k-1}\mid\lambda_n\mid^{k-1}\mid\lambda_n\mid\left(\frac{p_n}{P_n}\right)^k\mid t_n\mid^k\\
&= O(1)\sum_{n=1}^{m}\mid\lambda_n\mid\theta_n^{k-1}\left(\frac{1}{X_v}\right)^{k-1}\left(\frac{p_n}{P_n}\right)^k\mid t_n\mid^k\\
&= O(1)\sum_{n=1}^{m-1}\Delta\mid\lambda_n\mid\sum_{v=1}^{n}\theta_v^{k-1}\left(\frac{p_v}{P_v}\right)^k\frac{\mid t_v\mid^k}{X_v^{k-1}}\\
&+ O(1)\mid\lambda_m\mid\sum_{n=1}^{m}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\frac{\mid t_n\mid^k}{X_n^{k-1}}\\
&= O(1)\sum_{n=1}^{m-1}\mid A_n\mid X_n + O(1)\mid\lambda_m\mid X_m\\
&= O(1)\quad as\quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of the theorem and Lemma 3. 3. Now, when $k>1$ applying Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, as in $T_{n,1}$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\theta_n^{k-1}\mid T_{n,2}\mid^k &= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}}\left\{\sum_{v=1}^{n-1}p_v\mid\lambda_v\mid^k\mid t_v\mid^k\right\}\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}\\
&= O(1)\sum_{v=1}^{m}p_v\mid\lambda_v\mid^{k-1}\mid\lambda_v\mid\mid t_v\mid^k\sum_{n=v+1}^{m+1}\left(\frac{\theta_n p_n}{P_n}\right)^{k-1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{\theta_v p_v}{P_v}\right)^{k-1}p_v\mid t_v\mid^k\mid\lambda_v\mid\left(\frac{1}{X_v}\right)^{k-1}\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}\theta_v^{k-1}\left(\frac{p_v}{P_v}\right)^k\mid\lambda_v\mid\left(\frac{1}{X_v}\right)^{k-1}\mid t_v\mid^k\\
&= O(1)\sum_{v=1}^{m}\mid\lambda_v\mid\theta_v^{k-1}\left(\frac{p_v}{P_v}\right)^k\frac{\mid t_v\mid^k}{X_v^{k-1}} = O(1)\quad as\quad m\to\infty.
\end{aligned}$$

Again, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\theta_n^{k-1}\mid T_{n,3}\mid^k &= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}}\left\{\sum_{v=1}^{n-1}\frac{P_v}{v}\mid\Delta\lambda_v\mid^k v^k\mid t_v\mid^k\right\}\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\frac{P_v}{v}\right\}^{k-1}\\
&= O(1)\sum_{v=1}^{m}\frac{P_v}{v}\mid t_v\mid^k v^k\mid A_v\mid^{k-1}\mid A_v\mid\sum_{n=v+1}^{m+1}\left(\frac{\theta_n p_n}{P_n}\right)^{k-1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{\theta_v p_v}{P_v}\right)^{k-1}v^{k-1}\left(\frac{1}{vX_v}\right)^{k-1}\mid A_v\mid\mid t_v\mid^k
\end{aligned} \tag{14}$$

$$
\begin{aligned}
&= O(1)\left(\frac{\theta_1 p_1}{P_1}\right)^{k-1}\sum_{v=1}^{m} v \mid A_v \mid \frac{\mid t_v \mid^k}{vX_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v\mid A_v \mid)\sum_{i=1}^{v}\frac{\mid t_i \mid^k}{iX_i^{k-1}} + O(1)m\mid A_m \mid \sum_{v=1}^{m}\frac{\mid t_v \mid^k}{vX_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}\mid \Delta(v \mid A_v \mid) \mid X_v + O(1)m\mid A_m \mid X_m\\
&= O(1)\sum_{v=1}^{m-1}\mid (v+1)\mid \Delta A_v \mid - A_v \mid X_v + O(1)m \mid A_m \mid X_m\\
&= O(1)\sum_{v=1}^{m-1} v \mid \Delta A_v \mid X_v + O(1)\sum_{v=1}^{m-1}\mid A_v \mid X_v + O(1)m\mid A_m \mid X_m\\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}
$$

by virtue of the hypotheses of the theorem and Lemma 3.4. Finally, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1}\theta_n^{k-1}\mid T_{n,4}\mid^k &\leq \sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}P_v \mid \lambda_{v+1}\mid^k \mid t_v \mid^k \frac{1}{v}\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\frac{P_v}{v}\right\}^{k-1}\\
&= O(1)\sum_{v=1}^{m}P_v \mid \lambda_{v+1}\mid^{k-1}\mid \lambda_{v+1}\mid\mid t_v \mid^k \frac{1}{v}\sum_{n=v+1}^{m+1}\left(\frac{\theta_n p_n}{P_n}\right)^{k-1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}P_v\left(\frac{1}{X_v}\right)^{k-1}\mid \lambda_{v+1}\mid\mid t_v \mid^k \frac{1}{v}\sum_{n=v+1}^{m+1}\left(\frac{\theta_n p_n}{P_n}\right)^{k-1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}\mid \lambda_{v+1}\mid\left(\frac{\theta_v p_v}{P_v}\right)^{k-1}\frac{\mid t_v \mid^k}{vX_v^{k-1}}\\
&= O(1)\left(\frac{\theta_1 p_1}{P_1}\right)^{k-1}\sum_{v=1}^{m}\mid \lambda_{v+1}\mid \frac{\mid t_v \mid^k}{vX_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}\Delta \mid \lambda_{v+1}\mid \sum_{r=1}^{v}\frac{\mid t_r \mid^k}{rX_r^{k-1}} + O(1)\mid \lambda_{m+1}\mid \sum_{v=1}^{m}\frac{\mid t_v \mid^k}{vX_v^{k-1}}\\
&= O(1)\sum_{v=2}^{m}\mid \Delta\lambda_v \mid X_v + O(1)\mid \lambda_{m+1}\mid X_{m+1}\\
&= O(1)\sum_{v=1}^{m}\mid A_v \mid X_v + O(1)\mid \lambda_{m+1}\mid X_{m+1}\\
&= O(1) \quad as \quad m \to \infty
\end{aligned}
$$

by virtue of the hypotheses of the theorem and Lemma 3. 3. This completes the proof of Theorem 3. 1. If we set $\theta_n = \frac{P_n}{p_n}$, then we obtain the result in [6] under weaker conditions. If we take $p_n = 1$ for all values of n and $\theta_n = n$, then we get a new result concerning the $\mid C, 1 \mid_k$ summability factors of infinite series. Also, if we take $p_n = 1$ for all values of n then we have a new result dealing with the $\mid C, 1, \theta_n \mid_k$ summability factors of infinite series. Furthermore, if we take $\theta_n = n$, then we have another new result concerning the $\mid R, p_n \mid_k$ summability factors of infinite series.

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