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A Note on the System of Linear Recurrence Equations

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Abstract. We will find a solution to a system of 2^d linear recurrence equations. Each equation is of the form $x_{2k}(n+1) = x_k(n)$ or $x_{2k+1}(n+1) = x_k(n) + x_{2^{d-1}+k}(n)$. This kind of system is connected with counting restricted permutations.

1. Introduction

The study of restricted permutations has a long history. Probably the most well known example is derangement problem or "le Problème des Rencontres" (see [4]). "Today, most of the restricted permutations considered in the literature deal with pattern avoidance. For an exhaustive survey of such studies, see [5]. For a related topic of pattern avoidance in compositions and words see [3] and in set partitions see [7]. Study of permutation patterns has applications in counting different combinatorial structures, computer science, statistical mechanics and computational biology [5, Ch.2,3].

Another type of restricted permutations is a generalization of the derangement problem. Detailed historical introduction to restricted permutations of this kind can be found in [1, 2]. Let p be a *permutation* of the set $\mathbb{N}_n = \{1, 2, \ldots, n\}$. So, $p(i)$ refers to the value taken by the function p when evaluated at a point i. Mendelsohn, Lagrange, Lehmer, Tomescu and Stanley studied particular types of strongly restricted permutations satisfying the condition $|p(i) - i| \le d$, where d is 1, 2, or 3 (more information on their work can be found in [1]). In [1] we pursue more general, asymmetric cases and we end up with asymmetric cases with more forbidden positions.

In [1] we developed a technique for counting restricted permutations of $\mathbb{N}_n$ satisfying the conditions $-k \le p(i) - i \le r$ (for arbitrary natural numbers k and r) and $p(i) - i \notin I$ (for some set I). For a given k, r and I the technique produces a system of linear recurrence equations. When trying to determine the reduced system in a particular case, we get the following system of linear recurrence equations:

$$\begin{array}{rcl} x_{2k}(n+1) & = & x_k(n) \\ x_{2k+1}(n+1) & = & x_k(n) \quad + \quad x_{2^{d-1}+k}(n) \end{array} \tag{$*$}$$

for $k = 0, 1, \ldots, 2^{d-1} - 1$.

Purpose of this note is to solve the system $(*)$ (for other type of systems, we refer the reader to [6]). Solution of a special case of $(*)$ is given in Lemma 2.1, which is used to solve the general case of $(*)$ in Theorem 2.2.

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Now we will introduce some notations.

The number $n < 2^d$ in binary form is represented by $n_2 = (b_{d-1}b_{d-2}\dots b_1b_0)_2$, where $b_i \in \{0,1\}$ and $n = \sum_{s=0}^{d-1} b_s \cdot 2^s$.

Binary operation $\oplus$ is given by $x \oplus y = x + y \pmod{d}$.

For each position s, $0 \le s \le d-1$, let us introduce the function

$$f_s(a,b,r) = \begin{cases} 1, & a_s = b_{s\oplus r} \\ 0, & a_s = 1, b_{s\oplus r} = 0 \\ t, & a_s = 0, b_{s\oplus r} = 1, s < d-r \\ t+1, & a_s = 0, b_{s\oplus r} = 1, s \ge d-r. \end{cases}$$

2. Main Results

Lemma 2.1. *Suppose we have a system of 2^d linear recurrence equations which are of the form $x_{2k}(n+1) = x_k(n)$ and $x_{2k+1}(n+1) = x_k(n) + x_{2^{d-1}+k}(n)$ for $k = 0,1,\dots,2^{d-1}-1$, where $0 \le a,b \le 2^d - 1$ and for some a, $x_a(0) = 1$, while for all $b \ne a$, $x_b(0) = 0$.*

Then for $n = d \cdot t + r$, $0 \le r \le d-1$, the following equality is true:

$$x_b(n) = \prod_{s=0}^{d-1} f_s(a,b,r).$$

Proof. Let us prove that this solution satisfies the initial conditions and the equations of the first type and the second type of a given system.

1° the initial conditions

For $n = 0 = d \cdot 0 + 0$ we have that $r = 0$, so $s \oplus r = s \oplus 0 = s$.

If $b = a$ for each position applies $a_s = b_s$, therefore $f_s(a,a,0) = 1$, which implies that

$$x_a(0) = \prod_{s=0}^{d-1} 1 = 1.$$

If $b \ne a$, then there is a position s from where the binary forms a_2 and b_2 differ.

If $a_s = 1$ and $b_s = 0$, it immediately follows that $f_s(a,b,0) = 0$.

If $a_s = 0$ and $b_s = 1$, as is true for $s < d = d - r$, we have that $f_s(a,b,0) = t = 0$.

When $b \ne a$ in both cases we get that $f_s(a,b,0) = 0$ for some s, which implies that $x_b(0) = 0$.

2° $x_{2k}(n+1) = x_k(n)$

Let $b = 2k$, for $k < 2^{d-1}$.

For binary forms

$$(k)_2 = (b_{d-1}, b_{d-2}, \dots, b_1, b_0) \qquad \text{and} \qquad (2k)_2 = (b'_{d-1}, b'_{d-2}, \dots, b'_1, b'_0)$$

we have that $(2k)_2$ is obtained from $(k)_2$ with cyclic shift to the left by one position, i.e. $b'_{s\oplus 1} = b_s$. Also, with increasing n to $n+1$ we have that the remainder of the division with d increases by 1 modulo d, i.e. $r' = r \oplus 1$. Now we will prove the equality $x_{2k}(n+1) = x_k(n)$ by considering the following cases:

- If $a_s = b_{s\oplus r} \Rightarrow a_s = b_{s\oplus r} = b'_{(s\oplus r)\oplus 1} = b'_{s\oplus(r\oplus 1)} \Rightarrow f_s(a,k,r) = f_s(a,2k,r\oplus 1) = 1$.
- If $a_s = 1$, $b_{s\oplus r} = 0 \Rightarrow a_s = 1$, $0 = b_{s\oplus r} = b'_{(s\oplus r)\oplus 1} = b'_{s\oplus(r\oplus 1)} \Rightarrow f_s(a,k,r) = f_s(a,2k,r\oplus 1) = 0$.

- If $a_s = 0, b_{s\oplus r} = 1, s < d-r-1 \Rightarrow a_s = 0, 1 = b_{s\oplus r} = b'_{s\oplus(r\oplus 1)}, s < d-(r\oplus 1) \Rightarrow f_s(a,k,r) = f_s(a,2k,r\oplus 1) = t$.
- If $a_s = 0, b_{s\oplus r} = 1, s \geq d-r, r \neq d-1 \Rightarrow a_s = 0, 1 = b_{s\oplus r} = b'_{s\oplus(r\oplus 1)}, s \geq d-(r+1)$
$$\Rightarrow f_s(a,k,r) = f_s(a,2k,r\oplus 1) = t+1.$$
- If $r = d-1$, then $r\oplus 1 = 0$, and also previous equality holds, but with different reasoning: if $a_s = 0, b_{s\oplus(d-1)} = 1, s \geq d-(d-1) = 1, f_s(a,k,r) = t+1 \Rightarrow a_s = 0, 1 = b_{s\oplus r} = b'_{s\oplus(r\oplus 1)} = b'_{s\oplus 0}, s \leq d-0 = d$, then $f_s(a,2k,r\oplus 1) = t' = t+1$, because $n' = n+1 = d\cdot t + (d-1) + 1 = d\cdot(t+1)+0$ and again we get that $f_s(a,k,r) = f_s(a,2k,r\oplus 1) = t+1$.
- If $a_s = 0, b_{s\oplus r} = 1, s = d-r-1 \Rightarrow 1 = b_{s\oplus r} = b'_{s\oplus(r\oplus 1)}$. On the other hand we have that $b'_{s\oplus(r\oplus 1)} = b'_0 = 0$, since this is the last digit in the binary form of even number $2k$. Thus, we get that this case is not possible.

As in all cases, we get that $f_s(a,k,r) = f_s(a,2k,r\oplus 1)$, which entails that

$$x_k(n) = \prod_{s=0}^{d-1} f_s(a,k,r) = \prod_{s=0}^{d-1} f_s(a,2k,r\oplus 1) = x_{2k}(n+1).$$

3° $\underline{x_{2k+1}(n+1) = x_k(n) + x_{2^{d-1}+k}(n)}$

Let $b = 2k+1$, for $k < 2^{d-1}$.

For binary forms $(k)_2 = (b_{d-1}, b_{d-2}, \ldots, b_0)$,

$$(2^{d-1}+k)_2 = (b''_{d-1}, b''_{d-2}, \ldots, b''_0), \qquad (2k+1)_2 = (b'_{d-1}, b'_{d-2}, \ldots, b'_0)$$

it is true that $b_{d-1} = 0$, $b'_{d-1} = 1$, $b_s = b'_s = 0$ for $s < d-1$, while $(2k+1)_2$ is obtained from $(2^{d-1}+k)_2$ with cyclic shift to the left by one position, i.e. $b'_{s\oplus 1} = b''_s$. The same as before we have that $r' = r\oplus 1$. Now we will prove the equality $x_{2k+1}(n+1) = x_k(n) + x_{2^{d-1}+k}(n)$ by considering the following cases:

- If $a_s = b_{s\oplus r}, r \neq d-1 \Rightarrow a_s = b_{s\oplus r} = b''_{s\oplus r} = b'_{(s\oplus r)\oplus 1} = b'_{s\oplus(r\oplus 1)}$
$$\Rightarrow f_s(a,k,r) = f_s(a,2^{d-1}+k,r) = f_s(a,2k+1,r+1) = 1.$$
- If $a_s = 1, b_{s\oplus r} = 0, r \neq d-1 \Rightarrow a_s = 1, 0 = b_{s\oplus r} = b''_{s\oplus r} = b'_{(s\oplus r)\oplus 1} = b'_{s\oplus(r\oplus 1)}$
$$\Rightarrow f_s(a,k,r) = f_s(a,2^{d-1}+k,r) = f_s(a,2k+1,r+1) = 0.$$
- If $s = d-r-1, a_s = 1$, then we have
$a_s = 1, b_{s\oplus r} = b_{d-1} = 0 \Rightarrow f_s(a,k,r) = 0$;
$a_s = 1, b''_{s\oplus r} = b''_{d-1} = 1 \Rightarrow f_s(a,2^{d-1}+k,r) = 1$;
$a_s = 1, 1 = b''_{s\oplus r} = b'_{(s\oplus r)\oplus 1} = b'_0 \Rightarrow f_s(a,2k+1,r\oplus 1) = 1$.
- If $a_s = 0, b_{s\oplus r} = 1, s < d-r-1 \Rightarrow a_s = 0, 1 = b_{s\oplus r} = b''_{s\oplus r} = b'_{s\oplus(r\oplus 1)}, s < d-(r\oplus 1)$
$$\Rightarrow f_s(a,k,r) = f_s(a,2^{d-1}+k,r) = f_s(a,2k+1,r\oplus 1) = t.$$
- If $a_s = 0, b_{s\oplus r} = 1, s \geq d-r \Rightarrow a_s = 0, 1 = b_{s\oplus r} = b''_{s\oplus r} = b'_{s\oplus(r\oplus 1)}, s \geq d-(r+1)$
$$\Rightarrow f_s(a,k,r) = f_s(a,2^{d-1}+k,r) = f_s(a,2k+1,r\oplus 1) = t+1.$$
- If $s = d-r-1 < d-r, a_s = 0$, then we have
$a_s = 0, b_{s\oplus r} = b_{d-1} = 0 \Rightarrow f_s(a,k,r) = 1$;
$a_s = 0, b''_{s\oplus r} = b''_{d-1} = 1 \Rightarrow f_s(a,2^{d-1}+k,r) = t$;
$a_s = 0, 1 = b''_{s\oplus r} = b'_{(s\oplus r)\oplus 1} = b'_0, s \leq d-0 \Rightarrow f_s(a,2k+1,r\oplus 1) = t' = t+1$, because $n' = n+1 = d\cdot t + (d-1) + 1 = d\cdot(t+1)+0$.

For $s = d-r-1$ and $a_{d-r-1} = 1$ we get $f_s(a,k,r) = 0 \Rightarrow x_k(n) = 0$, and since $f_s(a, 2^{d-1}+k, r) = f_s(a, 2k+1, r\oplus 1)$ for all positions $s \neq d-r-1$, we have that $x_{2^{d-1}+k}(n) = x_{2k+1}(n+1)$, which entails equality $x_{2k+1}(n+1) = x_k(n)+x_{2^{d-1}+k}(n)$. For $s = d-r-1$ and $a_{d-r-1} = 0$ we get $f_s(a,k,r) = 1$, $f_s(a, 2^{d-1}+k, r) = t$, $f_s(a, 2k+1, r\oplus 1) = t+1$, while $f_s(a,k,r) = f_s(a, 2^{d-1}+k, r) = f_s(a, 2k+1, r\oplus 1)$ for all positions $s \neq d-r-1$, and we have that

$$\begin{aligned}
x_k(n) + x_{2^{d-1}+k}(n) &= \prod_{s=0}^{d-1} f_s(a,k,r) + \prod_{s=0}^{d-1} f_s(a, 2^{d-1}+k, r)\\
&= f_{d-r-1}(a,k,r)\cdot \prod_{\substack{0\le s\le d-1\\ s\neq d-r-1}} f_s(a,k,r) + f_{d-r-1}(a,2^{d-1}+k,r)\cdot \prod_{\substack{0\le s\le d-1\\ s\neq d-r-1}} f_s(a,2^{d-1}+k,r)\\
&= 1\cdot \prod_{\substack{0\le s\le d-1\\ s\neq d-r-1}} f_s(a,2k+1,r\oplus 1) + t\cdot \prod_{\substack{0\le s\le d-1\\ s\neq d-r-1}} f_s(a,2k+1,r\oplus 1)\\
&= (1+t)\cdot \prod_{\substack{0\le s\le d-1\\ s\neq d-r-1}} f_s(a,2k+1,r\oplus 1) = \prod_{s=0}^{d-1} f_s(a,2k+1,r\oplus 1) = x_{2k+1}(n+1).
\end{aligned}$$

In both cases we get that $x_{2k+1}(n+1) = x_k(n) + x_{2^{d-1}+k}(n)$. □

We will illustrate this Theorem later, in Example 3.1. Now, we move on to the general case of the system (∗).

Theorem 2.2. *Suppose we have a system of 2^d linear recurrence equations of the form $x_{2k}(n+1) = x_k(n)$ and $x_{2k+1}(n+1) = x_k(n) + x_{2^{d-1}+k}(n)$ for $k = 0,1,\ldots,2^{d-1}-1$, with initial conditions $x_0(0) = y_0$, $x_1(0) = y_1$, ..., $x_{2^d-1}(0) = y_{2^d-1}$, for arbitrary real numbers $y_0, y_1, \ldots, y_{2^d-1}$.*
Then for $n = d\cdot t + r$, $0 \le r \le d-1$, the following equality is true:

$$x_b(n) = \sum_{a=0}^{2^d-1}\left(y_a\cdot \prod_{s=0}^{d-1} f_s(a,b,r)\right).$$

Proof. This result is a direct consequence of Lemma 2.1 and the basic properties of the system of linear recurrence equations. □

3. Examples

Example 3.1. *We will now illustrate Lema 2.1, for the case $d = 3$ and $a = 4$. Then we have the system*

$$\begin{aligned}
x_0(n+1) &= x_0(n), & x_1(n+1) &= x_0(n) + x_4(n),\\
x_2(n+1) &= x_1(n), & x_3(n+1) &= x_1(n) + x_5(n),\\
x_4(n+1) &= x_2(n), & x_5(n+1) &= x_2(n) + x_6(n),\\
x_6(n+1) &= x_3(n), & x_7(n+1) &= x_3(n) + x_7(n),
\end{aligned}$$

with initial conditions $x_4(0) = 1$ and $x_b(0) = 0$ for $b \neq 4$, $0 \le b \le 2^d - 1 = 7$.

Solution. We will take case analysis on all values of b.

- For $a = 4$ and $b = 0$ binary form $0_2 = 000$ has more zeros than binary form $4_2 = 100$, so by the Pigeonhole principle at least one position will be $a_s = 1$ and $b_{s\oplus r} = 0$. Then for each n the equality $x_0(n) = 0$ is satisfied. These conclusions are valid whenever the binary form $(b)_2$ has more zeros than $(a)_2$!

- For $a = 4_2 = 100$ and $b = 1_2 = 001$ when $r = 0$ or $r = 2$ we will have a position s such that $a_s = 1$ and $b_{s\oplus r} = 0$ (for $r = 0$, i.e. when there is no movement, $a_2 = 1$ and $b_2 = 0$; for $r = 2$, i.e. when moving to the right by two positions, $a_2 = 1$ and $b_{2\oplus 2} = b_1 = 0$). Then for $n \equiv 0 \pmod 3$ and $n \equiv 2 \pmod 3$ it is true that $x_0(n) = 0$.
When $r = 1$ we have that
$$\begin{array}{lcl} a_0 = b_{0\oplus 1} = b_1 = 0 & \Rightarrow & f_0(a, b, 1) = 1, \\ a_1 = b_{1\oplus 1} = b_2 = 0 & \Rightarrow & f_1(a, b, 1) = 1, \\ a_2 = b_{2\oplus 1} = b_0 = 1 & \Rightarrow & f_2(a, b, 1) = 1 \end{array}$$
and we have that $x_b(n) = x_1(n) = f_0(a, b, 1) \cdot f_1(a, b, 1) \cdot f_2(a, b, 1) = 1 \cdot 1 \cdot 1 = 1$ for $n \equiv 1 \pmod 3$.
Thus, we have shown that:
$$x_1(n) = \begin{cases} 0, & n = 3t \\ 1, & n = 3t + 1 \\ 0, & n = 3t + 2. \end{cases}$$

- For $a = 4_2 = 100$ and $b = 3_2 = 011$ when $r = 0$ we have that $a_2 = 1$ and $b_{2\oplus 0} = b_2 = 0$).
Then for $n \equiv 0 \pmod 3$ it is true that $x_3(n) = 0$.
When $r = 1$ we have that
$$\begin{array}{lcl} a_0 = 0,\ b_{0\oplus 1} = b_1 = 1 \text{ and } s = 0 < 3 - 1 = d - r & \Rightarrow & f_0(a, b, 1) = t, \\ a_1 = b_{1\oplus 1} = b_2 = 0 & \Rightarrow & f_1(a, b, 1) = 1, \\ a_2 = 1,\ b_{2\oplus 1} = b_0 = 1 & \Rightarrow & f_2(a, b, 1) = 1 \end{array}$$
and we have that $x_b(n) = x_3(n) = f_0(a, b, 1) \cdot f_1(a, b, 1) \cdot f_2(a, b, 1) = t \cdot 1 \cdot 1 = t$ for $n \equiv 1 \pmod 3$.
When $r = 2$ we have that
$$\begin{array}{lcl} a_0 = 0,\ b_{0\oplus 2} = b_2 = 0 & \Rightarrow & f_0(a, b, 1) = 1, \\ a_1 = 0,\ b_{1\oplus 2} = b_0 = 1 \text{ and } s = 1 \geqslant 3 - 2 = d - r & \Rightarrow & f_1(a, b, 1) = t + 1, \\ a_2 = 1,\ b_{2\oplus 1} = b_0 = 1 & \Rightarrow & f_2(a, b, 1) = 1 \end{array}$$
and we have that $x_b(n) = x_3(n) = 1 \cdot (t + 1) \cdot 1 = t + 1$ for $n \equiv 2 \pmod 3$.
Thus, we have shown that:
$$x_3(n) = \begin{cases} 0, & n = 3t \\ t, & n = 3t + 1 \\ t + 1, & n = 3t + 2. \end{cases}$$

- For $a = 4_2 = 100$ and $b = 7_2 = 111$ when $r = 0$ we have $x_7(n) = t \cdot t \cdot 1 = t^2$ for $n \equiv 1 \pmod 3$.
When $r = 1$ we have that $x_7(n) = t \cdot t \cdot 1 = t^2$ for $n \equiv 1 \pmod 3$.
When $r = 2$ we have that $x_7(n) = t \cdot (t + 1) \cdot 1 = t(t + 1)$ for $n \equiv 2 \pmod 3$.
Thus, we have shown that:
$$x_7(n) = \begin{cases} t^2, & n = 3t \\ t^2, & n = 3t + 1 \\ t(t + 1), & n = 3t + 2. \end{cases}$$

- Analogously we obtain:
$$x_2(n) = \begin{cases} 0, & n = 3t \\ 0, & n = 3t + 1 \\ 1, & n = 3t + 2, \end{cases} \qquad x_4(n) = \begin{cases} 1, & n = 3t \\ 0, & n = 3t + 1 \\ 0, & n = 3t + 2, \end{cases}$$
$$x_5(n) = \begin{cases} t, & n = 3t \\ t, & n = 3t + 1 \\ 0, & n = 3t + 2, \end{cases} \qquad x_6(n) = \begin{cases} t, & n = 3t \\ 0, & n = 3t + 1 \\ t, & n = 3t + 2. \end{cases}$$

All these sequences can be found in [8]: x_0 is A000004, x_1 is shifted A079978, x_2 and x_4 are A079978, x_3 is A087509, x_5 is shifted A087508, x_6 is shifted A087509, x_7 is A008133. ■

This particular example can be solved by using generating functions, such as in [6]. Although generating functions and then Cramers method can be used to solve the system (∗) in general, we think that results in Lemma 2.1 and Theorem 2.2 are more straightforward.

The following discussion illustrates the connection between the system considered in the paper, and restricted permutations from [1].

Let C_{md+1-q} denote the number of combinations where the smallest element is equal to $md+1-q$, for $q = 0, 1, \ldots, md$, and can be obtained from the initial combination $(r+1, r+2, \ldots, r+k+1)$ using techniques developed in [1] (those techniques count the number of permutations that satisfy $p(i) - i \in S$ and $S = \{-d, -d+1, \ldots, md\} \setminus \{-d, 0, md\}$). Then, C_{md+1-q} is equal to

$$C_{md+1-q} = \sum_{b=0}^{2^d-1} x_b(q) = \sum_{b=0}^{2^d-1} \prod_{s=0}^{d-1} f_s(2^{d-1}, b, r),$$

where $q = d \cdot t + r$.

Example 3.2. *Let us illustrate these considerations for the case $d = 3$ and $m = 2$ (when $k = d = 3$ and $r = md = 6$).*

Solution. Then we deal with the permutations that satisfy $p(i) - i \in S$, $S = \{-3, -2, \ldots, 5, 6\} \setminus \{-3, 0, 6\} = \{-2, -1, 1, 2, 3, 4, 5\}$, i.e. $I = \{-3, 0, 6\}$ and $r + 1 - I = 7 - I = \{10, 7, 1\}$. The number of such permutations is given in sequence A224810 at [8].
The set C consists of all combinations of the set $\mathbb{N}_{k+r+1} = \{1, 2 \ldots, 10\}$, with $k+1 = 4$ elements and containing a number $k + r + 1 = 10$. The set C has $|C| = \binom{9}{3} = 84$ elements, but most of them are not relevant to the technique developed in [1], because they cannot be generated starting from the initial combination $(7, 8, 9, 10)$.
In Example 3.1 we get the values of all sequences x_b that occur in the previous theorem.

For $q = 3t$ we have that

$$\begin{aligned} C_{md+1-q} &= x_0(q) + x_1(q) + x_2(q) + x_3(q) + x_4(q) + x_5(q) + x_6(q) + x_7(q) \\ &= 0 + 0 + 0 + 0 + 1 + t + t + t^2 = (t+1)^2, \end{aligned}$$

for $q = 3t + 1$ we have that

$$C_{md+1-q} = 0 + 1 + 0 + t + 0 + t + 0 + t^2 = (t+1)^2,$$

for $q = 3t + 2$ we have that

$$C_{md+1-q} = 0 + 0 + 1 + (t+1) + 0 + 0 + t + t(t+1) = (t+1)(t+2).$$

Thus, we find that for combinations starting with $md + 1 - q$ the following equality is satisfied:

$$C_{md+1-q} = \begin{cases} (t+1)^2, & q = 3t \\ (t+1)^2, & q = 3t+1 \\ (t+1)(t+2), & q = 3t+2. \end{cases}$$

This sequence is A008133 at [8].

For $q = 0$ we have $(0+1)^2 = 1$ combination that begins with $md + 1 - q = 7$. This is the initial combination $(7, 8, 9, 10)$.
For $q = 1$ we have $(0+1)^2 = 1$ combination that begins with 6: $(6, 7, 8, 10)$.

For $q = 2$ we have $(0+1)\cdot(0+2) = 2$ combinations starting with 5: $(5,7,9,10)$, $(5,6,7,10)$.

For $q = 3$ we have $(1+1)^2 = 4$ combinations starting with 4: $(4,8,9,10)$, $(4,6,8,10)$, $(4,5,9,10)$, $(4,5,6,10)$.

For $q = 4$ we have $(1+1)^2 = 4$ combinations starting with 3: $(3,7,8,10)$, $(3,5,7,10)$, $(3,4,8,10)$, $(3,4,5,10)$.

For $q = 5$ we have $(1+1)\cdot(1+2) = 6$ combinations starting with 2: $(2,7,9,10)$, $(2,6,7,10)$, $(2,4,9,10)$, $(2,4,6,10)$, $(2,3,7,10)$, $(2,3,4,10)$.

For $q = 6$ we have $(2+1)^2 = 9$ combinations starting with 1: $(1,8,9,10)$, $(1,6,8,10)$, $(1,5,9,10)$, $(1,5,6,10)$, $(1,3,8,10)$, $(1,3,5,10)$, $(1,2,9,10)$, $(1,2,6,10)$, $(1,2,3,10)$.

Altogether we have

$$1+1+2+4+4+6+9 = 27 = (m+1)^d$$

combinations which occur in the technique developed in [1]. We get a $(m+1)^d \times (m+1)^d$ matrix as the matrix of the reduced system of linear recurrence equations. Furthermore, the generating function corresponding to the restricted permutations is a rational function $P(z)/Q(z)$. Also, the denominator $Q(z)$ is of degree less than or equal to $(m+1)^d$, i.e. $\deg Q(z) \leq (m+1)^d$, which is significantly less than $|C| = \binom{(m+1)d}{d}$, the total number of combinations that occur in technique developed in [1].

In this particular case, we have that $\deg Q(z) = 24 \leq 27 = (m+1)^d$, because

$$A(z) = \frac{1+z^3-z^4-z^5+z^6-2z^7-z^8-z^9-2z^{10}-z^{12}-z^{13}-z^{15}}{1-z+z^3-2z^4+2z^6-4z^7-2z^9-2z^{10}-4z^{12}+2z^{13}-2z^{15}+4z^{16}+2z^{18}+2z^{19}+z^{21}+z^{22}+z^{24}}.$$

The denominator of $A(z)$ is $(z-1)(z^2+z+1)(z^3+z-1)(z^{18}+3z^{15}+7z^{12}+9z^9+7z^6+3z^3+1)$ and the numerator is $2-(z+1)(z^2-z+1)(z^{12}+z^{10}+z^7+z^6+z^5+z^4-2z^3+1)$. It is significantly less than $|C| = \binom{(m+1)d}{d} = \binom{9}{3} = 84$. The sequence corresponding to $A(z)$ is A224810 in [8]. ■

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