Approximation of Viscosity Zero Points of Accretive Operators in a Banach Space

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Abstract. In this paper, zero points of m-accretive operators are investigated based on a viscosity iterative algorithm with double computational errors. Strong convergence theorems for zero points of m-accretive operators are established in a Banach space.

1. Introduction

Zero points of accretive operators have been investigated based on iterative algorithms recently. Interest in accretive operators stems mainly from their firm connection with equations of evolution is an important class of nonlinear operators. It is known that many physically significant problems can be modelled by initial value problems of the form

$$x'(t) + Ax(t) = 0, \quad x(0) = x_0, \tag{1.1}$$

where A is an accretive operator in an appropriate Banach space. Typical examples where such evolution equations occur can be found in the heat, wave or Schrödinger equations. If $x(t)$ is dependent of t, then (1.1) is reduced to

$$Au = 0, \tag{1.2}$$

whose solutions correspond to the equilibrium points of the system (1.1). Consequently, considerable research efforts have been devoted, especially within the past 40 years or so, to methods for finding approximate solutions of the equation (1.2). An early fundamental result in the theory of accretive operators, due to Browder [1], states that the initial value problem (1.1) is solvable if A is locally Lipschitz and accretive on E.

The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, we study zero points of m-accretive operators based on a viscosity iterative algorithm. Strong convergence theorems for zero points of m-accretive operators are established in a Banach space. It is proved that the zero point is also a solution to some variational inequality.

2. Preliminaries

In what follows, we always assume that E is a Banach space with the dual E^*. Let $U_E = \{x \in E : \|x\| = 1\}$. E is said to be *smooth* or said to be have a *Gâteaux differentiable norm* if the limit $\lim_{t \to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U_E$. E is said to have a *uniformly Gâteaux differentiable norm* if for each $y \in U_E$, the limit is attained uniformly for all $x \in U_E$. E is said to be *uniformly smooth* or said to have a *uniformly Fréchet differentiable norm* if the limit is attained uniformly for $x, y \in U_E$. Let C be a nonempty closed convex subset of a real Banach space E and E^* be the dual space of E. Let $\langle \cdot, \cdot \rangle$ denote the pairing between E and E^*. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}$$

for all $x \in E$. In the sequel, we use j to denote the single-valued normalized duality mapping. It is known that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping J is single valued and uniformly norm to weak* continuous on each bounded subset of E.

Recall that a closed convex subset C of a Banach space E is said to have *normal structure* if for each bounded closed convex subset K of C which contains at least two points, there exists an element x of K which is not a diametral point of K, i.e., $\sup\{\|x - y\| : y \in K\} < d(K)$, where $d(K)$ is the diameter of K. It is well known that a closed convex subset of uniformly convex Banach space has the normal structure and a compact convex subset of a Banach space has the normal structure; see [2] for more details.

Let $T : C \to C$ be a mapping. In this paper, we use $F(T)$ to denote the set of fixed points of T. Recall that T is said to be *contractive* if there exits a constant $\alpha \in (0, 1)$ such that

$$\|Tx - Ty\| \le \alpha\|x - y\|, \quad \forall x, y \in C.$$

For such a case, we also call T an α-*contraction*. T is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

T is said to be *pseudocontractive* if there exists some $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \le \|x - y\|^2, \quad \forall x, y \in C.$$

Let D be a nonempty subset of C. Let $Q : C \to D$. Q is said to be *contraction* if $Q^2 = Q$; *sunny* if for each $x \in C$ and $t \in (0, 1)$, we have $Q(tx + (1 - t)Qx) = Qx$; *sunny nonexpansive retractction* if Q sunny, nonexpansive, and contraction. K is said to be a *nonexpansive retract* of C if there exists a nonexpansive retraction from C onto D.

The following result, which was established in [3,4], describes a characterization of sunny nonexpansive retractions on a smooth Banach space.

Let E be a smooth Banach space and C be a nonempty subset of E. Let $Q : E \to C$ be a retraction and j be the normalized duality mapping on E. Then the following are equivalent:

(1) Q is sunny and nonexpansive;
(2) $\|Qx - Qy\|^2 \le \langle x - y, j(Qx - Qy) \rangle, \forall x, y \in E$;
(3) $\langle x - Qx, j(y - Qx) \rangle \le 0, \forall x \in E, y \in C$.

Let I denote the identity operator on E. An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \ne \emptyset\}$ and range $R(A) = \cup\{Az : z \in D(A)\}$ is said to be *accretive* if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists $j(x_1 - x_2) \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j(x_1 - x_2) \rangle \ge 0$. An accretive operator A is said to be *m-accretive* if $R(I + rA) = E$ for all $r > 0$. In a real Hilbert space, an operator A is *m*-accretive if and only if A is maximal monotone. In this paper, we use $A^{-1}(0)$ to denote the set of zero points of A.

For an accretive operator A, we can define a nonexpansive single-valued mapping $J_r : R(I + rA) \to D(A)$ by $J_r = (I + rA)^{-1}$ for each $r > 0$, which is called the *resolvent* of A.

One of classical methods of studying the problem $0 \in Ax$, where $A \subset E \times E$ is an accretive operator, is the following:

$$x_0 \in E, \quad x_{n+1} = J_{r_n} x_n, \quad \forall n \ge 0,$$

where $J_{r_n} = (I + r_n A)^{-1}$ and $\{r_n\}$ is a sequence of positive real numbers.

The classical method was first proposed by Martinet [5] and generalized by Rockaffellar [6] and [7]. This method and its dual version in the context of convex programming have been extensively studied; see, for instance, [8-10] and the references therein. This method is known to yield as special cases decomposition methods such as the method of partial inverses, the Douglas-Rachford splitting method, and the alternating direction method of multipliers. Regularization methods recently have been investigated for treating zero points of accretive operators; for [11-25] and the references therein. However, as pointed in [26], the ideal form of the above method is often impractical since, in may cases, to solve the problem exactly is either impossible or the same difficult as the original inclusion problem. Therefore, one of the most interesting and important problems in the theory of accretive operators is to find an efficient iterative methods to compute approximately zeros of A.

In this paper, zero points of m-accretive operators are investigated based on a viscosity iterative algorithm with double computational errors. Strong convergence theorems for zero points of m-accretive operators are established in a reflexive Banach space.

In order to prove our main results, we also need the following lemmas.

Lemma 2.1 [27] *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnegative real sequences satisfying*

$$a_{n+1} \le (1 - t_n)a_n + b_n + c_n, \quad \forall n \ge 0,$$

where $\{t_n\}$ is a sequence in $(0, 1)$. Assume that the following conditions are satisfied
  (a) $\sum_{n=0}^{\infty} t_n = \infty$ and $b_n = o(t_n)$;
  (b) $\sum_{n=0}^{\infty} c_n < \infty$.
Then $\lim_{n \to \infty} a_n = 0$.

Lemma 2.2 [28] *Let E be a Banach space, and A an m-accretive operator. For $\lambda > 0$, $\mu > 0$, and $x \in E$, we have*

$$J_\lambda x = J_\mu\Big(\frac{\mu}{\lambda}x + \Big(1 - \frac{\mu}{\lambda}\Big)J_\lambda x\Big),$$

where $J_\lambda = (I + \lambda A)^{-1}$ and $J_\mu = (I + \mu A)^{-1}$.

Lemma 2.3 [29] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E, and $\{\beta_n\}$ be a sequence in $(0, 1)$ with*

$$0 < \liminf_{n \to \infty} \beta_n \le \limsup_{n \to \infty} \beta_n < 1.$$

Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n, \forall n \ge 1$ and

$$\limsup_{n \to \infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n \to \infty} \|y_n - x_n\| = 0$.

Lemma 2.4 [30] *Let E a real reflexive Banach space with the uniformly Gâteaux differentiable norm and the normal structure, and C be a nonempty closed convex subset of E. Let $S : C \to C$ be a nonexpansive mapping with a fixed point, and $T : C \to C$ be a fixed contraction with the coefficient $\alpha \in (0, 1)$. Let $\{x_t\}$ be a sequence generated by the following $x_t = tTx_t + (1 - t)Sx_t$, where $t \in (0, 1)$. Then $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point x^* of T, which is the unique solution in $F(T)$ to the following variational inequality $\langle Tx^* - x^*, j(x^* - p)\rangle \ge 0, \forall p \in F(S)$.*

3. Main Results

Now, we are in a position to state our main result.

Theorem 3.1. *Let E be a real reflexive Banach space with the uniformly Gâteaux differentiable norm and let A be an m-accretive operators in E. Assume that $C := \overline{D(A)}$ is convex and has the normal structure. Let $T : C \to C$ be a fixed*

α-contraction. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha_n'\}$, $\{\beta_n'\}$ and $\{\gamma_n'\}$ be real number sequences in $(0,1)$. Let Q_C be the sunny nonexpansive retraction from E onto C and $\{x_n\}$ be a sequence generated in the following manner: $x_0 \in C$ and

$$\begin{cases} y_n = \alpha_n' x_n + \beta_n' J_{r_n}(x_n + e_{n+1}) + \gamma_n' Q_C(f_n), \\ x_{n+1} = \alpha_n T x_n + \beta_n x_n + \gamma_n y_n, \quad \forall n \geq 0, \end{cases}$$

where $\{e_n\}$ is a sequence in E, $\{f_n\}$ is a bounded sequence in E, $\{r_n\}$ is a positive real numbers sequence, and $J_{r_n} = (I + r_n A)^{-1}$. Assume that $A^{-1}(0)$ is not empty and the above control sequences satisfy the following restrictions:
  (a) *$\alpha_n + \beta_n + \gamma_n = \alpha_n' + \beta_n' + \gamma_n' = 1$;*
  (b) *$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \alpha_n' = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;*
  (c) *$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;*
  (d) *$\sum_{n=1}^{\infty} \|e_n\| < \infty$ and $\sum_{n=0}^{\infty} \gamma_n' < \infty$;*
  (e) *$\lim_{n\to\infty} |\beta_n' - \beta_{n+1}'| = 0$;*
  (f) *$r_n \geq \epsilon$ for each $n \geq 1$ and $\lim_{n\to\infty} |r_n - r_{n+1}| = 0$.*
Then the sequence $\{x_n\}$ converges strongly to $\bar{x}$, which is the unique solution to the following variational inequality

$$\langle T\bar{x} - \bar{x}, j(p - \bar{x}) \rangle \leq 0, \quad \forall p \in A^{-1}(0).$$

Proof. First, we prove that $\{x_n\}$ is bounded. Fixing $p \in A^{-1}(0)$, we see that

$$\begin{aligned} \|y_0 - p\| &= \|\alpha_0'(x_0 - p) + \beta_0'(J_{r_0}(x_0 + e_1) - p) + \gamma_0'(Q_C(f_0) - p)\| \\ &\leq \alpha_0'\|x_0 - p\| + \beta_0'\|J_{r_0}(x_0 + e_1) - p\| + \gamma_0'\|Q_C(f_0) - p\| \\ &\leq (1 - \gamma_0')\|x_0 - p\| + \|e_1\| + \gamma_0'\|f_0 - p\|. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_1 - p\| &\leq \alpha_0\|Tx_0 - p\| + \beta_0\|x_0 - p\| + \gamma_0\|y_0 - p\| \\ &\leq \alpha_0\alpha\|x_0 - p\| + \alpha_0\|Tp - p\| + \beta_0\|x_0 - p\| + \gamma_0\|y_0 - p\| \\ &\leq \alpha_0\alpha\|x_0 - p\| + \alpha_0\|Tp - p\| + \beta_0\|x_0 - p\| + (1 - \gamma_0')\gamma_0\|x_0 - p\| \\ &\quad + \|e_1\| + \gamma_0\gamma_0'\|f_0 - p\| \\ &\leq \left(1 - \alpha_0(1 - \alpha)\right)\|x_0 - p\| + \alpha_0\|Tp - p\| + \|e_1\| + \gamma_0'\|f_0 - p\|. \end{aligned}$$

Put $M_1 = \max\{\|x_0 - p\|, \frac{\|Tp-p\|}{1-\alpha}\} < \infty$. Next, we prove that

$$\|x_n - p\| \leq M_1 + \sum_{i=1}^{n} \|e_i\| + \sum_{i=1}^{n-1} \gamma_i'\|f_i\|. \tag{3.1}$$

It is easy to see that (3.1) holds for $n = 1$. We assume that the result holds for some n. Notice that

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n'(x_n - p) + \beta_n'(J_{r_n}(x_n + e_{n+1}) - p) + \gamma_n'(Q_C(f_n) - p)\| \\ &\leq \alpha_n'\|x_n - p\| + \beta_n'\|J_{r_n}(x_n + e_{n+1}) - p\| + \gamma_n'\|Q_C(f_n) - p\| \\ &\leq (1 - \gamma_n')\|x_n - p\| + \|e_{n+1}\| + \gamma_n'\|f_n - p\|. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n\|Tx_n - p\| + \beta_n\|x_n - p\| + \gamma_n\|y_n - p\| \\ &\leq \alpha_n\alpha\|x_n - p\| + \alpha_n\|Tp - p\| + \beta_n\|x_n - p\| + \gamma_n\|y_n - p\| \\ &\leq \alpha_n\alpha\|x_n - p\| + \alpha_n\|Tp - p\| + \beta_n\|x_n - p\| + (1 - \gamma_n')\gamma_n\|x_n - p\| \\ &\quad + \|e_{n+1}\| + \gamma_n\gamma_n'\|f_n - p\| \\ &\leq \left(1 - \alpha_n(1 - \alpha)\right)\|x_n - p\| + \alpha_n\|Tp - p\| + \|e_{n+1}\| + \gamma_n'\|f_n - p\| \\ &\leq M_1 + \sum_{i=1}^{n+1} \|e_i\| + \sum_{i=1}^{n} \gamma_i'\|f_i\|. \end{aligned}$$

This yields that the sequence $\{x_n\}$ is bounded. Next, we show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.2}$$

It follows from Lemma 2.2 that

$$
\begin{aligned}
&\|J_{r_n}(x_n + e_{n+1}) - J_{r_{n+1}}(x_{n+1} + e_{n+2})\| \\
&= \left\| J_{r_{n+1}}\Big(\frac{r_{n+1}}{r_n}(x_n + e_{n+1}) + \Big(1 - \frac{r_{n+1}}{r_n}\Big)J_{r_n}(x_n + e_{n+1})\Big) - J_{r_{n+1}}(x_{n+1} + e_{n+2}) \right\| \\
&\le \left\| \frac{r_{n+1}}{r_n}\big((x_n + e_{n+1}) - (x_{n+1} + e_{n+2})\big) + \Big(1 - \frac{r_{n+1}}{r_n}\Big)\big(J_{r_n}(x_n + e_{n+1}) - (x_{n+1} + e_{n+2})\big) \right\| \\
&= \left\| \big((x_n + e_{n+1}) - (x_{n+1} + e_{n+2})\big) + \Big(1 - \frac{r_{n+1}}{r_n}\Big)\big(J_{r_n}(x_n + e_{n+1}) - (x_n + e_{n+1})\big) \right\| \\
&\le \|x_n - x_{n+1}\| + \|e_{n+1}\| + \|e_{n+2}\| + \frac{M_2}{\epsilon}|r_{n+1} - r_n|,
\end{aligned}
\tag{3.3}
$$

where M_2 is an appropriate constant such that

$$M_2 \ge \sup_{n\ge 1}\{\|J_{r_n}(x_n + e_{n+1}) - (x_n + e_{n+1})\|\}.$$

Notice that

$$
\begin{aligned}
y_{n+1} - y_n &= \alpha'_{n+1}(x_{n+1} - x_n) + x_n(\alpha'_{n+1} - \alpha'_n) \\
&+ \beta'_{n+1}\big(J_{r_{n+1}}(x_{n+1} + e_{n+2}) - J_{r_n}(x_n + e_{n+1})\big) + J_{r_n}(x_n + e_{n+1})(\beta'_{n+1} - \beta'_n) \\
&+ \gamma'_{n+1}(Q_C(f_{n+1}) - Q_C(f_n)) + Q_C(f_n)(\gamma'_{n+1} - \gamma'_n),
\end{aligned}
$$

which yields that

$$
\begin{aligned}
\|y_{n+1} - y_n\| &\le \alpha'_{n+1}\|x_{n+1} - x_n\| + \|x_n\||\alpha'_{n+1} - \alpha'_n| \\
&+ \beta'_{n+1}\|J_{r_{n+1}}(x_{n+1} + e_{n+2}) - J_{r_n}(x_n + e_{n+1})\| + \|J_{r_n}(x_n + e_{n+1})\||\beta'_{n+1} - \beta'_n| \\
&+ \gamma'_{n+1}\|Q_C(f_{n+1}) - Q_C(f_n)\| + \|Q_C(f_n)\||\gamma'_{n+1} - \gamma'_n|.
\end{aligned}
\tag{3.4}
$$

Substituting (3.3) into (3.4), we find that

$$
\begin{aligned}
\|y_{n+1} - y_n\| &\le \|x_{n+1} - x_n\| + \|x_n\||\alpha'_{n+1} - \alpha'_n| \\
&+ \|e_{n+1}\| + \|e_{n+2}\| + \frac{M_2}{\epsilon}|r_{n+1} - r_n| \\
&+ \|J_{r_n}(x_n + e_{n+1})\||\beta'_{n+1} - \beta'_n| \\
&+ \gamma'_{n+1}\|Q_C(f_{n+1}) - Q_C(f_n)\| + \|Q_C(f_n)\||\gamma'_{n+1} - \gamma'_n|.
\end{aligned}
\tag{3.5}
$$

Put $g_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}$. That is,

$$x_{n+1} = (1 - \beta_n)g_n + \beta_n x_n, \quad n \ge 0. \tag{3.6}$$

Now, we compute $\|g_{n+1} - g_n\|$. Note that

$$
\begin{aligned}
g_{n+1} - g_n &= \frac{\alpha_{n+1}Tx_{n+1} + \gamma_{n+1}y_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n Tx_n + \gamma_n y_n}{1 - \beta_n} \\
&= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}Tx_{n+1} + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}y_{n+1} - \frac{\alpha_n}{1 - \beta_n}Tx_n - \frac{\gamma_n}{1 - \beta_n}y_n \\
&= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(Tx_{n+1} - y_{n+1}) + y_{n+1} - \frac{\alpha_n}{1 - \beta_n}(Tx_n - y_n) - y_n.
\end{aligned}
$$

This yields that

$$\|g_{n+1} - g_n\| \le \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|Tx_{n+1} - y_{n+1}\| + \|y_{n+1} - y_n\| + \frac{\alpha_n}{1 - \beta_n}\|Tx_n - y_n\|. \tag{3.7}$$

Substituting (3.5) into (3.7), we find that

$$\|g_{n+1} - g_n\| - \|x_{n+1} - x_n\|$$
$$\le \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|Tx_{n+1} - y_{n+1}\| + \|x_n\||\alpha'_{n+1} - \alpha'_n|$$
$$+ \|e_{n+1}\| + \|e_{n+2}\| + \frac{M_2}{\epsilon}|r_{n+1} - r_n| + \|J_{r_n}(x_n + e_{n+1})\||\beta'_{n+1} - \beta'_n|$$
$$+ \gamma'_{n+1}\|Q_C(f_{n+1}) - Q_C(f_n)\| + \|Q_C(f_n)\||\gamma'_{n+1} - \gamma'_n| + \frac{\alpha_n}{1 - \beta_n}\|Tx_n - y_n\|.$$

From the restrictions (b)-(f), we see that

$$\limsup_{n \to \infty}\left(\|g_{n+1} - g_n\| - \|x_n - x_{n+1}\|\right) \le 0.$$

It follows from Lemma 2.3 that

$$\lim_{n \to \infty}\|g_n - x_n\| = 0.$$

In view of (3.6), we have

$$x_{n+1} - x_n = (1 - \beta_n)(g_n - x_n).$$

Hence, we find that

$$\lim_{n \to \infty}\|x_{n+1} - x_n\| = 0. \tag{3.8}$$

Notice that

$$y_n - x_n = \frac{x_{n+1} - x_n + \alpha_n\left(x_n - f(x_n)\right)}{\gamma_n}.$$

In view of the restrictions (b) and (c), we find from (3.8) that

$$\lim_{n \to \infty}\|y_n - x_n\| = 0. \tag{3.9}$$

Notice that

$$\|x_n - J_{r_n}(x_n + e_{n+1})\|$$
$$\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - J_{r_n}(x_n + e_{n+1})\|$$
$$\le \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - y_n\| + \beta_n\|x_n - y_n\| + \|y_n - J_{r_n}(x_n + e_{n+1})\|$$
$$\le \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - y_n\| + \beta_n\|x_n - y_n\|$$
$$+ \alpha'_n\|x_n - J_{r_n}(x_n + e_{n+1})\| + \gamma'_n\|Q_C(f_n) - J_{r_n}(x_n + e_{n+1})\|.$$

In view of the restrictions (b) and (d), we find from (3.8) and (3.9) that

$$\lim_{n \to \infty}\|x_n - J_{r_n}(x_n + e_{n+1})\| = 0. \tag{3.10}$$

Notice that

$$\|x_n - J_{r_n}x_n\| \le \|x_n - J_{r_n}(x_n + e_{n+1})\| + \|J_{r_n}(x_n + e_{n+1}) - J_{r_n}x_n\|$$
$$\le \|x_n - J_{r_n}(x_n + e_{n+1})\| + \|e_{n+1}\|.$$

Since $\sum_{n=1}^{\infty}\|e_n\| < \infty$, we see from (3.10) that

$$\lim_{n \to \infty}\|x_n - J_{r_n}x_n\| = 0.$$

Take a fixed number r such that $\epsilon > r > 0$. In view of Lemma 2.2, we obtain that

$$
\begin{aligned}
\|J_{r_n}x_n - J_r x_n\| &= \|J_r\left(\frac{r}{r_n}x_n + (1 - \frac{r}{r_n})J_{r_n}x_n\right) - J_r x_n\| \\
&\leq \|(1 - \frac{r}{r_n})(J_{r_n}x_n - x_n)\| \\
&\leq \|J_{r_n}x_n - x_n\|.
\end{aligned}
\tag{3.11}
$$

Note that

$$
\begin{aligned}
\|x_n - J_r x_n\| &\leq \|x_n - J_{r_n}x_n\| + \|J_{r_n}x_n - J_r x_n\| \\
&\leq 2\|x_n - J_{r_n}x_n\|.
\end{aligned}
$$

Using (3.11), we see that

$$
\lim_{n \to \infty} \|x_n - J_r x_n\| = 0. \tag{3.12}
$$

Next, we claim that $\limsup_{n \to \infty}\langle \bar{x} - T\bar{x}, j(x_n - \bar{x})\rangle \leq 0$, where $\bar{x} = \lim_{t \to 0} z_t$, and z_t solves the fixed point equation $z_t = tTz_t + (1 - t)J_r z_t$, $\forall t \in (0, 1)$, from which it follows that

$$
\|z_t - x_n\| = \|(1 - t)(J_r z_t - x_n) + t(Tz_t - x_n)\|.
$$

For any $t \in (0, 1)$, we see that

$$
\begin{aligned}
\|z_t - x_n\|^2 &= (1 - t)\langle J_r z_t - x_n, j(z_t - x_n)\rangle + t\langle Tz_t - x_n, j(z_t - x_n)\rangle \\
&= (1 - t)\left(\langle J_r z_t - J_r x_n, j(z_t - x_n)\rangle + \langle J_r x_n - x_n, j(z_t - x_n)\rangle\right) \\
&\quad + t\langle Tz_t - z_t, j(z_t - x_n)\rangle + t\langle z_t - x_n, j(z_t - x_n)\rangle \\
&\leq (1 - t)\left(\|z_t - x_n\|^2 + \|J_r x_n - x_n\|\|z_t - x_n\|\right) \\
&\quad + t\langle Tz_t - z_t, j(z_t - x_n)\rangle + t\|z_t - x_n\|^2 \\
&\leq \|z_t - x_n\|^2 + \|J_r x_n - x_n\|\|z_t - x_n\| + t\langle Tz_t - z_t, j(z_t - x_n)\rangle.
\end{aligned}
$$

It follows that

$$
\langle z_t - Tz_t, j(z_t - x_n)\rangle \leq \frac{1}{t}\|J_r x_n - x_n\|\|z_t - x_n\| \quad \forall t \in (0, 1).
$$

In view of (3.12), we see that

$$
\limsup_{n \to \infty}\langle z_t - Tz_t, j(z_t - x_n)\rangle \leq 0. \tag{3.13}
$$

Since $z_t \to \bar{x}$, as $t \to 0$ and the fact that j is strong to weak* uniformly continuous on bounded subsets of E, we see that

$$
\begin{aligned}
&|\langle T\bar{x} - \bar{x}, j(x_n - \bar{x})\rangle - \langle z_t - Tz_t, j(z_t - x_n)\rangle| \\
&\leq |\langle T\bar{x} - \bar{x}, j(x_n - \bar{x})\rangle - \langle T\bar{x} - \bar{x}, j(x_n - z_t)\rangle| \\
&\quad + |\langle T\bar{x} - \bar{x}, j(x_n - z_t)\rangle - \langle z_t - Tz_t, j(z_t - x_n)\rangle| \\
&\leq |\langle T\bar{x} - \bar{x}, j(x_n - \bar{x}) - j(x_n - z_t)\rangle| + |\langle T\bar{x} - \bar{x} + z_t - Tz_t, J(x_n - z_t)\rangle| \\
&\leq \|T\bar{x} - \bar{x}\|\|j(x_n - \bar{x}) - j(x_n - z_t)\| + \|T\bar{x} - \bar{x} + z_t - Tz_t\|\|x_n - z_t\| \to 0, \quad \text{as } t \to 0.
\end{aligned}
$$

Hence, for any $\epsilon > 0$, there exists $\lambda > 0$ such that $\forall t \in (0, \lambda)$ the following inequality holds

$$
\langle T\bar{x} - \bar{x}, j(x_n - \bar{x})\rangle \leq \langle z_t - Tz_t, j(z_t - x_n)\rangle + \epsilon.
$$

This implies that

$$
\limsup_{n \to \infty}\langle T\bar{x} - \bar{x}, j(x_n - \bar{x})\rangle \leq \limsup_{n \to \infty}\langle z_t - Tz_t, j(z_t - x_n)\rangle + \epsilon.
$$

Since ϵ is arbitrary and (3.13), we see that $\limsup_{n\to\infty}\langle T\bar{x} - \bar{x}, j(x_n - \bar{x})\rangle \leq 0$. This implies that

$$\limsup_{n\to\infty}\langle T\bar{x} - \bar{x}, j(x_{n+1} - \bar{x})\rangle \leq 0. \tag{3.14}$$

Finally, we show that $x_n \to \bar{x}$ as $n \to \infty$. Note that

$$\begin{aligned}
\|y_n - \bar{x}\| &\leq \alpha_n'\|x_n - \bar{x}\| + \beta_n'\|J_{r_n}(x_n + e_{n+1}) - \bar{x}\| + \gamma_n'\|Q_C f_n - \bar{x}\| \\
&\leq (\alpha_n' + \beta_n')\|x_n - \bar{x}\| + \|e_{n+1}\| + \gamma_n'\|f_n - \bar{x}\| \\
&\leq \|x_n - \bar{x}\| + \|e_{n+1}\| + \gamma_n' M_2,
\end{aligned} \tag{3.15}$$

where $M_2 = \sup_{n\geq 1}\|f_n - \bar{x}\|$. Put $\delta_n = \|e_{n+1}\| + \gamma_n' M_2$. In view of the restriction (d), we see that $\sum_{n=1}^{\infty} \delta_n < \infty$. Notice that

$$\begin{aligned}
\|x_{n+1} - \bar{x}\|^2 &\leq \alpha_n\langle Tx_n - \bar{x}, j(x_{n+1} - \bar{x})\rangle + \beta_n\|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\| \\
&\quad + \gamma_n\|y_n - \bar{x}\|\|x_{n+1} - \bar{x}\| \\
&\leq \alpha_n\langle Tx_n - \bar{x}, j(x_{n+1} - \bar{x})\rangle + \frac{\beta_n}{2}(\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2) \\
&\quad + \frac{\gamma_n}{2}(\|y_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2).
\end{aligned}$$

It follows from (3.15) that

$$\begin{aligned}
\|x_{n+1} - \bar{x}\|^2 &\leq 2\alpha_n\langle Tx_n - \bar{x}, j(x_{n+1} - \bar{x})\rangle + \beta_n\|x_n - \bar{x}\|^2 + \gamma_n\|y_n - \bar{x}\|^2 \\
&\leq 2\alpha_n\langle Tx_n - \bar{x}, j(x_{n+1} - \bar{x})\rangle + \beta_n\|x_n - \bar{x}\|^2 + \gamma_n\|x_n - \bar{x}\|^2 \\
&\quad + \gamma_n\delta_n^2 + 2\gamma_n\delta_n\|x_n - \bar{x}\| \\
&\leq (1 - \alpha_n)\|x_n - \bar{x}\|^2 + 2\alpha_n\langle Tx_n - \bar{x}, j(x_{n+1} - \bar{x})\rangle + \delta_n M_3,
\end{aligned} \tag{3.16}$$

where M_3 is an appropriate constant such that $M_3 \geq \sup_{n\geq 0}\{\delta_n + 2\|x_n - \bar{x}\|\}$. Let $\mu_n = \max\{\langle Tx_n - \bar{x}, j(x_{n+1} - \bar{x})\rangle, 0\}$. Next, we show that $\lim_{n\to\infty}\mu_n = 0$. Indeed, from (3.14), for any give $\epsilon > 0$, there exists a positive integer n_1 such that

$$\langle Tx_n - \bar{x}, j(x_{n+1} - \bar{x})\rangle < \epsilon, \quad \forall n \geq n_1.$$

This implies that $0 \leq \mu_n < \epsilon, \forall n \geq n_1$. Since $\epsilon > 0$ is arbitrary, we see that $\lim_{n\to\infty}\mu_n = 0$. It follows from (3.16) that

$$\|x_{n+1} - \bar{x}\|^2 \leq (1 - \alpha_n)\|x_n - \bar{x}\|^2 + 2\alpha_n\mu_n + \delta_n M_3.$$

Put $a_n = \|x_n - \bar{x}\|^2$, $t_n = \alpha_n$, $b_n = 2\alpha_n\mu_n$ and $c_n = \delta_n M_3$ for every $n \geq 0$. In view of Lemma 2.1, we find the desired conclusion. $\square$

If the mapping T maps every point in C to a fixed element, then we have the following results.

Corollary 3.2. *Let E be a real reflexive Banach space with the uniformly Gâteaux differentiable norm and A be an m-accretive operators in E. Assume that $C := \overline{D(A)}$ is convex and has the normal structure. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha_n'\}$, $\{\beta_n'\}$ and $\{\gamma_n'\}$ be real number sequences in $(0, 1)$. Let Q_C be the sunny nonexpansive retraction from E onto C and $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases}
x_0 \in C, \\
y_n = \alpha_n'x_n + \beta_n'J_{r_n}(x_n + e_{n+1}) + \gamma_n'Q_C(f_n), \\
x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n y_n, \quad \forall n \geq 0,
\end{cases}$$

where u is a fixed element in C, $\{e_n\}$ is a sequence in E, $\{f_n\}$ is a bounded sequence in E, $\{r_n\}$ is a positive real numbers sequence, and $J_{r_n} = (I + r_n A)^{-1}$. Assume that $A^{-1}(0)$ is not empty and the above control sequences satisfy the following restrictions:

(a) $\alpha_n + \beta_n + \gamma_n = \alpha_n' + \beta_n' + \gamma_n' = 1$;

(b) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \alpha'_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(c) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;
(d) $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *and* $\sum_{n=0}^{\infty} \gamma'_n < \infty$;
(e) $\lim_{n\to\infty} |\beta'_n - \beta'_{n+1}| = 0$;
(f) $r_n \ge \epsilon$ *for each* $n \ge 1$ *and* $\lim_{n\to\infty} |r_n - r_{n+1}| = 0$.
Then the sequence $\{x_n\}$ *converges strongly to* $\bar{x}$, *which is the unique solution to the following variational inequality*

$$\langle u - Q_{A^{-1}(0)}u, j(p - Q_{A^{-1}(0)}u)\rangle \le 0, \quad \forall p \in A^{-1}(0).$$

If $\gamma'_n = 0$, then Corollary 3.2 is reduced to the following.

Corollary 3.3. *Let E be a real reflexive Banach space with the uniformly Gâteaux differentiable norm and let A be an m-accretive operators in E. Assume that $C := \overline{D(A)}$ is convex and has the normal structure. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha_n'\}$, and $\{\beta_n'\}$ be real number sequences in $(0,1)$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in C, \\ y_n = \alpha_n' x_n + \beta_n' J_{r_n}(x_n + e_{n+1}), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n y_n, \quad \forall n \ge 0, \end{cases}$$

where u is a fixed element in C, $\{e_n\}$ is a sequence in E, $\{r_n\}$ is a positive real numbers sequence, and $J_{r_n} = (I + r_n A)^{-1}$. Assume that $A^{-1}(0)$ is not empty and the above control sequences satisfy the following restrictions:
    (a) $\alpha_n + \beta_n + \gamma_n = \alpha_n' + \beta_n' = 1$;
    (b) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \alpha'_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
    (c) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;
    (d) $\sum_{n=1}^{\infty} \|e_n\| < \infty$;
    (e) $\lim_{n\to\infty} |\beta'_n - \beta'_{n+1}| = 0$;
    (f) $r_n \ge \epsilon$ *for each* $n \ge 1$ *and* $\lim_{n\to\infty} |r_n - r_{n+1}| = 0$.
Then the sequence $\{x_n\}$ *converges strongly to* $\bar{x}$, *which is the unique solution to the following variational inequality*

$$\langle u - Q_{A^{-1}(0)}u, j(p - Q_{A^{-1}(0)}u)\rangle \le 0, \quad \forall p \in A^{-1}(0).$$

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