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Mixed-type Reverse Order Law for Moore-Penrose Inverse of Products of Three Elements in Ring with Involution

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Abstract. In this paper we establish some results concerning the mixed-type reverse order laws for the Moore-Penrose inverse of various products of three elements in rings with involution.

1. Introduction

Let R be an associative ring with unity and an involution $a \mapsto a^*$ satisfying $(a^*)^* = a$, $(a+b)^* = a^* + b^*$, $(ab)^* = b^*a^*$. An element $a \in R$ has Moore-Penrose inverse, if there exists b such that the following equations hold [11]:

$$(1)\ \ aba = a, \qquad (2)\ \ bab = b, \qquad (3)\ \ (ab)^* = ab, \qquad (4)\ \ (ba)^* = ba.$$

In this case, b is unique and denoted by $a^\dagger$. The set of all Moore-Penrose invertible elements of R is denoted by $R^\dagger$.

The well-known reverse order law for the ordinary inverses states that $(ab)^{-1} = b^{-1}a^{-1}$, where a and b are invertible in R. However, this formula cannot trivially be extended to the Moore-Penrose inverse of ab. Many authors studied this problem and gave some equivalent conditions for $(ab)^\dagger = b^\dagger a^\dagger$, as well as $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ in settings of matrices, C^*-algebra and rings (see, e.g., [1]-[10] and [12]). In 2007, Y. Tian [13] investigated necessary and sufficient conditions for a group of mixed-type reverse order laws to hold for the Moore-Penrose inverse of a triple matrix product. Recently, N. Č. Dinčić and D. S. Djordjević [2] studied mixed-type reverse order law for various products of three operators on Hilbert spaces. Motivated by [13] and [2], we consider mixed-type reverse order law for Moore-Penrose inverse of products of three elements in rings with involution.

Rank formulas played an important role in [13], while [2] adopted the matrix representation of operators with respect to the orthogonal decomposition of Hilbert spaces. In contrast to the above papers, we present a purely ring theoretical proof of some equivalent conditions related to the mixed-type reverse order law for the Moore-Penrose inverse. Thus, some known results from [2] are extend to more general settings.

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2. Main Results

We start with some simple lemmas on Moore-Penrose inverse that will be used later on in the paper.

Lemma 2.1. *The following hold true for any $a \in R^\dagger$.*
(i) $a^*a, aa^* \in R^\dagger$. *Moreover,* $(a^*a)^\dagger = a^\dagger(a^*)^\dagger$, $(aa^*)^\dagger = (a^*)^\dagger a^\dagger$ *and* $a^\dagger = (a^*a)^\dagger a^* = a^*(aa^*)^\dagger$.
(ii) $(aa^*)^\dagger(aa^*) = (aa^*)(aa^*)^\dagger$ *and* $(a^*a)^\dagger(a^*a) = (a^*a)(a^*a)^\dagger$.
(iii) $(aa^*)^n \in R^\dagger$ *and* $[(aa^*)^n]^\dagger = [(aa^*)^\dagger]^n$*, where n is any positive integer.*
(iv) $(aa^*)^n a \in R^\dagger$ *and* $[(aa^*)^n a]^\dagger = a^\dagger[(aa^*)^\dagger]^n = a^\dagger[(a^\dagger)^*a^\dagger]^n$ *for any positive integer n.*

Proof. (i) is straightforward to check.
(ii) The first equality holds since $(aa^*)^\dagger(aa^*) = ((aa^*)^\dagger(aa^*))^* = (aa^*)^*((aa^*)^\dagger)^* = (aa^*)(aa^*)^\dagger$. The second can be verified similarly.
(iii) follows by (ii).
(iv) $[(aa^*)^n a]^\dagger = a^\dagger[(aa^*)^\dagger]^n$ follows by (i), (ii) and (iii). $a^\dagger[(aa^*)^\dagger]^n = a^\dagger[(a^\dagger)^*a^\dagger]^n$ follows by $(aa^*)^\dagger = (a^*)^\dagger a^\dagger$ in (i). □

Lemma 2.2. *Let $a \in R$ and $b \in R^\dagger$ be such that $a^* = a$ and $bR \subseteq aR$. Then $abR = bR$ if and only if $abb^\dagger = bb^\dagger a$.*

Proof. If $abR = bR$, there exists $r \in R$ such that $ab = br$. Then $abb^\dagger = brb^\dagger = bb^\dagger brb^\dagger = bb^\dagger abb^\dagger$ and $bb^\dagger a = (bb^\dagger)^*a^* = (abb^\dagger)^* = (bb^\dagger abb^\dagger)^* = (bb^\dagger)^*a^*(bb^\dagger)^* = bb^\dagger abb^\dagger = abb^\dagger$.
Conversely, if $abb^\dagger = bb^\dagger a$ then we have $abR \subseteq bR$ since $ab = abb^\dagger b = bb^\dagger ab$. By hypothesis $bR \subseteq aR$, there exists $r' \in R$ such that $b = ar'$. Consequently, $b = bb^\dagger b = bb^\dagger ar' = abb^\dagger r'$. This implies $bR \subseteq abR$. Therefore, $abR = bR$. □

Lemma 2.3. *Let $b \in R$ and $a \in R^\dagger$ be such that $a^* = a$. Then $a^\dagger bR = bR$ if and only if $abR = bR$.*

Proof. If $a^\dagger bR = bR$, there exist $r, r' \in R$ such that $b = a^\dagger br$ and $a^\dagger b = br'$. Then we get $ab = aa^\dagger br = aa^\dagger a^\dagger brr = (aa^\dagger)^*a^\dagger brr = (a^\dagger)^*a^*a^\dagger brr = a^\dagger aa^\dagger brr = a^\dagger brr = br'rr$. This implies $abR \subseteq bR$. Simultaneously, we have $bR \subseteq abR$ since $b = a^\dagger br = a^\dagger aa^\dagger br = (a^\dagger a)^*a^\dagger br = aa^\dagger a^\dagger br = aa^\dagger b = abr'$. This shows $bR = abR$.
Conversely, if $abR = bR$ then $(a^\dagger)^\dagger bR = abR = bR$. By the above argument, we have $a^\dagger bR = bR$. □

Next, we prove the mixed-type reverse order law for the MP-inverse of various products of three elements. In what follows, let $a_1, a_2, a_3 \in R$ and $m = a_1a_2a_3$.

Theorem 2.4. *Suppose that $a_1, a_3, m, a_1^\dagger ma_3^\dagger \in R^\dagger$. Then the following statements are equivalent:*
(i) $m^\dagger = a_3^\dagger(a_1^\dagger ma_3^\dagger)^\dagger a_1^\dagger$.
(ii) $a_1a_1^*mR = mR$ *and* $a_3^*a_3m^*R = m^*R$.

Proof. (i)⇒(ii) By hypothesis, we have the following equation

$$\begin{aligned} m^\dagger &= a_3^\dagger(a_1^\dagger ma_3^\dagger)^\dagger a_1^\dagger \\ &= a_3^\dagger(a_1^\dagger ma_3^\dagger)^\dagger(a_1^\dagger ma_3^\dagger)(a_1^\dagger ma_3^\dagger)^\dagger a_1^\dagger \\ &= a_3^\dagger(a_1^\dagger ma_3^\dagger)^*((a_1^\dagger ma_3^\dagger)^\dagger)^*(a_1^\dagger ma_3^\dagger)^\dagger a_1^\dagger. \end{aligned} \tag{1}$$

Multiplying (1) by a_3 from the left-hand side, we get

$$\begin{aligned} a_3m^\dagger &= a_3a_3^\dagger(a_1^\dagger ma_3^\dagger)^*((a_1^\dagger ma_3^\dagger)^\dagger)^*(a_1^\dagger ma_3^\dagger)^\dagger a_1^\dagger \\ &= a_3a_3^\dagger(a_3^\dagger)^*(a_1^\dagger m)^*((a_1^\dagger ma_3^\dagger)^\dagger)^*(a_1^\dagger ma_3^\dagger)^\dagger a_1^\dagger \\ &= (a_1^\dagger ma_3^\dagger)^*((a_1^\dagger ma_3^\dagger)^\dagger)^*(a_1^\dagger ma_3^\dagger)^\dagger a_1^\dagger \\ &= (a_1^\dagger ma_3^\dagger)^\dagger(a_1^\dagger ma_3^\dagger)(a_1^\dagger ma_3^\dagger)^\dagger a_1^\dagger \\ &= (a_1^\dagger ma_3^\dagger)^\dagger((a_1^\dagger ma_3^\dagger)^\dagger)^*(a_1^\dagger ma_3^\dagger)^*a_1^\dagger. \end{aligned} \tag{2}$$

Multiplying (2) by a_1 from the right-hand side, we get

$$\begin{aligned} a_3m^\dagger a_1 &= (a_1^\dagger ma_3^\dagger)^\dagger((a_1^\dagger ma_3^\dagger)^\dagger)^*(a_1^\dagger ma_3^\dagger)^*a_1^\dagger a_1 \\ &= (a_1^\dagger ma_3^\dagger)^\dagger((a_1^\dagger ma_3^\dagger)^\dagger)^*(a_1^\dagger ma_3^\dagger)^* = (a_1^\dagger ma_3^\dagger)^\dagger, \end{aligned} \tag{3}$$

whence

$$(a_1^\dagger ma_3^\dagger)(a_1^\dagger ma_3^\dagger)^\dagger = a_1^\dagger ma_3^\dagger a_3m^\dagger a_1 = a_1^\dagger mm^\dagger a_1.$$

It follows that $a_1^\dagger mm^\dagger a_1 = (a_1^\dagger mm^\dagger a_1)^*$. Multiplying it by a_1 from the left-hand side and a_1^* from the right-hand side, we get $a_1a_1^\dagger mm^\dagger a_1a_1^* = a_1(a_1^\dagger mm^\dagger a_1)^*a_1^*$. Note that $a_1a_1^\dagger mm^\dagger a_1a_1^* = mm^\dagger a_1a_1^*$ and $a_1(a_1^\dagger mm^\dagger a_1)^*a_1^* = a_1a_1^*(m^\dagger)^*m^*(a_1^\dagger)^*a_1^* = a_1a_1^*mm^\dagger$. So we have

$$mm^\dagger a_1a_1^* = a_1a_1^*mm^\dagger. \tag{4}$$

Similarly, one can verify

$$(a_1^\dagger ma_3^\dagger)^\dagger(a_1^\dagger ma_3^\dagger) = a_3m^\dagger a_1(a_1^\dagger ma_3^\dagger) = a_3m^\dagger ma_3^\dagger,$$

from which we can see that $a_3m^\dagger ma_3^\dagger = (a_3m^\dagger ma_3^\dagger)^*$. Multiplying it by a_3^* from the left-hand side and a_3 from the right-hand side, we get

$$a_3^*a_3m^\dagger ma_3^\dagger a_3 = a_3^*(a_3m^\dagger ma_3^\dagger)^*a_3.$$

This yields

$$a_3^*a_3m^\dagger m = m^\dagger ma_3^*a_3. \tag{5}$$

Since $m = a_1a_2a_3 = (a_1a_1^*)((a_1^\dagger)^*a_2a_3)$ and $m^* = a_3^*a_2^*a_1^* = (a_3^*a_3)(a_3^\dagger a_2^*a_1^*)$, it follows that $mR \subseteq (a_1a_1^*)R$ and $m^*R \subseteq (a_3^*a_3)R$. By (4), (5) and Lemma 2.2, we have $a_1a_1^*mR = mR$ and $a_3^*a_3m^*R = m^*R$.

(ii)⇒(i) By hypothesis and Lemma 2.2, we have $mm^\dagger a_1a_1^* = a_1a_1^*mm^\dagger$ and $a_3^*a_3m^\dagger m = m^\dagger ma_3^*a_3$. Multiply the first equation from the left-hand side by $a_1^\dagger$ and from the right-hand side by $(a_1^\dagger)^*$, and multiply the second equation from the left-hand side by $(a_3^\dagger)^*$ and from the right-hand side by $a_3^\dagger$, then we get $a_1^\dagger mm^\dagger a_1 = a_1^*mm^\dagger(a_1^\dagger)^*$ and $a_3m^\dagger ma_3^\dagger = (a_3^\dagger)^*m^\dagger ma_3^*$.

Now, it is straightforward to check $(a_1^\dagger ma_3^\dagger)^\dagger = a_3m^\dagger a_1$. Finally, we have

$$\begin{aligned} a_3^\dagger(a_1^\dagger ma_3^\dagger)^\dagger a_1^\dagger &= a_3^\dagger a_3m^\dagger a_1a_1^\dagger = a_3^\dagger a_3m^\dagger mm^\dagger a_1a_1^\dagger = (m^\dagger ma_3^\dagger a_3)^*m^\dagger a_1a_1^\dagger \\ &= m^\dagger mm^\dagger a_1a_1^\dagger = m^\dagger(a_1a_1^\dagger mm^\dagger)^* = m^\dagger(mm^\dagger)^* = m^\dagger. \end{aligned}$$

□

If a_1 and a_3 have MP-inverse then $a_1a_2a_3 = (a_1^\dagger)^*(a_1^*a_1a_2a_3a_3^*)(a_3^\dagger)^*$. Substituting $\widetilde{a_1} = (a_1^\dagger)^*$, $\widetilde{a_2} = a_1^*a_1a_2a_3a_3^*$ and $\widetilde{a_3} = (a_3^\dagger)^*$ for a_1, a_2 and a_3 respectively in Theorem 2.4, we can establish another representation for $m^\dagger$ under suitable conditions.

Theorem 2.5. *Suppose that $a_1, a_3, m, a_1^*ma_3^* \in R^\dagger$. Then the following statements are equivalent:*

(i) $m^\dagger = a_3^*(a_1^*ma_3^*)^\dagger a_1^*$.

(ii) $a_1a_1^*mR = mR$ *and* $a_3^*a_3m^*R = m^*R$.

Proof. By hypothesis, $m = a_1a_2a_3 = (a_1^\dagger)^*(a_1^*a_1a_2a_3a_3^*)(a_3^\dagger)^*$. Let

$$\widetilde{a_1} = (a_1^\dagger)^*, \ \widetilde{a_2} = a_1^*a_1a_2a_3a_3^*, \ \widetilde{a_3} = (a_3^\dagger)^*.$$

Then we have $m = \widetilde{a_1}\widetilde{a_2}\widetilde{a_3}$, $\widetilde{a_1}, \widetilde{a_3} \in R^\dagger$ and $(\widetilde{a_1})^\dagger m(\widetilde{a_3})^\dagger = a_1^*ma_3^* \in R^\dagger$.

According to Theorem 2.4, we know that the following conditions are equivalent:

(i′) $m^\dagger = (\widetilde{a_3})^\dagger((\widetilde{a_1})^\dagger m(\widetilde{a_3})^\dagger)^\dagger(\widetilde{a_1})^\dagger$;

(ii′) $\widetilde{a_1}(\widetilde{a_1})^*mR = mR$ and $(\widetilde{a_3})^*\widetilde{a_3}m^*R = m^*R$.

Note that $(\widetilde{a_3})^\dagger((\widetilde{a_1})^\dagger m(\widetilde{a_3})^\dagger)^\dagger(\widetilde{a_1})^\dagger = a_3^*(a_1^*ma_3^*)^\dagger a_1^*$, $\widetilde{a_1}(\widetilde{a_1})^* = (a_1^\dagger)^*a_1^\dagger = (a_1a_1^*)^\dagger$ and $(\widetilde{a_3})^*\widetilde{a_3} = a_3^\dagger(a_3^\dagger)^* = (a_3^*a_3)^\dagger$. Thus (i′) and (ii′) can be restated as follows:

(i) $m^\dagger = a_3^*(a_1^* m a_3^*)^\dagger a_1^*$;
(ii') $(a_1a_1^*)^\dagger mR = mR$ and $(a_3^*a_3)^\dagger m^*R = m^*R$.
By Lemma 2.3, (ii') is equivalent to (ii). Therefore, the result follows. □

Using a similar method as in Theorem 2.5, we obtain the following result based on Theorem 2.4.

Theorem 2.6. *If $a_1, a_3, m, ((a_1a_1^*)^\dagger)^k m((a_3^*a_3)^\dagger)^l \in R^\dagger$, then the following statements are equivalent for all positive integers k and l:*
(i) $m^\dagger = ((a_3^*a_3)^\dagger)^l(((a_1a_1^*)^\dagger)^k m((a_3^*a_3)^\dagger)^l)^\dagger((a_1a_1^*)^\dagger)^k$.
(ii) $(a_1a_1^*)^{2k}mR = mR$ *and* $(a_3^*a_3)^{2l}m^*R = m^*R$.

Proof. By Lemma 2.1(iii), we have $((a_3^*a_3)^\dagger)^l = ((a_3^*a_3)^l)^\dagger$ and $((a_1a_1^*)^\dagger)^k = ((a_1a_1^*)^k)^\dagger$. Thus condition (i) can be restated as
(i) $m^\dagger = ((a_3^*a_3)^l)^\dagger(((a_1a_1^*)^k)^\dagger m((a_3^*a_3)^l)^\dagger)^\dagger((a_1a_1^*)^k)^\dagger$.
Note that $m = (a_1a_1^*)^k((a_1a_1^*)^\dagger)^{k-1}(a_1^*)^\dagger a_2(a_3^\dagger)^*((a_3^*a_3)^\dagger)^{l-1}(a_3^*a_3)^l$. We define $\widetilde{a_1} = (a_1a_1^*)^k$, $\widetilde{a_2} = ((a_1a_1^*)^\dagger)^{k-1}(a_1^*)^\dagger a_2(a_3^\dagger)^*((a_3^*a_3)^\dagger)^{l-1}$ and $\widetilde{a_3} = (a_3^*a_3)^l$. Then $m = \widetilde{a_1}\widetilde{a_2}\widetilde{a_3}$. By Lemma 2.1(iii), we have $\widetilde{a_1}, \widetilde{a_3} \in R^\dagger$. In addition, $(\widetilde{a_1})^\dagger m(\widetilde{a_3})^\dagger = ((a_1a_1^*)^\dagger)^k m((a_3^*a_3)^\dagger)^l \in R^\dagger$. In view of Theorem 2.4, we know that the following are equivalent:
(i') $m^\dagger = (\widetilde{a_3})^\dagger((\widetilde{a_1})^\dagger m(\widetilde{a_3})^\dagger)^\dagger(\widetilde{a_1})^\dagger$;
(ii') $\widetilde{a_1}(\widetilde{a_1})^* mR = mR$ and $(\widetilde{a_3})^*\widetilde{a_3}m^*R = m^*R$.
But (i') is just a restatement of (i) while (ii') coincides with (ii) since $\widetilde{a_1}(\widetilde{a_1})^* = (a_1a_1^*)^{2k}$ and $(\widetilde{a_3})^*\widetilde{a_3} = (a_3^*a_3)^{2l}$. □

Taking $k = l = 1$ in Theorem 2.6, we obtain the following corollary, which will be used in the next section.

Corollary 2.7. *Let $a_1, a_3, m, (a_1a_1^*)^\dagger m(a_3^*a_3)^\dagger \in R^\dagger$. Then the following statements are equivalent:*
(i) $m^\dagger = (a_3^*a_3)^\dagger((a_1a_1^*)^\dagger m(a_3^*a_3)^\dagger)^\dagger(a_1a_1^*)^\dagger$.
(ii) $(a_1a_1^*)^2mR = mR$ *and* $(a_3^*a_3)^2m^*R = m^*R$.

Remark 2.8. Since $(a_1a_1^*)^\dagger m(a_3^*a_3)^\dagger = (a_1a_1^*)^\dagger a_1a_2a_3(a_3^*a_3)^\dagger = (a_1^*)^\dagger a_2(a_3^*)^\dagger = (a_3^\dagger a_2^* a_1^\dagger)^*$, the equality in Corollary 2.7(i) can be written as $m^\dagger = (a_3^*a_3)^\dagger((a_3^\dagger a_2^* a_1^\dagger)^\dagger)^*(a_1a_1^*)^\dagger$.

From Theorem 2.5, we have the following result.

Theorem 2.9. *Let $a_1, a_3, m, (a_1a_1^*)^k m(a_3^*a_3)^l \in R^\dagger$. Then the following statements are equivalent for all positive integers k and l:*
(i) $m^\dagger = (a_3^*a_3)^l((a_1a_1^*)^k m(a_3^*a_3)^l)^\dagger(a_1a_1^*)^k$.
(ii) $(a_1a_1^*)^{2k}mR = mR$ *and* $(a_3^*a_3)^{2l}m^*R = m^*R$.

Proof. By Lemma 2.1, it is easy to check

$$m = (a_1a_1^*)^k(((a_1a_1^*)^\dagger)^{k-1}(a_1^\dagger)^*a_2(a_3^\dagger)^*((a_3^*a_3)^\dagger)^{l-1})(a_3^*a_3)^l.$$

Let $\widetilde{a_1} = (a_1a_1^*)^k$, $\widetilde{a_2} = ((a_1a_1^*)^\dagger)^{k-1}(a_1^*)^\dagger a_2(a_3^\dagger)^*((a_3^*a_3)^\dagger)^{l-1}$, $\widetilde{a_3} = (a_3^*a_3)^l$. Then $m = \widetilde{a_1}\widetilde{a_2}\widetilde{a_3}$. By Lemma 2.1 again, we have $\widetilde{a_1}, \widetilde{a_3} \in R^\dagger$. Simultaneously, $(\widetilde{a_1})^* m(\widetilde{a_3})^* = (a_1a_1^*)^k m(a_3^*a_3)^l \in R^\dagger$ by hypothesis.
Now, Theorem 2.5 ensures that the following are equivalent:
(i') $m^\dagger = (\widetilde{a_3})^*((\widetilde{a_1})^* m(\widetilde{a_3})^*)^\dagger(\widetilde{a_1})^*$;
(ii') $\widetilde{a_1}(\widetilde{a_1})^* mR = mR$ and $(\widetilde{a_3})^*\widetilde{a_3}m^*R = m^*R$.
It is easy to see that (i') and (ii') coincide with (i) and (ii), respectively. Therefore, (i) and (ii) are equivalent. □

As a particular case of Theorem 2.9, we have the following corollary.

Corollary 2.10. *The following are equivalent provided that a_1, a_3, m, $a_1a_1^* m a_3^*a_3 \in R^\dagger$.*
(i) $m^\dagger = a_3^*a_3(a_1a_1^* m a_3^*a_3)^\dagger a_1a_1^*$.
(ii) $(a_1a_1^*)^2mR = mR$ *and* $(a_3^*a_3)^2m^*R = m^*R$.

Theorem 2.11. *Suppose that* $a_1, a_3, m, ((a_1a_1^*)^k a_1)^\dagger m((a_3a_3^*)^l a_3)^\dagger \in R^\dagger$. *Then the following statements are equivalent for all positive integers k and l:*

(i) $m^\dagger = ((a_3a_3^*)^l a_3)^\dagger(((a_1a_1^*)^k a_1)^\dagger m((a_3a_3^*)^l a_3)^\dagger)^\dagger((a_1a_1^*)^k a_1)^\dagger$.

(ii) $(a_1a_1^*)^{2k+1}mR = mR$ *and* $(a_3^*a_3)^{2l+1}m^*R = m^*R$.

Proof. First, we recompose m as $m = ((a_1a_1^*)^k a_1)(((a_1^*a_1)^\dagger)^k a_2((a_3a_3^*)^\dagger)^l)((a_3a_3^*)^l a_3)$. Then set $\widetilde{a_1} = (a_1a_1^*)^k a_1$, $\widetilde{a_2} = ((a_1^*a_1)^\dagger)^k a_2((a_3a_3^*)^\dagger)^l$, $\widetilde{a_3} = (a_3a_3^*)^l a_3$. By Lemma 2.1(iv), we have $\widetilde{a_1}, \widetilde{a_3} \in R^\dagger$. Moreover, $(\widetilde{a_1})^\dagger m(\widetilde{a_3})^\dagger = ((a_1a_1^*)^k a_1)^\dagger m((a_3a_3^*)^l a_3)^\dagger \in R^\dagger$.

By Theorem 2.4, we know that

$$m^\dagger = (\widetilde{a_3})^\dagger((\widetilde{a_1})^\dagger m(\widetilde{a_3})^\dagger)^\dagger(\widetilde{a_1})^\dagger \quad \Leftrightarrow \quad \widetilde{a_1}(\widetilde{a_1})^* mR = mR \text{ and } (\widetilde{a_3})^*\widetilde{a_3}m^*R = m^*R.$$

Thus, the result follows from the following facts:

(1) $(\widetilde{a_3})^\dagger((\widetilde{a_1})^\dagger m(\widetilde{a_3})^\dagger)^\dagger(\widetilde{a_1})^\dagger = ((a_3a_3^*)^l a_3)^\dagger(((a_1a_1^*)^k a_1)^\dagger m((a_3a_3^*)^l a_3)^\dagger)^\dagger((a_1a_1^*)^k a_1)^\dagger$;

(2) $\widetilde{a_1}(\widetilde{a_1})^* = (a_1a_1^*)^{2k+1}$ and $(\widetilde{a_3})^*\widetilde{a_3} = (a_3^*a_3)^{2l+1}$. □

The following corollary is a special case of Theorem 2.11.

Corollary 2.12. *Let* $a_1, a_3, m, (a_1a_1^*a_1)^\dagger m(a_3a_3^*a_3)^\dagger \in R^\dagger$. *Then the following statements are equivalent:*

(i) $m^\dagger = (a_3a_3^*a_3)^\dagger((a_1a_1^*a_1)^\dagger m(a_3a_3^*a_3)^\dagger)^\dagger(a_1a_1^*a_1)^\dagger$.

(ii) $(a_1a_1^*)^3 mR = mR$ *and* $(a_3^*a_3)^3 m^*R = m^*R$.

Theorem 2.13. *Suppose that* a_1, a_3, m, $((a_1a_1^*)^k a_1)^* m((a_3a_3^*)^l a_3)^* \in R^\dagger$. *Then the following conditions are equivalent for any positive integers k and l:*

(i) $m^\dagger = ((a_3a_3^*)^l a_3)^*(((a_1a_1^*)^k a_1)^* m((a_3a_3^*)^l a_3)^*)^\dagger((a_1a_1^*)^k a_1)^*$.

(ii) $(a_1a_1^*)^{2k+1}mR = mR$ *and* $(a_3^*a_3)^{2l+1}m^*R = m^*R$.

Proof. Let $\widetilde{a_1} = (a_1a_1^*)^k a_1$, $\widetilde{a_2} = ((a_1^*a_1)^\dagger)^k a_2((a_3a_3^*)^\dagger)^l$, and $\widetilde{a_3} = (a_3a_3^*)^l a_3$. Then $m = \widetilde{a_1}\widetilde{a_2}\widetilde{a_3}$. As a consequence of Lemma 2.1(iv), we have $\widetilde{a_1}, \widetilde{a_3} \in R^\dagger$. Moreover, $(\widetilde{a_1})^* m(\widetilde{a_3})^* = ((a_1a_1^*)^k a_1)^* m((a_3a_3^*)^l a_3)^* \in R^\dagger$.

By Theorem 2.5, $m^\dagger = (\widetilde{a_3})^*((\widetilde{a_1})^* m(\widetilde{a_3})^*)^\dagger(\widetilde{a_1})^*$ if and only if $\widetilde{a_1}(\widetilde{a_1})^* mR = mR$ and $(\widetilde{a_3})^*\widetilde{a_3}m^*R = m^*R$. It can be verified that

$$(\widetilde{a_3})^*((\widetilde{a_1})^* m(\widetilde{a_3})^*)^\dagger(\widetilde{a_1})^* = ((a_3a_3^*)^l a_3)^*(((a_1a_1^*)^k a_1)^* m((a_3a_3^*)^l a_3)^*)^\dagger((a_1a_1^*)^k a_1)^*,$$

$\widetilde{a_1}(\widetilde{a_1})^* = (a_1a_1^*)^{2k+1}$ and $(\widetilde{a_3})^*\widetilde{a_3} = (a_3^*a_3)^{2l+1}$. This completes the proof. □

By taking $k = l = 1$ in Theorem 2.13, we obtain the following corollary.

Corollary 2.14. *Let* $a_1, a_3, m, (a_1a_1^*a_1)^* m(a_3a_3^*a_3)^* \in R^\dagger$. *Then the following statements are equivalent:*

(i) $m^\dagger = (a_3a_3^*a_3)^*((a_1a_1^*a_1)^* m(a_3a_3^*a_3)^*)^\dagger(a_1a_1^*a_1)^*$.

(ii) $(a_1a_1^*)^3 mR = mR$ *and* $(a_3^*a_3)^3 m^*R = m^*R$.

3. Some Equivalencies

In this section, whenever we write $a^\dagger$ we will assume $a \in R$ has Moore-Penrose inverse. The results presented in previous section are connected as follows.

Theorem 3.1. *The following statements are equivalent:*
(i) $m^\dagger = a_3^\dagger(a_1^\dagger m a_3^\dagger)^\dagger a_1^\dagger$.
(ii) $m^\dagger = a_3^*(a_1^* m a_3^*)^\dagger a_1^*$.
(iii) $a_3^\dagger(a_1^\dagger m a_3^\dagger)^\dagger a_1^\dagger = (a_3^* a_3)^\dagger((a_1 a_1^*)^\dagger m (a_3^* a_3)^\dagger)^\dagger (a_1 a_1^*)^\dagger$.
(iv) $a_3^*(a_1^\dagger m a_3^\dagger)^\dagger a_1^* = a_3^* a_3 m^\dagger a_1 a_1^*$.
(v) $(a_1^\dagger m a_3^\dagger)^\dagger = a_3 m^\dagger a_1$.
(vi) $a_3^*(a_1^* m a_3^*)^\dagger a_1^* = a_3^* a_3 (a_1 a_1^* m a_3^* a_3)^\dagger a_1 a_1^*$.
(vii) $a_1 a_1^* mR = mR$ *and* $a_3^* a_3 m^* R = m^* R$.

Proof. (i)⇔(ii)⇔(vii) follows from Theorem 2.4 and 2.5.
(iii)⇒(vii) First we have

$$\begin{aligned}
& a_3^\dagger(a_1^\dagger m a_3^\dagger)^\dagger a_1^\dagger \\
= \; & (a_3^* a_3)^\dagger((a_1 a_1^*)^\dagger m (a_3^* a_3)^\dagger)^\dagger (a_1 a_1^*)^\dagger \\
= \; & (a_3^* a_3)^\dagger((a_1 a_1^*)^\dagger m (a_3^* a_3)^\dagger)^\dagger (a_1 a_1^*)^\dagger m (a_3^* a_3)^\dagger ((a_1 a_1^*)^\dagger m (a_3^* a_3)^\dagger)^\dagger (a_1 a_1^*)^\dagger \\
= \; & (a_3^* a_3)^\dagger((a_1 a_1^*)^\dagger m (a_3^* a_3)^\dagger)^* (((a_1 a_1^*)^\dagger m (a_3^* a_3)^\dagger)^\dagger)^* ((a_1 a_1^*)^\dagger m (a_3^* a_3)^\dagger)^\dagger (a_1 a_1^*)^\dagger. \qquad (6)
\end{aligned}$$

Multiplying (6) by $a_3^* a_3$ from the left-hand side, we have

$$\begin{aligned}
& (a_3^* a_3)(a_3^\dagger(a_1^\dagger m a_3^\dagger)^\dagger a_1^\dagger) \\
= \; & ((a_1 a_1^*)^\dagger m (a_3^* a_3)^\dagger)^* (((a_1 a_1^*)^\dagger m (a_3^* a_3)^\dagger)^\dagger)^* ((a_1 a_1^*)^\dagger m (a_3^* a_3)^\dagger)^\dagger (a_1 a_1^*)^\dagger \\
= \; & ((a_1 a_1^*)^\dagger m (a_3^* a_3)^\dagger)^\dagger (a_1 a_1^*)^\dagger \\
= \; & ((a_1 a_1^*)^\dagger m (a_3^* a_3)^\dagger)^\dagger (((a_1 a_1^*)^\dagger m (a_3^* a_3)^\dagger)^\dagger)^* ((a_1 a_1^*)^\dagger m (a_3^* a_3)^\dagger)^* (a_1 a_1^*)^\dagger. \qquad (7)
\end{aligned}$$

Multiplying (7) by $a_1 a_1^*$ from the right-hand side, we have

$$\begin{aligned}
& (a_3^* a_3)(a_3^\dagger(a_1^\dagger m a_3^\dagger)^\dagger a_1^\dagger)(a_1 a_1^*) \\
= \; & ((a_1 a_1^*)^\dagger m (a_3^* a_3)^\dagger)^\dagger ((a_1 a_1^*)^\dagger m ((a_3^* a_3)^\dagger)^\dagger)^* ((a_1 a_1^*)^\dagger m (a_3^* a_3)^\dagger)^* \\
= \; & ((a_1 a_1^*)^\dagger m (a_3^* a_3)^\dagger)^\dagger.
\end{aligned}$$

Hence $((a_1 a_1^*)^\dagger m (a_3^* a_3)^\dagger)^\dagger = (a_3^* a_3)(a_3^\dagger(a_1^\dagger m a_3^\dagger)^\dagger a_1^\dagger)(a_1 a_1^*) = a_3^*(a_1^\dagger m a_3^\dagger)^\dagger a_1^*$. This implies

$$\begin{aligned}
((a_1 a_1^*)^\dagger m (a_3^* a_3)^\dagger)((a_1 a_1^*)^\dagger m (a_3^* a_3)^\dagger)^\dagger &= ((a_1 a_1^*)^\dagger m (a_3^* a_3)^\dagger) a_3^*(a_1^\dagger m a_3^\dagger)^\dagger a_1^* \\
&= (a_1^\dagger)^*(a_1^\dagger m a_3^\dagger)(a_1^\dagger m a_3^\dagger)^\dagger a_1^*.
\end{aligned}$$

By the condition (3) in the definition of MP-inverse, we have

$$\left[(a_1^\dagger)^*(a_1^\dagger m a_3^\dagger)(a_1^\dagger m a_3^\dagger)^\dagger a_1^*\right]^* = (a_1^\dagger)^*(a_1^\dagger m a_3^\dagger)(a_1^\dagger m a_3^\dagger)^\dagger a_1^*.$$

Multiplying it from the left-hand side by a_1^* and from the right-hand side by a_1, we obtain

$$\begin{aligned}
a_1^*\left[(a_1^\dagger)^*(a_1^\dagger m a_3^\dagger)(a_1^\dagger m a_3^\dagger)^\dagger a_1^*\right]^* a_1 &= a_1^*(a_1^\dagger)^*(a_1^\dagger m a_3^\dagger)(a_1^\dagger m a_3^\dagger)^\dagger a_1^* a_1, \\
a_1^* a_1((a_1^\dagger m a_3^\dagger)^\dagger)^*(a_1^\dagger m a_3^\dagger)^* a_1^\dagger a_1 &= (a_1^\dagger m a_3^\dagger)(a_1^\dagger m a_3^\dagger)^\dagger a_1^* a_1, \\
a_1^* a_1((a_1^\dagger m a_3^\dagger)^\dagger)^*(m a_3^\dagger)^*(a_1^\dagger)^* a_1^\dagger a_1 &= (a_1^\dagger m a_3^\dagger)(a_1^\dagger m a_3^\dagger)^\dagger a_1^* a_1
\end{aligned}$$

and

$$a_1^* a_1 (a_1^\dagger m a_3^\dagger)(a_1^\dagger m a_3^\dagger)^\dagger = (a_1^\dagger m a_3^\dagger)(a_1^\dagger m a_3^\dagger)^\dagger a_1^* a_1. \qquad (8)$$

Then we have $a_1^*a_1(a_1^\dagger ma_3^\dagger) = a_1^*a_1(a_1^\dagger ma_3^\dagger)(a_1^\dagger ma_3^\dagger)^\dagger(a_1^\dagger ma_3^\dagger)$ and by (8) we get

$$a_1^*ma_3^\dagger = (a_1^\dagger ma_3^\dagger)(a_1^\dagger ma_3^\dagger)^\dagger(a_1^*a_1)(a_1^\dagger ma_3^\dagger) \tag{9}$$

Multiplying (9) from the left-hand side by a_1 and from the right-hand side by a_3, we get $a_1a_1^*m = a_1(a_1^\dagger ma_3^\dagger)(a_1^\dagger ma_3^\dagger)^\dagger(a_1^*a_1)a_1^\dagger m = ma_3^\dagger(a_1^\dagger ma_3^\dagger)^\dagger a_1^*m$. Consequently, it follows that $a_1a_1^*mR \subseteq mR$.

By (3.3), we also have

$$\begin{aligned} a_1^\dagger ma_3^\dagger &= (a_1^\dagger ma_3^\dagger)(a_1^\dagger ma_3^\dagger)^\dagger(a_1^\dagger ma_3^\dagger) \\ &= (a_1^\dagger ma_3^\dagger)(a_1^\dagger ma_3^\dagger)^\dagger(a_1^*a_1)(a_1^\dagger(a_1^\dagger)^*a_2a_3a_3^\dagger) \\ &= (a_1^*a_1)(a_1^\dagger ma_3^\dagger)(a_1^\dagger ma_3^\dagger)^\dagger(a_1^\dagger(a_1^\dagger)^*a_2a_3a_3^\dagger) \end{aligned} \tag{10}$$

Multiplying (10) by a_1 from the left-hand side and a_3 from the right-hand side, one can see

$$m = a_1a_1^*ma_3^\dagger(a_1^\dagger ma_3^\dagger)^\dagger a_1^\dagger(a_1^\dagger)^*a_2a_3,$$

which induces $mR \subseteq a_1a_1^*mR$. Thus, $mR = a_1a_1^*mR$.

Similarly, we have

$$\begin{aligned} ((a_1a_1^*)^\dagger m(a_3^*a_3)^\dagger)^\dagger((a_1a_1^*)^\dagger m(a_3^*a_3)^\dagger) &= a_3^*(a_1^\dagger ma_3^\dagger)^\dagger a_1^*((a_1a_1^*)^\dagger m(a_3^*a_3)^\dagger) \\ &= a_3^*(a_1^\dagger ma_3^\dagger)^\dagger(a_1^\dagger ma_3^\dagger)(a_3^\dagger)^*. \end{aligned}$$

By the condition (4) in the definition of MP-inverse, we have

$$(a_3^*(a_1^\dagger ma_3^\dagger)^\dagger(a_1^\dagger ma_3^\dagger)(a_3^\dagger)^*)^* = a_3^*(a_1^\dagger ma_3^\dagger)^\dagger(a_1^\dagger ma_3^\dagger)(a_3^\dagger)^*$$

and hence

$$a_3^\dagger(a_1^\dagger ma_3^\dagger)^*((a_1^\dagger ma_3^\dagger)^\dagger)^*a_3 = a_3^*(a_1^\dagger ma_3^\dagger)^\dagger(a_1^\dagger ma_3^\dagger)(a_3^\dagger)^*. \tag{11}$$

Multiplying (11) by a_3 from the left-hand side and a_3^* from the right-hand side, we get

$$a_3a_3^\dagger(a_1^\dagger ma_3^\dagger)^*((a_1^\dagger ma_3^\dagger)^\dagger)^*a_3a_3^* = a_3a_3^*(a_1^\dagger ma_3^\dagger)^\dagger(a_1^\dagger ma_3^\dagger)(a_3^\dagger)^*a_3^*,$$

i.e.,

$$(a_1^\dagger ma_3^\dagger)^\dagger(a_1^\dagger ma_3^\dagger)a_3a_3^* = a_3a_3^*(a_1^\dagger ma_3^\dagger)^\dagger(a_1^\dagger ma_3^\dagger). \tag{12}$$

This implies

$$\begin{aligned} a_3m^*(a_1^\dagger)^* &= a_3a_3^*(a_1^\dagger ma_3^\dagger)^* \\ &= a_3a_3^*(a_1^\dagger ma_3^\dagger)^\dagger(a_1^\dagger ma_3^\dagger)(a_1^\dagger ma_3^\dagger)^* \\ &= (a_1^\dagger ma_3^\dagger)^\dagger(a_1^\dagger ma_3^\dagger)a_3a_3^*(a_1^\dagger ma_3^\dagger)^*. \end{aligned} \tag{13}$$

Multiplying (13) by a_3^* from the left-hand side and a_1^* from the right-hand side, we get

$$a_3^*a_3m^*(a_1^\dagger)^*a_1^* = a_3^*(a_1^\dagger ma_3^\dagger)^*((a_1^\dagger ma_3^\dagger)^*)^\dagger a_3a_3^*(a_1^\dagger ma_3^\dagger)^*a_1^*$$

and

$$a_3^*a_3m^* = m^*(a_1^\dagger)^*((a_1^\dagger ma_3^\dagger)^*)^\dagger a_3a_3^*(a_1^\dagger ma_3^\dagger)^*a_1^*,$$

from which one can see that $a_3^*a_3m^*R \subseteq m^*R$.

By (12) we also have

$$\begin{aligned} (a_1^\dagger ma_3^\dagger)^* &= (a_1^\dagger ma_3^\dagger)^\dagger(a_1^\dagger ma_3^\dagger)(a_1^\dagger ma_3^\dagger)^* \\ &= (a_1^\dagger ma_3^\dagger)^\dagger(a_1^\dagger ma_3^\dagger)(a_3a_3^*)((a_3^\dagger)^*a_3^\dagger(a_2)^*a_1^\dagger a_1) \\ &= (a_3a_3^*)(a_1^\dagger ma_3^\dagger)^\dagger(a_1^\dagger ma_3^\dagger)((a_3^\dagger)^*a_3^\dagger(a_2)^*a_1^\dagger a_1). \end{aligned} \tag{14}$$

Multiplying (14) from the left-hand side by a_3^* and from the right-hand side by a_1^*, we obtain

$$a_3^*(a_1^\dagger m a_3^\dagger)^* a_1^* = a_3^*(a_3a_3^*)(a_1^\dagger m a_3^\dagger)^*((a_1^\dagger m a_3^\dagger)^\dagger)^*((a_3^\dagger)^* a_3^\dagger (a_2)^* a_1^\dagger a_1)a_1^*.$$

Since $m^* = a_3^*(a_3^\dagger)^* m^* (a_1^\dagger)^* a_1^* = a_3^*(a_1^\dagger m a_3^\dagger)^* a_1^*$ and

$$\begin{aligned} & a_3^* a_3 m^* (a_1^\dagger)^*((a_1^\dagger m a_3^\dagger)^\dagger)^*(a_3^\dagger)^* a_3^\dagger (a_2)^* a_1^* \\ = \;& a_3^* a_3 a_3^* (a_3^\dagger)^* m^* (a_1^\dagger)^*((a_1^\dagger m a_3^\dagger)^\dagger)^*(a_3^\dagger)^* a_3^\dagger (a_2)^* a_1^* \\ = \;& a_3^*(a_3a_3^*)(a_1^\dagger m a_3^\dagger)^*((a_1^\dagger m a_3^\dagger)^\dagger)^*((a_3^\dagger)^* a_3^\dagger (a_2)^* a_1^\dagger a_1)a_1^*, \end{aligned}$$

it follows that $m^*R \subseteq a_3^* a_3 m^* R$. So we have $m^*R = a_3^* a_3 m^* R$.

(vii)⇒(iii) By hypothesis, we have $(a_1a_1^*)^2 mR = mR$ and $(a_3^*a_3)^2 m^*R = m^*R$. In view of Theorem 2.4 and Corollary 2.7, it follows that $m^\dagger = a_3^\dagger (a_1^\dagger m a_3^\dagger)^\dagger a_1^\dagger$ and $m^\dagger = (a_3^*a_3)^\dagger((a_1a_1^*)^\dagger m (a_3^*a_3)^\dagger)^\dagger (a_1a_1^*)^\dagger$. Hence

$$a_3^\dagger (a_1^\dagger m a_3^\dagger)^\dagger a_1^\dagger = (a_3^*a_3)^\dagger((a_1a_1^*)^\dagger m (a_3^*a_3)^\dagger)^\dagger (a_1a_1^*)^\dagger.$$

(vi)⇔(vii) Let $\widetilde{a_1} = (a_1^\dagger)^*$, $\widetilde{a_2} = a_1^* a_1 a_2 a_3 a_3^*$ and $\widetilde{a_3} = (a_3^\dagger)^*$. Then $m = \widetilde{a_1}\widetilde{a_2}\widetilde{a_3}$. From the proof of (iii)⇔(vii), one can see the following conditions are equivalent:

(vi′) $(\widetilde{a_3})^\dagger((\widetilde{a_1})^\dagger m (\widetilde{a_3})^\dagger)^\dagger (\widetilde{a_1})^\dagger = (\widetilde{a_3}^*\widetilde{a_3})^\dagger((\widetilde{a_1}\widetilde{a_1}^*)^\dagger m (\widetilde{a_3}^*\widetilde{a_3})^\dagger)^\dagger (\widetilde{a_1}\widetilde{a_1}^*)^\dagger$;

(vii′) $\widetilde{a_1}(\widetilde{a_1})^* mR = mR$ and $(\widetilde{a_3})^*\widetilde{a_3} m^*R = m^*R$,

where (vi′) coincides with (vi) since $(\widetilde{a_1})^\dagger = a_1^*$, $(\widetilde{a_3})^\dagger = a_3^*$, $(\widetilde{a_1})^* = a_1^\dagger$ and $(\widetilde{a_3})^* = a_3^\dagger$. Moreover, (vii′) can be translated into

(vii′′) $(a_1a_1^*)^\dagger mR = mR$ and $(a_3^*a_3)^\dagger m^*R = m^*R$,

which is equivalent to (vii) by Lemma 2.3.

(iv)⇒(v) Suppose that $a_3^*(a_1^\dagger m a_3^\dagger)^\dagger a_1^* = a_3^* a_3 m^\dagger a_1 a_1^*$. Multiplying this equation by $(a_3^\dagger)^*$ from the left-hand side and by $(a_1^\dagger)^*$ from the right-hand side, we obtain

$$(a_3^\dagger)^* a_3^* (a_1^\dagger m a_3^\dagger)^\dagger a_1^* (a_1^\dagger)^* = a_3 a_3^\dagger (a_1^\dagger m a_3^\dagger)^\dagger a_1^\dagger a_1 = (a_3^\dagger)^* a_3^* a_3 m^\dagger a_1 a_1^* (a_1^\dagger)^*.$$

Note the fact: $p(ap)^\dagger = (ap)^\dagger$ and $(pa)^\dagger p = (pa)^\dagger$, where p is a orthogonal projection.
Since $a_3 a_3^\dagger$ and $a_1^\dagger a_1$ are orthogonal projections, it follows that $a_3 a_3^\dagger (a_1^\dagger m a_3^\dagger)^\dagger a_1^\dagger a_1 = (a_1^\dagger m a_3^\dagger)^\dagger$. Therefore, $(a_1^\dagger m a_3^\dagger)^\dagger = (a_3^\dagger)^* a_3^* a_3 m^\dagger a_1 a_1^* (a_1^\dagger)^* = a_3 m^\dagger a_1$.

(v)⇒(iv) is obvious.

(v)⇔(vii) By the proof of the Theorem 2.4, one can verify the following equivalence:

$$\begin{aligned} & (a_1^\dagger m a_3^\dagger)^\dagger = a_3 m^\dagger a_1 \\ \Leftrightarrow \;& a_1 a_1^* m m^\dagger = m m^\dagger a_1 a_1^* \text{ and } a_3^* a_3 m^\dagger m = m^\dagger m a_3^* a_3 \\ \Leftrightarrow \;& a_1 a_1^* mR = mR \text{ and } a_3^* a_3 m^* R = m^* R. \end{aligned}$$

This completes the proof. □

Theorem 3.2. *The following statements are equivalent:*

(i) $m^\dagger = (a_3^*a_3)^\dagger((a_1a_1^*)^\dagger m (a_3^*a_3)^\dagger)^\dagger (a_1a_1^*)^\dagger$.

(ii) $m^\dagger = a_3^* a_3 (a_1 a_1^* m a_3^* a_3)^\dagger a_1 a_1^*$.

(iii) $a_3^\dagger (a_1^\dagger m a_3^\dagger)^\dagger a_1^\dagger = a_3^* (a_1^* m a_3^*)^\dagger a_1^*$.

(iv) $(a_1a_1^*)^2 mR = mR$ *and* $(a_3^*a_3)^2 m^*R = m^*R$.

Proof. (i)⇔(iv) See Corollary 2.7.

(ii)⇔(iv) See Corollary 2.10.

(iii)⇒(iv) Multiplying the equation in (iii) by a_3 from the left-hand side and a_1 from the right side, we obtain $a_3 a_3^* (a_1^* m a_3^*)^\dagger a_1^* a_1 = a_3 a_3^\dagger (a_1^\dagger m a_3^\dagger)^\dagger a_1^\dagger a_1$. Since $a_3 a_3^\dagger$ and $a_1^\dagger a_1$ are orthogonal projections, we have $a_3 a_3^\dagger (a_1^\dagger m a_3^\dagger)^\dagger a_1^\dagger a_1 = (a_1^\dagger m a_3^\dagger)^\dagger$. Therefore,

$$(a_1^\dagger m a_3^\dagger)^\dagger = a_3 a_3^* (a_1^* m a_3^*)^\dagger a_1^* a_1. \quad (15)$$

From which it follows that

$$\begin{aligned}(a_1^\dagger m a_3^\dagger)(a_1^\dagger m a_3^\dagger)^\dagger &= (a_1^\dagger m a_3^\dagger)(a_3 a_3^*(a_1^* m a_3^*)^\dagger a_1^* a_1) \\ &= a_1^\dagger m a_3^*(a_1^* m a_3^*)^\dagger a_1^* a_1 \\ &= (a_1^* a_1)^\dagger (a_1^* m a_3^*)(a_1^* m a_3^*)^\dagger a_1^* a_1.\end{aligned}$$

By the condition (3) in the definition of MP-inverse, we have

$$((a_1^* a_1)^\dagger (a_1^* m a_3^*)(a_1^* m a_3^*)^\dagger a_1^* a_1)^* = (a_1^* a_1)^\dagger (a_1^* m a_3^*)(a_1^* m a_3^*)^\dagger a_1^* a_1.$$

Multiplying it by $a_1^* a_1$ from the left-hand side and $a_1^* a_1$ from the right-hand side, we get

$$a_1^* a_1 a_1^* a_1((a_1^* m a_3^*)^\dagger)^*(a_1^* m a_3^*)^*(a_1^* a_1)^\dagger a_1^* a_1 = a_1^* a_1 (a_1^* a_1)^\dagger (a_1^* m a_3^*)(a_1^* m a_3^*)^\dagger a_1^* a_1 a_1^* a_1.$$

Hence $(a_1^* a_1)^2 (a_1^* m a_3^*)(a_1^* m a_3^*)^\dagger = (a_1^* m a_3^*)(a_1^* m a_3^*)^\dagger (a_1^* a_1)^2$. Consequently,

$$\begin{aligned}(a_1^* a_1)^2 (a_1^* m a_3^*) &= (a_1^* a_1)^2 (a_1^* m a_3^*)(a_1^* m a_3^*)^\dagger (a_1^* m a_3^*) \\ &= (a_1^* m a_3^*)(a_1^* m a_3^*)^\dagger (a_1^* a_1)^2 (a_1^* m a_3^*). \qquad (16)\end{aligned}$$

Multiplying (16) by $(a_1^*)^\dagger$ from the left-hand side and $(a_3^*)^\dagger$ from the right-hand side, we get

$$(a_1^*)^\dagger (a_1^* a_1)^2 (a_1^* m a_3^*)(a_3^\dagger)^* = (a_1^*)^\dagger (a_1^* m a_3^*)(a_1^* m a_3^*)^\dagger (a_1^* a_1)^2 (a_1^* m a_3^*)(a_3^\dagger)^*,$$

which means $(a_1 a_1^*)^2 m = m a_3^*(a_1^* m a_3^*)^\dagger ((a_1^* a_1)^2 a_1^* m$. This guarantees $(a_1 a_1^*)^2 mR \subseteq mR$.

From (15) it also follows that

$$\begin{aligned}a_1^* m a_3^* &= (a_1^* m a_3^*)(a_1^* m a_3^*)^\dagger (a_1^* m a_3^*) \\ &= (a_1^* m a_3^*)(a_1^* m a_3^*)^\dagger (a_1^* a_1)^2 (a_1^\dagger (a_1^\dagger)^* a_2 a_3 a_3^*) \\ &= (a_1^* a_1)^2 (a_1^* m a_3^*)(a_1^* m a_3^*)^\dagger (a_1^\dagger (a_1^\dagger)^* a_2 a_3 a_3^*).\end{aligned}$$

Whence

$$\begin{aligned}m &= (a_1^\dagger)^* a_1^* m a_3^* (a_3^\dagger)^* \\ &= (a_1^\dagger)^* (a_1^* a_1)^2 (a_1^* m a_3^*)(a_1^* m a_3^*)^\dagger (a_1^\dagger (a_1^\dagger)^* a_2 a_3 a_3^*)(a_3^\dagger)^* \\ &= (a_1 a_1^*)^2 m a_3^* (a_1^* m a_3^*)^\dagger a_1^\dagger (a_1^\dagger)^* a_2 a_3.\end{aligned}$$

This implies $mR \subseteq (a_1 a_1^*)^2 mR$. So we have $mR = (a_1 a_1^*)^2 mR$.

By a similar argument, one can show $(a_3^* a_3)^2 m^* R = m^* R$.

(iv)⇒(iii) First, we claim that $(a_1^* a_1)^2 a_1^* m a_3^* R = a_1^* m a_3^* R$. Indeed, since $(a_1 a_1^*)^2 mR = mR$, there exist $r_1, r_2 \in R$ such that $(a_1 a_1^*)^2 m = m r_1$ and $m = (a_1 a_1^*)^2 m r_2$. This induces $a_1^*(a_1 a_1^*)^2 m = a_1^* m r_1$, $(a_1^* a_1)^2 a_1^* m a_3^* = a_1^* m r_1 a_3^* = a_1^* m a_3^* (a_3^\dagger)^* r_1 a_3^*$ and $a_1^* m a_3^* = a_1^* (a_1 a_1^*)^2 m r_2 a_3^* = (a_1^* a_1)^2 a_1^* m a_3^* (a_3^\dagger)^* r_2 a_3^*$. Thus, $(a_1^* a_1)^2 a_1^* m a_3^* R = a_1^* m a_3^* R$.

Similarly, it follows that $(a_3 a_3^*)^2 (a_1^* m a_3^*)^* R = (a_1^* m a_3^*)^* R$.

Now, in view of Lemma 2.2, we have

$$(a_1^* a_1)^2 (a_1^* m a_3^*)(a_1^* m a_3^*)^\dagger = (a_1^* m a_3^*)(a_1^* m a_3^*)^\dagger (a_1^* a_1)^2$$

and

$$(a_3 a_3^*)^2 (a_1^* m a_3^*)^\dagger (a_1^* m a_3^*) = (a_1^* m a_3^*)^\dagger (a_1^* m a_3^*)(a_3 a_3^*)^2.$$

Based on these two equations, one can verify

$$\begin{aligned}(a_1^* a_1)(a_1^* m a_3^*)(a_1^* m a_3^*)^\dagger (a_1^* a_1)^\dagger &= (a_1^* a_1)^\dagger (a_1^* a_1)^2 (a_1^* m a_3^*)(a_1^* m a_3^*)^\dagger (a_1^* a_1)^\dagger \\ &= (a_1^* a_1)^\dagger (a_1^* m a_3^*)(a_1^* m a_3^*)^\dagger (a_1^* a_1)^2 (a_1^* a_1)^\dagger \\ &= (a_1^* a_1)^\dagger (a_1^* m a_3^*)(a_1^* m a_3^*)^\dagger (a_1^* a_1)\end{aligned}$$

and

$$(a_3a_3^*)(a_1^*ma_3^*)^\dagger(a_1^*ma_3^*)(a_3a_3^*)^\dagger = (a_3a_3^*)^\dagger(a_1^*ma_3^*)^\dagger(a_1^*ma_3^*)(a_3a_3^*).$$

Combining these with the fact that $a_1^\dagger = (a_1^*a_1)^\dagger a_1^*$ and $a_3^\dagger = a_3^*(a_3a_3^*)^\dagger$, it is easy to check that $(a_1^\dagger ma_3^\dagger)^\dagger = a_3a_3^*(a_1^*ma_3^*)^\dagger a_1^*a_1$. Therefore, $a_3^\dagger(a_1^\dagger ma_3^\dagger)^\dagger a_1^\dagger = a_3^\dagger a_3a_3^*(a_1^*ma_3^*)^\dagger a_1^*a_1a_1^\dagger = a_3^*(a_1^*ma_3^*)^\dagger a_1^*$. □

Theorem 3.3. *The following statements are equivalent:*

(i) $m^\dagger = (a_3a_3^*a_3)^\dagger((a_1a_1^*a_1)^\dagger m(a_3a_3^*a_3)^\dagger)^\dagger(a_1a_1^*a_1)^\dagger$.

(ii) $m^\dagger = (a_3a_3^*a_3)^*((a_1a_1^*a_1)^*m(a_3a_3^*a_3)^*)^\dagger(a_1a_1^*a_1)^*$.

(iii) $a_3^\dagger(a_1^\dagger ma_3^\dagger)^\dagger a_1^\dagger = a_3^*a_3(a_1a_1^*ma_3^*a_3)^\dagger a_1a_1^*$.

(iv) $(a_1^\dagger ma_3^\dagger)^\dagger = a_3a_3^*a_3(a_1a_1^*ma_3^*a_3)^\dagger a_1a_1^*a_1$.

(v) $(a_1a_1^*)^3mR = mR$ *and* $(a_3^*a_3)^3m^*R = m^*R$.

Proof. (i)⇔(ii)⇔(v) follows from Corollary 2.12 and 2.14.

(iii)⇒(iv) By hypothesis, it is clear that

$$a_3a_3^*a_3(a_1a_1^*ma_3^*a_3)^\dagger a_1a_1^*a_1 = a_3a_3^\dagger(a_1^\dagger ma_3^\dagger)^\dagger a_1^\dagger a_1.$$

Moreover, we have $a_3a_3^\dagger(a_1^\dagger ma_3^\dagger)^\dagger a_1^\dagger a_1 = (a_1^\dagger ma_3^\dagger)^\dagger$ since $a_3a_3^\dagger$ and $a_1^\dagger a_1$ are orthogonal projections. Therefore, $(a_1^\dagger ma_3^\dagger)^\dagger = a_3a_3^*a_3(a_1a_1^*ma_3^*a_3)^\dagger a_1a_1^*a_1$.

(iv)⇒(v) By Lemma 2.1(iv), we have

$$\begin{aligned}(a_1^\dagger ma_3^\dagger)(a_1^\dagger ma_3^\dagger)^\dagger &= (a_1^\dagger ma_3^\dagger)a_3a_3^*a_3(a_1a_1^*ma_3^*a_3)^\dagger a_1a_1^*a_1\\ &= a_1^\dagger ma_3^*a_3(a_1a_1^*ma_3^*a_3)^\dagger a_1a_1^*a_1\\ &= (a_1a_1^*a_1)^\dagger(a_1a_1^*ma_3^*a_3)(a_1a_1^*ma_3^*a_3)^\dagger a_1a_1^*a_1.\end{aligned}$$

By the condition (3) of the definition of MP-inverse, we get

$$((a_1a_1^*a_1)^\dagger(a_1a_1^*ma_3^*a_3)(a_1a_1^*ma_3^*a_3)^\dagger a_1a_1^*a_1)^* = (a_1a_1^*a_1)^\dagger(a_1a_1^*ma_3^*a_3)(a_1a_1^*ma_3^*a_3)^\dagger a_1a_1^*a_1.$$

Multiply it by $a_1a_1^*a_1$ from the left-hand side and $(a_1a_1^*a_1)^*$ from the right-hand side, we can see

$$\begin{aligned}&a_1a_1^*a_1(a_1a_1^*a_1)^*((a_1a_1^*ma_3^*a_3)^\dagger)^*(a_1a_1^*ma_3^*a_3)^*\\ =\ &(a_1a_1^*ma_3^*a_3)(a_1a_1^*ma_3^*a_3)^\dagger a_1a_1^*a_1(a_1a_1^*a_1)^*,\end{aligned}$$

which can be simplified as

$$(a_1a_1^*)^3(a_1a_1^*ma_3^*a_3)(a_1a_1^*ma_3^*a_3)^\dagger = (a_1a_1^*ma_3^*a_3)(a_1a_1^*ma_3^*a_3)^\dagger(a_1a_1^*)^3. \tag{17}$$

Consequently, we have

$$\begin{aligned}(a_1a_1^*)^3(a_1a_1^*ma_3^*a_3) &= (a_1a_1^*)^3(a_1a_1^*ma_3^*a_3)(a_1a_1^*ma_3^*a_3)^\dagger(a_1a_1^*ma_3^*a_3)\\ &= (a_1a_1^*ma_3^*a_3)(a_1a_1^*ma_3^*a_3)^\dagger(a_1a_1^*)^3(a_1a_1^*ma_3^*a_3).\end{aligned}$$

Multiplying it by $a_3^\dagger(a_3^\dagger)^*$ from the right-hand side, we get

$$(a_1a_1^*)^3(a_1a_1^*ma_3^*a_3)(a_3)^\dagger(a_3^\dagger)^* = (a_1a_1^*ma_3^*a_3)(a_1a_1^*ma_3^*a_3)^\dagger(a_1a_1^*)^3(a_1a_1^*ma_3^*a_3)(a_3)^\dagger(a_3^\dagger)^*,$$

i.e.,

$$(a_1a_1^*)^4m = (a_1a_1^*ma_3^*a_3)(a_1a_1^*ma_3^*a_3)^\dagger(a_1a_1^*)^4m.$$

Multiplying it by $(a_1^\dagger)^*a_1^\dagger$ from the left-hand side, we have $(a_1a_1^*)^3m = ma_3^*a_3(a_1a_1^*ma_3^*a_3)^\dagger(a_1a_1^*)^4m$. Hence $(a_1a_1^*)^3mR \subseteq mR$.

On the other hand, (17) induces

$$\begin{aligned} a_1a_1^*ma_3^*a_3 &= (a_1a_1^*ma_3^*a_3)(a_1a_1^*ma_3^*a_3)^\dagger(a_1a_1^*ma_3^*a_3) \\ &= (a_1a_1^*ma_3^*a_3)(a_1a_1^*ma_3^*a_3)^\dagger(a_1a_1^*)^3(a_1a_1^*)^\dagger(a_1^\dagger)^*a_2a_3a_3^*a_3 \\ &= (a_1a_1^*)^3(a_1a_1^*ma_3^*a_3)(a_1a_1^*ma_3^*a_3)^\dagger(a_1a_1^*)^\dagger(a_1^\dagger)^*a_2a_3a_3^*a_3. \end{aligned}$$

Multiplying it by $(a_1^\dagger)^*a_1^\dagger$ from the left-hand side and $a_3^\dagger(a_3^\dagger)^*$ from the right-hand side, we get $m = (a_1a_1^*)^3ma_3^*a_3(a_1a_1^*ma_3^*a_3)^\dagger(a_1a_1^*$ from which we can see $mR \subseteq (a_1a_1^*)^3mR$. Hence, $(a_1a_1^*)^3mR = mR$.

The equality $(a_3^*a_3)^3m^*R = m^*R$ can be proved in a similar way.

(v)⇒(iii) First, we claim that $(a_1a_1^*)^3a_1a_1^*ma_3^*a_3R = a_1a_1^*ma_3^*a_3R$. Indeed, since $(a_1a_1^*)^3mR = mR$, there exist $r_1, r_2 \in R$ such that $(a_1a_1^*)^3m = mr_1$ and $m = (a_1a_1^*)^3mr_2$. Hence

$$(a_1a_1^*)^3a_1a_1^*ma_3^*a_3 = a_1a_1^*mr_1a_3^*a_3 = a_1a_1^*ma_3^*a_3(a_3^*a_3)^\dagger r_1a_3^*a_3$$

and

$$a_1a_1^*ma_3^*a_3 = a_1a_1^*(a_1a_1^*)^3mr_2a_3^*a_3 = (a_1a_1^*)^3a_1a_1^*ma_3^*a_3(a_3^*a_3)^\dagger r_2a_3^*a_3.$$

Now, $(a_1a_1^*)^3a_1a_1^*ma_3^*a_3R = a_1a_1^*ma_3^*a_3R$ is clear.

Simultaneously, a similar argument shows $(a_3^*a_3)^3(a_1a_1^*m^*a_3^*a_3)^*R = (a_1a_1^*m^*a_3^*a_3)^*R$ from $(a_3^*a_3)^3m^*R = m^*R$.

By Lemma 2.2, we know that

$$(a_1a_1^*)^3(a_1a_1^*ma_3^*a_3)(a_1a_1^*ma_3^*a_3)^\dagger = (a_1a_1^*ma_3^*a_3)(a_1a_1^*ma_3^*a_3)^\dagger(a_1a_1^*)^3 \tag{18}$$

and

$$(a_3^*a_3)^3(a_1a_1^*ma_3^*a_3)^\dagger(a_1a_1^*ma_3^*a_3) = (a_1a_1^*ma_3^*a_3)^\dagger(a_1a_1^*ma_3^*a_3)(a_3^*a_3)^3. \tag{19}$$

Then by (18), we have

$$\begin{aligned} &(a_1a_1^*a_1)^*(a_1a_1^*ma_3^*a_3)(a_1a_1^*ma_3^*a_3)^\dagger((a_1a_1^*a_1)^\dagger)^* \\ =\ &(a_1a_1^*a_1)^\dagger(a_1a_1^*)^3(a_1a_1^*ma_3^*a_3)(a_1a_1^*ma_3^*a_3)^\dagger((a_1a_1^*a_1)^\dagger)^* \\ =\ &(a_1a_1^*a_1)^\dagger(a_1a_1^*ma_3^*a_3)(a_1a_1^*ma_3^*a_3)^\dagger(a_1a_1^*)^3((a_1a_1^*a_1)^\dagger)^* \\ =\ &(a_1a_1^*a_1)^\dagger(a_1a_1^*ma_3^*a_3)(a_1a_1^*ma_3^*a_3)^\dagger(a_1a_1^*a_1). \end{aligned}$$

Similarly, by (19), we have

$$\begin{aligned} &(a_3a_3^*a_3)(a_1a_1^*ma_3^*a_3)^\dagger(a_1a_1^*ma_3^*a_3)(a_3a_3^*a_3)^\dagger \\ =\ &((a_3a_3^*a_3)^\dagger)^*(a_1a_1^*ma_3^*a_3)^\dagger(a_1a_1^*ma_3^*a_3)(a_3a_3^*a_3)^*. \end{aligned}$$

By Lemma 2.1(iv), we have $a_1^\dagger = (a_1a_1^*a_1)^\dagger a_1a_1^*$ and $a_3^\dagger = a_3^*a_3(a_3a_3^*a_3)^\dagger$. Consequently, it is not hard to check $(a_1^\dagger ma_3^\dagger)^\dagger = a_3a_3^*a_3(a_1a_1^*ma_3^*a_3)^\dagger a_1a_1^*a_1$. Therefore, $a_3^\dagger(a_1^\dagger ma_3^\dagger)^\dagger a_1^\dagger = a_3^\dagger a_3a_3^*a_3(a_1a_1^*ma_3^*a_3)^\dagger a_1a_1^*a_1a_1^\dagger = a_3^*a_3(a_1a_1^*ma_3^*a_3)^\dagger a_1a_1^*$. □

We conclude this section by a corollary which follows from Theorem 2.6 and 2.9.

Corollary 3.4. *The following statements are equivalent:*
(i) $m^\dagger = ((a_3^*a_3)^\dagger)^2(((a_1a_1^*)^2)^\dagger m((a_3^*a_3)^2)^\dagger)^\dagger((a_1a_1^*)^\dagger)^2$.
(ii) $m^\dagger = (a_3^*a_3)^2((a_1a_1^*)^2m(a_3^*a_3)^2)^\dagger(a_1a_1^*)^2$.
(iii) $(a_1a_1^*)^4mR = mR$ *and* $(a_3^*a_3)^4m^*R = m^*R$.

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