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Weighted Lacunary Statistical Convergence in Locally Solid Riesz Spaces

Metin Başarır[a], Şükran Konca[b]

[a]*Department of Mathematics, Sakarya University, 54187, Sakarya, Turkey.*
[b]*Department of Mathematics, Bitlis Eren University, 13000, Bitlis, Turkey.*

Abstract. In this paper we introduce the concepts of weighted lacunary statistical τ-convergence, weighted lacunary statistical τ-bounded by combining both of the definitions of lacunary sequence and Nörlund-type mean, using a new lacunary sequence which has been defined by Basarir and Konca [3]. We also prove some topological results related to these concepts in the framework of locally solid Riesz spaces.

1. Introduction

A Riesz space is an ordered vector space which is a lattice at the same time. A locally solid Riesz space is a Riesz space equipped with a linear topology that has a base consisting of solid sets. The Riesz space was first introduced by F. Riesz in 1928, at the International Mathematical Congress in Bologna, Italy [26]. Soon after, in the mid-thirties, H. Freudenthal [13] and L. V. Kantorovich [16] independently set up the axiomatic foundation and derived a number of properties dealing with the lattice structure of Riesz spaces. From then on the growth of the subject was rapid. In the forties and early fifties the Japanese school led by H. Nakano, T. Ogasawara and K. Yosida, and the Russian school, led by L. V. Kantorovich, A. I. Judin, and B. Z. Vulikh, made fundamental contributions. At the same time a number of books started to appear on the field. The general theory of topological Riesz spaces seems somehow to have been neglected. The recent book by D. H. Fremlin [12] is partially devoted to this subject. Riesz spaces play an important role in analysis, measure theory, operator theory and optimization. They also provide the natural framework for any modern theory of integration. Further, they have some applications in economics [2].

The idea of statistical convergence was initially given by Zygmund in 1935 [30]. The concept was formally introduced by Fast [11] and Steinhaus [28] and later on by Schoenberg [29], and also independently by Buck [8]. Many years later, it was investigated from varied points of view, for example; in summability theory [9], [14], [23], topological groups [9]-[10], topological spaces [19], locally convex spaces [20]. In 1993, Fridy and Orhan [15] introduced the concept of lacunary statistical convergence.

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Email addresses: basarir@sakarya.edu.tr (Metin Başarır), skonca@beu.edu.tr (Şükran Konca)

Recently, Albayrak and Pehlivan [1] have introduced the concept of statistical τ-convergence in a locally solid Riesz space endowed with the topology τ. Mohiuddine and Alghamdi [21] have introduced the concept of lacunary statistical τ-convergence in a locally solid Riesz space and established some results.

Moricz and Orhan [22] have defined the concept of statistical summability $(\overline{N}, p_n)$. Later on, Karakaya and Chishti [17] have used $(\overline{N}, p_n)$-summability to generalize the concept of statistical convergence and have called this new method weighted statistical convergence. Mursaleen et. al. [24] have altered the definition of weighted statistical convergence and have found its relation with the concept of statistical summability $(\overline{N}, p_n)$. Related articles can be seen in [4]-[7].

In this paper we introduce the concepts of weighted lacunary statistical τ-convergence, weighted lacunary statistical τ-bounded by combining both of the definitions of lacunary sequence and Nörlund-type mean, using a new lacunary sequence which has been defined by Basarir and Konca [3]. We also prove some topological results related to these concepts in the framework of locally solid Riesz spaces. Further, we establish some inclusion relations between the set of weighted lacunary statistically τ-convergent sequences with the set of lacunary statistically τ-convergent sequences and with the set of weighted statistically τ-convergent sequences in locally solid Riesz spaces.

2. Definitions and Preliminaries

In this section, we recall some basic definitions and notations. Throughout the paper, we mean the "Riesz transformation" by "Nörlund-type transformation" and take $(x_k - \zeta)$ instead of $(x_k - \zeta e)$, $e = (1, 1, 1, ...)$ for all $k \in \mathbb{N}$.

Let (p_k) be a sequence of positive real numbers and $P_n = p_1 + p_2 + ... + p_n$ for $n \in \mathbb{N}$. Then the Nörlund-type transformation of $x = (x_k)$ is defined as:

$$t_n := \frac{1}{P_n} \sum_{k=1}^{n} p_k x_k. \tag{1}$$

If the sequence (t_n) has a finite limit ζ then the sequence x is said to be Nörlund-type convergent to ζ. We denote the set of all Nörlund-type convergent sequences by $(\overline{N}, p_n)$. Let us note that if $P_n \to \infty$ as $n \to \infty$ then Nörlund-type mean is a regular summability method. Throughout the paper, let $P_n \to \infty$ as $n \to \infty$ and let $P_0 = p_0 = 0$. If $p_k = 1$ for all $k \in \mathbb{N}$ in (1) then Nörlund-type mean reduces to Cesáro mean. Moreover, if we select $p_k = \frac{1}{k}$ for all $k \geq 1$, then Nörlund-type mean reduces to $(H, 1)$-summability which can be seen in [25].

Let $\theta = (k_r)$ be the sequence of positive integers such that $(k_0) = 0$, $0 < k_r < k_{r+1}$ and $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. Then θ is called a lacunary sequence. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r.

The following notations which were defined in [3] will be used throughout the paper.

Let $\theta = (k_r)$ be a lacunary sequence, (p_k) be a sequence of positive real numbers such that $H_r := \sum_{k \in I_r} p_k$, $P_{k_r} := \sum_{k \in (0, k_r]} p_k$, $P_{k_{r-1}} := \sum_{k \in (0, k_{r-1}]} p_k$, $Q_r := \frac{P_{k_r}}{P_{k_{r-1}}}$, $P_0 = 0$ and the intervals determined by θ and (p_k) are denoted by $I'_r = (P_{k_{r-1}}, P_{k_r}]$. It is easy to see that $H_r = P_{k_r} - P_{k_{r-1}}$. If we take $p_k = 1$ for all $k \in \mathbb{N}$, then H_r, P_{k_r}, $P_{k_{r-1}}$, Q_r and I'_r reduce to h_r, k_r, k_{r-1}, q_r and I_r, respectively.

Throughout the paper, we assume that $P_n \to \infty$ as $n \to \infty$ such that $H_r \to \infty$ as $r \to \infty$. If $\theta = (k_r)$ is a lacunary sequence and $P_n \to \infty$ as $n \to \infty$ such that $H_r \to \infty$ as $r \to \infty$, then $\theta' = (P_{k_r})$ is a lacunary sequence, that is, $P_0 = 0$, $0 < P_{k_{r-1}} < P_{k_r}$ and $H_r = P_{k_r} - P_{k_{r-1}} \to \infty$ as $r \to \infty$.

The weighted lacunary density of $K \subset \mathbb{N}$ is denoted by $\delta_{(\overline{N},\theta)}(K) = lim_{r\to\infty}\frac{1}{H_r}$ $|K_r(\varepsilon)|$ if the limit exists. The sequence $x = (x_k)$ is said to be weighted lacunary statistically convergent to ζ if for every $\varepsilon > 0$, the set $K_r(\varepsilon) = \{k \in I'_r : p_k|x_k - \zeta| \geqslant \varepsilon\}$ has weighted density zero, i.e.

$$\lim_{r\to\infty}\frac{1}{H_r}|\{k \in I'_r : p_k|x_k - \zeta| \geqslant \varepsilon\}| = 0.$$

In this case, it is written $S_{(\overline{N},\theta)}$-lim$x = \zeta$. The set of all weighted lacunary statistically convergent sequences is denoted by $S_{(\overline{N},\theta)}$.

In the definition above, if we take $p_k = 1$ for all $k \in \mathbb{N}$, then we obtain the definition of lacunary statistical convergence (see in [15]). In case of $\theta = (k_r) = (2^r)$ for $r > 0$, the definition of weighted statistical convergence is obtained (see in [24]). If we choose $\theta = (k_r) = (2^r)$ for $r > 0$ and $p_k = \frac{1}{k}$ for all $k \geq 1$, then weighted lacunary density reduces to logarithmic density (see in [18]). If $p_k = 1$ for all $k \in \mathbb{N}$ and $\theta = (k_r) = (2^r)$ for $r > 0$, then the definition of usual statistical convergence is obtained.

A topological vector space (X, τ) is a vector space X, which has a linear topology τ, such that the algebraic operations of addition and scalar multiplication in X are continuous. Continuity of addition means that the function $f : X \times X \to X$ defined by $f(x, y) = x + y$ is continuous on $X \times X$, and continuity of scalar multiplication means that the function $f : \mathbb{C}\times X \to X$ defined by $f(\lambda, x) = \lambda x$ is continuous on $\mathbb{C}\times X$.

Every linear topology τ on a vector space X has a base $\mathcal{N}$ for the neighborhoods of θ satisfying the following properties:

(T1) Each $Y \in \mathcal{N}$ is a balanced set, that is, $\lambda x \in Y$ holds for all $x \in Y$ and every $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$.

(T2) Each $Y \in \mathcal{N}$ is an absorbing set, that is, for every $x \in X$, there exists $\lambda > 0$ such that $\lambda x \in Y$.

(T3) For each $y \in \mathcal{N}$, there exists some $E \in \mathcal{N}$ with $E + E \subseteq Y$.

Let X be a real vector space and $\leq$ be a partial order on this space. Then X is said to be an ordered vector space if it satisfies the following properties:

1. If $x, y \in X$ and $y \leq x$, then $y + z \leq x + z$ for each $z \in X$.
2. If $x, y \in X$ and $y \leq x$, then $\lambda y \leq \lambda x$ for each $\lambda \geq 0$.

If in addition X is a lattice with respect to the positive part of x by $x^+ = x \vee \theta$ =sup$\{x, \theta\}$, the negative part of x by $x^- = (-x) \vee \theta$ and the absolute value of x by $|x| = x \vee (-x)$, where θ is the zero element of X.

A subset of a Riesz space X is said to be solid if $y \in S$ and $|x| \leq |y|$ implies $x \in S$.

A linear topology τ on a Riesz space X is said to be locally solid if τ has a base at zero consisting of solid sets. A locally solid Riesz space (X, τ) is a Riesz space equipped with a locally solid topology τ [27].

Let (X, τ) be a locally solid Riesz space and $x = (x_k)$ be a sequence in X. It is said that $x = (x_k)$ is statistically τ-convergent to $\zeta \in X$ provided that, for every τ-neighborhood U of zero,

$$\lim_{n\to\infty}\frac{1}{n}|\{k \leq n : x_k - \zeta \notin U\}| = 0$$

holds. It is written as $S(\tau)\text{-}\lim_k x_k = \zeta$ and the set of all statistically τ-convergent sequences is denoted by $S(\tau)$ [1].

Let (X, τ) be a locally solid Riesz space and θ be a lacunary sequence. Then a sequence $x = (x_k)$ in X is said to be lacunary statistically τ-convergent (or $S_\theta(\tau)$-convergent) to the element $\zeta \in X$ if for every τ-neighborhood U of zero, $\delta_\theta(K_U) = 0$ where $K_U = \{k \in \mathbb{N} : x_k - \zeta \notin U\}$, i.e.,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : x_k - \zeta \notin U\}| = 0.$$

In this case, we write $S_\theta(\tau)\text{-}\lim_k x_k = \zeta$ [21].

We shall assume throughout this paper that the symbol $\mathcal{N}_{sol}$ will denote any base at zero consisting of solid sets and satisfying the conditions $(T1)$, $(T2)$, $(T3)$ in a locally solid topology.

3. Main Results

In this section, we define the concepts of weighted lacunary statistical τ-convergence, weighted lacunary statistical τ-bounded and also weighted statistical τ-convergence, which is a special case of weighted lacunary statistical τ-convergence, in the framework of locally solid Riesz spaces and prove some topological results related to these concepts. We also examine some inclusion relations between the set of weighted lacunary statistically τ-convergent sequences with the set of lacunary statistically τ-convergent sequences and with the set of weighted statistically τ-convergent sequences in locally solid Riesz spaces.

Definition 3.1. *Let (X, τ) be a locally solid Riesz space and θ be a lacunary sequence. Then a sequence $x = (x_k)$ in X is said to be weighted lacunary statistically τ-convergent (or $S_{(\overline{N},\theta)}(\tau)$-convergent) to the element $\zeta \in X$ if for every τ-neighborhood U of zero, the set $K_U(H_r) = \{k \in \mathbb{N} : p_k(x_k - \zeta) \notin U\}$ has weighted lacunary τ-density zero or shortly $\delta_{(\overline{N},\theta)}(K_U(H_r)) = 0$ i.e.,*

$$\lim_{r\to\infty} \frac{1}{H_r} |\{k \in I'_r : p_k(x_k - \zeta) \notin U\}| = 0. \tag{2}$$

In this case, we write $S_{(\overline{N},\theta)}(\tau)$-$lim_k x_k = \zeta$. We denote the set of all weighted lacunary statistically τ-convergent sequences by $S_{(\overline{N},\theta)}(\tau)$.

1. If we take $p_k = 1$ for all $k \in \mathbb{N}$ in (2) then we obtain the definition of lacunary statistical τ-convergence which can be seen in [21].

2. In case of $\theta = (k_r) = (2^r)$ for $r > 0$, the definition of weighted statistical τ-convergence reduces to a new concept named weighted statistical τ-convergence which can be seen as follows:

   Let (X, τ) be a locally solid Riesz space, then a sequence $x = (x_k)$ in X is said to be weighted statistically τ-convergent (or $S_{\overline{N}}(\tau)$-convergent) to the element $\zeta \in X$ if for every τ-neighborhood U of zero, the set $K_U(P_n) = \{k \in \mathbb{N} : p_k(x_k - \zeta) \notin U\}$ has weighted τ-density zero or shortly, $\delta_{\overline{N}}(K_U(P_n)) = 0$, i.e.,

   $$\lim_{n\to\infty} \frac{1}{P_n} |\{k \leq P_n : p_k(x_k - \zeta) \notin U\}| = 0.$$

   In this case, we write $S_{\overline{N}}(\tau)\text{-}\lim x = \zeta$. We denote the set of all weighted statistically τ-convergent sequences by $S_{\overline{N}}(\tau)$.

3. If $p_k = 1$ for all $k \in \mathbb{N}$ and $\theta = (k_r) = (2^r)$ for $r > 0$, then the definition of statistical τ-convergence is obtained (see in [1]).

4. If we choose $\theta = (k_r) = (2^r)$ for $r > 0$ and $p_k = \frac{1}{k}$ for all $k \geq 1$, then weighted lacunary τ-density reduces to logarithmic τ-density.

Definition 3.2. *Let (X, τ) be a locally solid Riesz space and θ be a lacunary sequence. Then we say that a sequence $x = (x_k)$ in X is weighted lacunary statistically τ-bounded or $S_{(\overline{N},\theta)}(\tau)$-bounded if for every τ-neighborhood U of zero there exists some $\alpha > 0$ such that $M_U = \{k \in \mathbb{N} : \alpha p_k x_k \notin U\}$ has weighted lacunary τ-density zero or $\delta_{(\overline{N},\theta)}(M_U) = 0$, i.e.,*

$$\lim_{r\to\infty} \frac{1}{H_r} |\{k \in I'_r : \alpha p_k x_k \notin U\}| = 0.$$

Theorem 3.3. *Let (X, τ) be a Hausdorff local solid Riesz space and θ be a lacunary sequence. Suppose that $x = (x_k)$ and $y = (y_k)$ are two sequences in X. Then the followings hold:*

1. *If $S_{(\overline{N},\theta)}(\tau)$-$lim_k\ x_k = \zeta_1$ and $S_{(\overline{N},\theta)}(\tau)$-$lim_k\ x_k = \zeta_2$ then $\zeta_1 = \zeta_2$.*
2. *If $S_{(\overline{N},\theta)}(\tau)$-$lim_k\ x_k = \zeta$, then $S_{(\overline{N},\theta)}(\tau)$-$lim_k\ \alpha x_k = \alpha\zeta$, $\alpha \in \mathbb{R}$.*
3. *If $S_{(\overline{N},\theta)}(\tau)$-$lim_k\ x_k = \zeta$ and $S_{(\overline{N},\theta)}(\tau)$-$lim_k\ y_k = \eta$, then $S_{(\overline{N},\theta)}(\tau)$-$lim_k\ (x_k + y_k) = \zeta + \eta$.*

Proof. 1. Suppose that $S_{(\overline{N},\theta)}(\tau)$-$\lim_k x_k = \zeta_1$ and $S_{(\overline{N},\theta)}(\tau)$-$\lim_k x_k = \zeta_2$. Let U be any τ-neighborhood of zero. Then there exists $Y \in \mathcal{N}_{sol}$ such that $Y \subseteq U$. Choose any $E \in \mathcal{N}_{sol}$ such that $E + E \subseteq Y$. We define the following sets:

$$K_1 = \{k \in \mathbb{N} : p_k(x_k - \zeta_1) \in E\},$$

$$K_2 = \{k \in \mathbb{N} : p_k(x_k - \zeta_2) \in E\}.$$

Since $S_{(\overline{N},\theta)}(\tau)$-$\lim_k x_k = \zeta_1$ and $S_{(\overline{N},\theta)}(\tau)$-$\lim_k x_k = \zeta_2$, we have $\delta_{(\overline{N},\theta)}(K_1)$ $= \delta_{(\overline{N},\theta)}(K_2) = 1$. Thus $\delta_{(\overline{N},\theta)}(K_1 \cap K_2) = 1$, and in particular $K_1 \cap K_2 \neq \emptyset$. Now, let $k \in K_1 \cap K_2$. Then

$$p_k(\zeta_1 - \zeta_2) = p_k(x_k - \zeta_2) + p_k(\zeta_1 - x_k) \in E + E \subseteq Y \subseteq U.$$

Hence, for every τ-neighborhood U of zero, we have $p_k(\zeta_1 - \zeta_2) \in U$. Since (p_k) is a sequence of positive reals and (X, τ) is Hausdorff, the intersection of all τ-neighborhoods U of zero is the singleton set $\{\theta\}$. Thus we get $\zeta_1 - \zeta_2 = \theta$, i.e., $\zeta_1 = \zeta_2$.

2. Let U be an arbitrary τ-neighborhood of zero and $S_{(\overline{N},\theta)}(\tau)$-$\lim_k x_k = \zeta$. Then there exists $Y \in \mathcal{N}_{sol}$ such that $Y \subseteq U$. Since $S_{(\overline{N},\theta)}(\tau)$-$\lim_k x_k = \zeta$, we have

$$\lim_{r\to\infty} \frac{1}{H_r} |\{k \in I'_r : p_k(x_k - \zeta) \in Y\}| = 1.$$

Since Y is balanced, $p_k(x_k - \zeta) \in Y$ implies $\alpha p_k(x_k - \zeta) \in Y$ for every $\alpha \in \mathbb{R}$ with $|\alpha| \leq 1$. Hence

$$\{k \in \mathbb{N} : p_k(x_k - \zeta) \in Y\} \subseteq \{k \in \mathbb{N} : \alpha p_k(x_k - \zeta) \in Y\} \subseteq \{k \in \mathbb{N} : \alpha p_k(x_k - \zeta) \in U\}.$$

Thus, we obtain

$$\lim_{r\to\infty} \frac{1}{H_r} |\{k \in I'_r : \alpha p_k(x_k - \zeta) \in Y\}| = 1,$$

for each τ-neighborhood U of zero. Now, let $|\alpha| > 1$ and $[|\alpha|]$ be the smallest integer greater than or equal to $|\alpha|$. There exists $E \in \mathcal{N}_{sol}$ such that $[|\alpha|]E \subseteq Y$. Since $S_{(\overline{N},\theta)}(\tau)\text{-}\lim_k x_k = \zeta$, we have $\delta(K) = 1$ where $K = \{k \in \mathbb{N} : p_k(x_k - \zeta) \in E\}$. Then we have,

$$|\alpha p_k(x_k - \zeta)| = |\alpha||p_k(x_k - \zeta)| \leq [|\alpha|]|p_k(x_k - \zeta)| \in [|\alpha|]E \subseteq Y \subseteq U.$$

Since the set Y is solid, we have $\alpha p_k(x_k - \zeta) \in Y$ and so $\alpha p_k(x_k - \zeta) \in U$. Thus we get,

$$\lim_{r\to\infty} \frac{1}{H_r}|\{k \in I'_r : \alpha p_k(x_k - \zeta) \in U\}| = 1,$$

for each τ-neighborhood U of zero. Hence, $S_{(\overline{N},\theta)}(\tau)\text{-}\lim_k \alpha x_k = \alpha\zeta$ for every $\alpha \in \mathbb{R}$.

3. Let U be an arbitrary τ-neighborhood of zero. Then there exists $Y \in \mathcal{N}_{sol}$ such that $Y \subseteq U$. Choose E in $\mathcal{N}_{sol}$ such that $E + E \subseteq Y$. Since $S_{(\overline{N},\theta)}(\tau)\text{-}\lim_k x_k = \zeta$ and $S_{(\overline{N},\theta)}(\tau)\text{-}\lim_k y_k = \eta$, we have $S_{(\overline{N},\theta)}(H_1) = 1 = S_{(\overline{N},\theta)}(H_2)$ where

$$H_1 = \{k \in \mathbb{N} : p_k(x_k - \zeta) \in E\},$$

$$H_2 = \{k \in \mathbb{N} : p_k(y_k - \eta) \in E\}.$$

Let $H = H_1 \cap H_2$. Hence, we have $S_{(\overline{N},\theta)}(H) = 1$ and

$$p_k((x_k + y_k) - (\zeta + \eta)) = p_k(x_k - \zeta) + p_k(y_k - \eta) \in E + E \subseteq Y \subseteq U.$$

Thus, we get

$$\lim_{r\to\infty} \frac{1}{H_r}|\{k \in I'_r : p_k((x_k + y_k) - (\zeta + \eta)) \in U\}| = 1.$$

Since U is arbitrary, we have $S_{(\overline{N},\theta)}(\tau)\text{-}\lim_k (x_k + y_k) = \zeta + \eta$.
□

Theorem 3.4. *Let (X, τ) be a locally solid Riesz space and θ be a lacunary sequence. If a sequence $x = (x_k)$ is weighted lacunary statistically τ-convergent and (p_k) is bounded, then the sequence $x = (x_k)$ is weighted lacunary statistically τ-bounded.*

Proof. Suppose $x = (x_k)$ is weighted lacunary statistically τ-convergent to the point $\zeta \in X$ and (p_k) is a bounded sequence. Let U be an arbitrary τ-neighborhood of zero. Then there exists $Y \in \mathcal{N}_{sol}$ such that $Y \subseteq U$. Let us choose $E \in \mathcal{N}_{sol}$ such that $E + E \subseteq Y$. Since $S_{(\overline{N},\theta)}(\tau)\text{-}\lim_k x_k = \zeta$, the set

$$K = \{k \in \mathbb{N} : p_k(x_k - \zeta) \notin E\}$$

has weighted lacunary τ-density zero. Since E is absorbing, there exists $\lambda > 0$ such that $\lambda\zeta \in E$. Let α be such that $0 < \alpha \leq 1$. Since (p_k) is bounded, then there exists a $M = \frac{\lambda}{\alpha} > 0$ such that $p_k \leq M$ for all $k \in \mathbb{N}$. Then we can write $\alpha p_k \leq \lambda$ for all $k \in \mathbb{N}$. Since E is solid and $|\alpha p_k \zeta| \leq |\lambda\zeta|$, we have $\alpha p_k \zeta \in E$. Since E is balanced, $p_k(x_k - \zeta) \in E$ implies $\alpha p_k(x_k - \zeta) \in E$. Then we have

$$\alpha p_k x_k = \alpha p_k(x_k - \zeta) + \alpha p_k \zeta \in E + E \subseteq Y \subseteq U$$

for each $k \in \mathbb{N}\backslash K$. Thus,

$$\lim_{r\to\infty} \frac{1}{H_r}|k \in I'_r : \alpha p_k x_k \notin U| = 0.$$

Hence, (x_k) is weighted lacunary statistically τ-bounded. □

Theorem 3.5. *Let (X, τ) be a locally solid Riesz space and θ be a lacunary sequence. If $x = (x_k)$, $y = (y_k)$ and $z = (z_k)$ are sequences such that*

1. $x_k \leq y_k \leq z_k$ *for all* $k \in \mathbb{N}$.
2. $S_{(\overline{N},\theta)}(\tau)$*-lim*$_k$ $x_k = \zeta = S_{(\overline{N},\theta)}(\tau)$*-lim*$_k$ z_k *then* $S_{(\overline{N},\theta)}(\tau)$*-lim*$_k$ $y_k = \zeta$.

Proof. Let U be an arbitrary τ-neighborhood of zero, then there exists $Y \in \mathcal{N}_{sol}$ such that $Y \subseteq U$. Choose $E \in \mathcal{N}_{sol}$ such that $E + E \subseteq Y$. From the condition (2), we have $S_{(\overline{N},\theta)}(K_1) = 1 = S_{(\overline{N},\theta)}(K_2)$, where

$$K_1 = \{k \in \mathbb{N} : p_k(x_k - \zeta) \in E\},$$

$$K_2 = \{k \in \mathbb{N} : p_k(z_k - \zeta) \in E\}.$$

Also, we get $S_{(\overline{N},\theta)}(K_1 \cap K_2) = 1$ and from (1) we have

$$p_k(x_k - \zeta) \leq p_k(y_k - \zeta) \leq p_k(z_k - \zeta)$$

for all $k \in \mathbb{N}$. This implies that for all $k \in K_1 \cap K_2$, we get

$$|p_k(y_k - \zeta)| \leq |p_k(z_k - \zeta)| + |p_k(x_k - \zeta)| \in E + E \subseteq Y.$$

Since Y is solid, we have $p_k(y_k - \zeta) \in Y \subseteq U$. Thus,

$$\lim_{r\to\infty} \frac{1}{H_r} |\{k \in I'_r : p_k(y_k - \zeta) \in U\}| = 1$$

for each τ-neighborhood U of zero. Hence, $S_{(\overline{N},\theta)}(\tau)$-$\lim_k y_k = \zeta$. □

Theorem 3.6. *Let (X, τ) be a locally solid Riesz space and $x = (x_k)$ be a sequence in X. For any lacunary sequence $\theta = (k_r)$, if $\liminf_r Q_r > 1$ then $S_{\overline{N}}(\tau) \subseteq S_{(\overline{N},\theta)}(\tau)$.*

Proof. Suppose that $\liminf_r Q_r > 1$, then there exists a $\delta > 0$ such that $Q_r \geq 1 + \delta$ for sufficiently large values of r, which implies that $\frac{H_r}{P_{k_r}} = 1 - \frac{P_{k_{r-1}}}{P_{k_r}} = 1 - \frac{1}{Q_r} \geq \frac{\delta}{1+\delta}$. Suppose that $S_{\overline{N}}(\tau)$-$\lim_k x_k = \zeta$. We prove that $S_{(\overline{N},\theta)}(\tau)$-$\lim_k x_k = \zeta$. Let U be an arbitrary τ-neighborhood of zero. Then for all $r > r_0$, we have

$$\begin{aligned}
\frac{1}{P_{k_r}} \left|\{k \leqslant P_{k_r} : \ p_k\ (x_k - \zeta) \notin U\}\right| &\geqslant \frac{1}{P_{k_r}} |\{P_{k_{r-1}} < k \leqslant P_{k_r} : p_k(x_k - \zeta) \notin U\}| \\
&= \frac{H_r}{P_{k_r}} \left(\frac{1}{H_r} \left|\{k \in I'_r : \ p_k\ (x_k - \zeta) \notin U\}\right| \right) \\
&\geqslant \frac{\delta}{1+\delta} . \frac{1}{H_r} \left|\{k \in I'_r : \ p_k\ (x_k - \zeta) \notin U\}\right|.
\end{aligned}$$

Since $S_{\overline{N}}(\tau)$-$\lim_k x_k = \zeta$, then the above inequality implies that $S_{(\overline{N},\theta)}(\tau)$-$\lim_k x_k = \zeta$. Hence, $S_{\overline{N}}(\tau) \subseteq S_{(\overline{N},\theta)}(\tau)$. □

Theorem 3.7. *Let (X, τ) be a locally solid Riesz space and $x = (x_k)$ be a sequence in X. For any lacunary sequence $\theta = (k_r)$, if $\limsup_r Q_r < \infty$, then $S_{(\overline{N},\theta)}(\tau) \subseteq S_{\overline{N}}(\tau)$.*

Proof. If $\limsup_r Q_r < \infty$, then there exists a $K > 0$ such that $Q_r \leq K$ for all $r \in \mathbb{N}$. Suppose that $S_{(\overline{N},\theta)}(\tau)$-$\lim_k$ $x_k = \zeta$. Let U be an arbitrary τ-neighborhood of zero. We write

$$N_r := \left|\{k \in I'_r : \ p_k\ (x_k - \zeta) \notin U\}\right|. \tag{3}$$

By (3) and from the definition of weighted lacunary statistical convergence, given $\varepsilon > 0$, there is a positive integer r_0 such that $\frac{N_r}{H_r} < \frac{\varepsilon}{2K}$ for all $r > r_0$. Now, let

$$M := \max\{N_r : 1 \leq r \leq r_0\} \tag{4}$$

and let n be any integer satisfying $k_{r-1} < n \leq k_r$, then we can write

$$\begin{aligned}\frac{1}{P_n}\left|\{k \leqslant P_n : p_k(x_k-\zeta) \notin U\}\right| &\leqslant \frac{1}{P_{k_{r-1}}}\left|\{k \leqslant P_{k_r} : p_k(x_k-\zeta) \notin U\}\right| \\ &= \frac{1}{P_{k_{r-1}}}\left(N_1+N_2+\ldots+N_{r_0}+N_{r_0+1}+\ldots+N_r\right) \\ &\leqslant \frac{M.r_0}{P_{k_{r-1}}}+\frac{1}{P_{k_{r-1}}}\frac{\varepsilon}{2K}\left(H_{r_0+1}+\ldots+H_r\right) \\ &= \frac{M.r_0}{P_{k_{r-1}}}+\frac{\varepsilon}{2K}\frac{\left(P_{k_r}-P_{k_{r_0}}\right)}{P_{k_{r-1}}} \\ &\leqslant \frac{M.r_0}{P_{k_{r-1}}}+\frac{\varepsilon}{2K}Q_r.\end{aligned}$$

Since $P_{k_{r-1}} \to \infty$ as $r \to \infty$, there exists a positive integer $r_1 \geq r_0$ such that $\frac{1}{P_{k_{r-1}}} < \frac{\varepsilon}{2r_0.M}$ for $r > r_1$. Hence for $r > r_1$

$$\frac{1}{P_n}\left|\{k \leqslant P_n : p_k(x_k-\zeta) \notin U\}\right| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

It follows that $S_{\overline{N}}(\tau)$-$\lim_k x_k = \zeta$. □

Corollary 3.8. *Let (X,τ) be a locally solid Riesz space and let $x=(x_k)$ be a sequence in X. For any lacunary sequence $\theta=(k_r)$, if $1 < liminf_r\, Q_r \leq limsup_r\, Q_r < \infty$, then $S_{(\overline{N},\theta)}(\tau) = S_{\overline{N}}(\tau)$ and $S_{(\overline{N},\theta)}(\tau)$-$lim_k\, x_k = S_{\overline{N}}(\tau)$-$lim_k\, x_k = \zeta$.*

Proof. It follows from Theorem 3.6 and Theorem 3.7.
□

Theorem 3.9. *Let (X,τ) be a locally solid Riesz space and $x=(x_k)$ be a sequence in X. For any lacunary sequence $\theta=(k_r)$ the following statements are true:*

1. *If $p_k \leq 1$ for all $k \in \mathbb{N}$, then $S_\theta(\tau) \subseteq S_{(\overline{N},\theta)}(\tau)$ and $S_\theta(\tau)$-$lim_k\, x_k = S_{(\overline{N},\theta)}(\tau)$–$lim_k\, x_k = \zeta$.*
2. *If $1 \leq p_k$ for all $k \in \mathbb{N}$ and $(\frac{H_r}{h_r})$ is upper bounded, then $S_{(\overline{N},\theta)}(\tau) \subseteq S_\theta(\tau)$ and $S_{(\overline{N},\theta)}(\tau)$-$lim_k\, x_k = S_\theta(\tau)$-$lim_k\, x_k = \zeta$.*

Proof.

1. If $p_k \leq 1$ for all $k \in \mathbb{N}$, then $H_r \leq h_r$ for all $r \in \mathbb{N}$. So, there exist M_1 and M_2 constants such that $0 < M_1 \leq \frac{H_r}{h_r} \leq M_2 \leq 1$ for all $r \in \mathbb{N}$. Assume that $S_\theta(\tau)$-$\lim_k x_k = \zeta$. Let U be an arbitrary τ-neighborhood of zero, then we have

$$\frac{1}{H_r}\left|\{k \in I'_r : p_k(x_k-\zeta) \notin U\}\right| \leqslant \frac{1}{M_1}.\frac{1}{h_r}\left|\{k \in I_r : x_k-\zeta \notin U\}\right|.$$

Hence, we obtain the result by taking the limit as $r \to \infty$.

2. Let $(\frac{H_r}{h_r})$ be upper bounded, then there exist M_1 and M_2 constants such that $1 \leq M_1 \leq \frac{H_r}{h_r} \leq M_2 < \infty$ for all $r \in \mathbb{N}$. If $1 \leq p_k$ for all $k \in \mathbb{N}$, then $h_r \leq H_r$ for all $r \in \mathbb{N}$. Assume that $S_{(\overline{N},\theta)}(\tau)$-$\lim_k x_k = \zeta$. Let U be an arbitrary τ-neighborhood of zero. Then we have

$$\frac{1}{h_r}\left|\{k \in I_r : (x_k-\zeta) \notin U\}\right| \leqslant M_2.\frac{1}{H_r}\left|\{k \in I'_r : p_k(x_k-\zeta) \notin U\}\right|.$$

Hence, the result is obtained by taking limit as $r \to \infty$.
□

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