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New Transformation Formulae of Quadratic ${}_7F_6$-Series

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Abstract. The modified Abel lemma on summation by parts with the " remainder term" is employed to investigate the partial sums of a quadratic ${}_7H_7$-series. Several unusual transformation formulae for these sums are established. As consequences, some new transformations of quadratic ${}_7F_6$-series are deduced, especially two of which respectively generalize two known ${}_3F_2(1)$-series summation formulae due to Watson and Whipple (1925).

1. Introduction and Motivation

For a complex x and an integer $n \in \mathbb{Z}$, define the rising shifted-factorial by

$$(x)_n = \Gamma(x+n)/\Gamma(x)$$

where the Γ-function is given by the Euler integral

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \quad \text{with} \quad \Re(t) > 0.$$

When $n \in \mathbb{N}_0$, it reduces the usual rising shifted-factorial

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{for} \quad n \in \mathbb{N}.$$

For the sake of brevity, the quotients of shifted factorials and Γ-function will be abbreviated respectively as

$$\begin{aligned}\begin{bmatrix}\alpha, \beta, \cdots, \gamma \\ A, B, \cdots, C\end{bmatrix}_n &= \frac{(\alpha)_n(\beta)_n\cdots(\gamma)_n}{(A)_n(B)_n\cdots(C)_n},\\ \Gamma\begin{bmatrix}\alpha, \beta, \cdots, \gamma \\ A, B, \cdots, C\end{bmatrix}_n &= \frac{\Gamma(\alpha)\Gamma(\beta)\cdots\Gamma(\gamma)}{\Gamma(A)\Gamma(B)\cdots\Gamma(C)}.\end{aligned}$$

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Following Bailey [1] and Slater [6], the unilateral and bilateral generalized hypergeometric series for an indeterminate z is defined by

$$ {}_{1+r}F_s\left[\begin{matrix} a_0, a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}\middle| z\right] = \sum_{k=0}^{\infty} \left[\begin{matrix} a_0, a_1, \cdots, a_r \\ 1, b_1, \cdots, b_s \end{matrix}\right]_k z^k, $$

$$ {}_rH_s\left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix}\middle| z\right] = \sum_{k=-\infty}^{\infty} \left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix}\right]_k z^k, $$

where $\{a_i\}$ and $\{b_j\}$ are complex parameters such that no zero factors appear in the denominators of the summands on the right hand sides. For the bilateral ${}_rH_s$-series, we shall denote by ${}_rH_s{}^n$ and ${}_rH_s{}^+$ the partial sums consisting of some and all terms with nonnegative indices respectively as follows

$$ {}_rH_s{}^n\left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix}\middle| z\right] = \sum_{k=0}^{n-1} \left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix}\right]_k z^k, $$

$$ {}_rH_s{}^+\left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix}\middle| z\right] = \sum_{k=0}^{\infty} \left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix}\right]_k z^k. $$

In this paper, we consider the most important case $r = s$ for the generalized series. If the numerator parameters and the denominator parameters can be paired up so that each column has the same sum, i.e.

$$ 1 + a_0 = a_1 + b_1 = \cdots = a_r + b_r $$

then the ${}_{1+r}F_r$-series is called *well-poised*. In particular, it is said to be *very well-poised* if we have $a_1 = 1 + a_0/2$. Similarly, we say the ${}_rH_r$-series is *well-poised* when

$$ a_1 + b_1 = a_2 + b_2 = \cdots = a_r + b_r $$

and is *very well-poised* if

$$ a_1 + b_1 = a_2 + b_2 = \cdots = a_r + b_r = 1 + 2b_1. $$

Instead, when the numerator parameters and the denominator parameters satisfy

$$ \begin{cases} a_1 + 2b_1 = a_2 + 2b_2 = \cdots = a_r + 2b_r = 2 + a_0, \\ 2c_1 + d_1 = 2c_2 + d_2 = \cdots = 2c_r + d_r = 1 + a_0, \end{cases} $$

we call the series

$$ {}_{1+2r}H_{1+2r}\left[\begin{matrix} 1 + \frac{a_0}{3}, a_1, \cdots, a_r, c_1, \cdots, c_r \\ \frac{a_0}{3}, \quad b_1, \cdots, b_r, d_1, \cdots, d_r \end{matrix}\middle| z\right] $$

quadratic. Letting $b_1 = 1$, the last series will be reduced to the *quadratic* ${}_{1+2r}F_{2r}$-series

$$ {}_{1+2r}F_{2r}\left[\begin{matrix} a_0, 1 + \frac{a_0}{3}, a_2, \cdots, a_r, c_1, \cdots, c_r \\ \frac{a_0}{3}, \quad b_2, \cdots, b_r, d_1, \cdots, d_r \end{matrix}\middle| z\right]. $$

Recently, Chu and Wang [2, 3] have utilized Abel's lemma on summation by parts to deal with the terminating hypergeometric series. The purpose of the present work is to explore the same lemma with the " remainder term" to investigate the following partial sum of the quadratic ${}_7H_7$-series

$$ Q_n(a, b, c, d, e) := {}_7H_7{}^n\left[\begin{matrix} 1 + \frac{a}{3}, & b, & d, & 1 + a - b - d, & c, & e, & \frac{1}{2} + a - c - e \\ \frac{a}{3}, & 1 + \frac{a-b}{2}, & 1 + \frac{a-d}{2}, & \frac{1+b+d}{2}, & 1 + a - 2c, & 1 + a - 2e, & 2c + 2e - a \end{matrix}\middle| 1\right]. $$

Some transformations from Q_n-sum to another quadratic series, well-poised series and other series as well as fast convergent series expressions for Q-sum will be established. These transformations and expressions will further lead to several summation and transformation formulae on ${}_7F_6$-series.

In order to facilitate the subsequent applications, we reproduce Abel's lemma on summation by parts with the " remainder term" as follows. For an arbitrary complex sequence $\{\tau_k\}$, define the backward and forward difference operators $\triangledown$ and $\vartriangle$, respectively, by

$$\triangledown\tau_k = \tau_k - \tau_{k-1} \quad \text{and} \quad \vartriangle\tau_k = \tau_k - \tau_{k+1}.$$

It should be pointed out that $\vartriangle$ is adopted for convenience in the present paper, which differs from the usual operator Δ only in the minus sign. Then **Abel's lemma** on summation by parts with the " remainder term" may be modified as follows:

$$\sum_{k=0}^{n-1} B_k \triangledown A_k = \Big\{A_{n-1}B_n - A_{-1}B_0\Big\} + \sum_{k=0}^{n-1} A_k \vartriangle B_k, \tag{1}$$

where $A_{n-1}B_n - A_{-1}B_0$ is the so-called " remainder term" which will help to derive transformations between series.

Proof. According to the definition of the backward difference, we have

$$\sum_{k=0}^{n-1} B_k \triangledown A_k = \sum_{k=0}^{n-1} B_k\Big\{A_k - A_{k-1}\Big\} = \sum_{k=0}^{n-1} A_k B_k - \sum_{k=0}^{n-1} A_{k-1}B_k.$$

Replacing k by $k+1$ for the last sum, we can reformulate the equation as follows:

$$\sum_{k=0}^{n-1} B_k \triangledown A_k = A_{n-1}B_n - A_{-1}B_0 + \sum_{k=0}^{n-1} A_k\Big\{B_k - B_{k+1}\Big\} = A_{n-1}B_n - A_{-1}B_0 + \sum_{k=0}^{n-1} A_k \vartriangle B_k,$$

which is exactly the formula stated in equation (1). □

Throughout the paper, if Ω_n is used to denote partial sum of some series, then the corresponding letter Ω without subscript will stand for the limit of Ω_n (if it exists of course) when $n \to \infty$.

2. Reciprocal Relation of Q-Sum

For two sequences given by

$$A_k = \left[\begin{matrix} 1+b, & 2+a-b-d, & 1+c, & \frac{1+a+d}{2}-c \\ 1+\frac{a-b}{2}, & \frac{1+b+d}{2}, & 1+a-2c, & 2+2c-d \end{matrix}\right]_k, B_k = \left[\begin{matrix} d-1, & 2+2c-d, & e, & \frac{1}{2}+a-c-e \\ 1+\frac{a-d}{2}, & \frac{a+d-1}{2}-c, & 1+a-2e, & 2c+2e-a \end{matrix}\right]_k;$$

it is almost trivial to compute the relations

$$\varpi := A_{-1}B_0 = \frac{(a-b)(a-2c)(b+d-1)(1+2c-d)}{2b\,c\,(1+a-b-d)(a+d-2c-1)},$$

$$\mathcal{R}_1 := \frac{A_{n-1}B_n}{A_{-1}B_0} = \frac{1+2c-d+n}{1+2c-d}\left[\begin{matrix} b, & d-1, & 1+a-b-d, & c, & e, & \frac{1}{2}+a-c-e \\ \frac{a-b}{2}, & 1+\frac{a-d}{2}, & \frac{b+d-1}{2}, & a-2c, & 1+a-2e, & 2c+2e-a \end{matrix}\right]_n;$$

and the following differences

$$\nabla A_k = \frac{(a+3k)(d-1+k)(a-b-2c)(b+d-2c-1)}{2b\,c\,(1+a-b-d)(a+d-2c-1)}\left[\begin{matrix} b, & 1+a-b-d, & c, & \frac{a+d-1}{2}-c \\ 1+\frac{a-b}{2}, & \frac{1+b+d}{2}, & 1+a-2c, & 2+2c-d \end{matrix}\right]_k,$$

$$\triangle B_k = \frac{(1+a+3k)(a-2c+k)(1+2c+2e-a-d)(d+2e-a-2)}{(2+a-d)(a+d-2c-1)(2c+2e-a)(2e-a-1)}\left[\begin{matrix} d-1, & 2+2c-d, & e, & \frac{1}{2}+a-c-e \\ 2+\frac{a-d}{2}, & \frac{1+a+d}{2}-c, & 2+a-2e, & 1+2c+2e-a \end{matrix}\right]_k.$$

By means of the modified Abel lemma on summation by parts, we can manipulate the following Q_n-series

$$Q_n(a,b,c,d,e)\times\frac{a(a-b-2c)(b+d-2c-1)(d-1)}{2b\,c\,(1+a-b-d)(a+d-2c-1)} = \sum_{k=0}^{n-1} B_k \nabla A_k = \varpi(\mathcal{R}_1-1)+\sum_{k=0}^{n-1} A_k \triangle B_k.$$

Writing the last partial sum explicitly

$$\sum_{k=0}^{n-1} A_k \triangle B_k = \frac{(1+a)(1+2c+2e-a-d)(d+2e-a-2)(a-2c)}{(2+a-d)(a+d-2c-1)(2c+2e-a)(2e-a-1)} \times \sum_{k=0}^{n-1}\frac{1+a+3k}{1+a}\left[\begin{matrix} 1+b, & d-1, & 2+a-b-d, & 1+c, & e, & \frac{1}{2}+a-c-e \\ 1+\frac{a-b}{2}, & 2+\frac{a-d}{2}, & \frac{1+b+d}{2}, & a-2c, & 2+a-2e, & 1+2c+2e-a \end{matrix}\right]_k,$$

we find the following recurrence relation.

Lemma 2.1 (Recurrence relation).

$$\begin{aligned} Q_n(a,b,c,d,e) &= Q_n(1+a,1+b,1+c,d-1,e) \\ &\times \frac{(1+a)\,b\,c\,(1+a-b-d)(c+e+\frac{1-a-d}{2})(1-e+\frac{a-d}{2})(2c-a)}{a(2c+2e-a)(c+\frac{b-a}{2})(c+\frac{1-b-d}{2})(1-d)(1+\frac{a-d}{2})(1+a-2e)} \\ &+ \frac{(b-a)(2c-a)(b+d-1)(1+2c-d)}{a(2c+b-a)(1+2c-b-d)(1-d)}\{1-\mathcal{R}_1(a,b,c,d,e)\}. \end{aligned}$$

Iterating the last relation m-times, we get the following transformation

$$\begin{aligned} Q_n(a,b,c,d,e) &= Q_n(m+a,m+b,m+c,d-m,e)\left[\begin{matrix} 1+a, b, & c, 1+a-b-d, 2c-a, c+e+\frac{1-a-d}{2}, 1-e+\frac{a-d}{2} \\ a, 1-d, 2c+2e-a, 1+a-2e, c+\frac{b-a}{2}, c+\frac{1-b-d}{2}, 1+\frac{a-d}{2} \end{matrix}\right]_m \\ &+ \frac{(b-a)(b+d-1)(2c-a)(1+2c-d)}{a(2c+b-a)(1+2c-b-d)(1-d)}\sum_{k=0}^{m-1}\frac{1+2c-d+3k}{1+2c-d}\Big\{1-\mathcal{R}_1(k+a,k+b,k+c,d-k,e)\Big\} \\ &\times\left[\begin{matrix} b, & c, & 1+a-b-d, & 1+2c-a, c+e+\frac{1-a-d}{2}, 1-e+\frac{a-d}{2} \\ 2-d, 2c+2e-a, 1+a-2e, 1+c+\frac{b-a}{2}, c+\frac{3-b-d}{2}, 1+\frac{a-d}{2} \end{matrix}\right]_k. \end{aligned}$$

Reformulating the $\mathcal{R}_1$-function by singling out k-factorials

$$\mathcal{R}_1(k+a,k+b,k+c,d-k,e)=\frac{1+2c-d+n+3k}{1+2c-d+3k}$$
$$\times\begin{bmatrix}b+k, & d-1-k, & 1+a-b-d+k, & c+k, & e, & \frac{1}{2}+a-c-e\\ \frac{a-b}{2}, & 1+\frac{a-d}{2}+k, & \frac{b+d-1}{2}, & a-2c-k, & 1+a-2e+k, & 2c+2e-a+k\end{bmatrix}_n$$
$$=\frac{1+2c-d+n+3k}{1+2c-d+3k}\begin{bmatrix}b, & d-1, & 1+a-b-d, & c, & e, & \frac{1}{2}+a-c-e\\ \frac{a-b}{2}, & 1+\frac{a-d}{2}, & \frac{b+d-1}{2}, & a-2c, & 1+a-2e, & 2c+2e-a\end{bmatrix}_n$$
$$\times\begin{bmatrix}n+b, 2-d, 1+a-b-d+n, n+c, 1+2c-a-n, 1+a-2e, 2c+2e-a\\ b, 2-d-n, 1+a-b-d, c, 1+2c-a, 1+a-2e+n, 2c+2e-a+n\end{bmatrix}_k$$

and then denoting by $Q'_m(a,b,c,d,e)$ another quadratic partial sum

$${}_7H_7{}^m\left[\begin{matrix}1+\frac{1+2c-d}{3}, & b, & 1+a-b-d, 1+2c-a, & c, c+e+\frac{1-a-d}{2}, 1-e+\frac{a-d}{2}\\ \frac{1+2c-d}{3}, & c+\frac{3-b-d}{2}, & 1+c+\frac{b-a}{2}, 1+\frac{a-d}{2}, & 2-d, 1+a-2e, 2c+2e-a\end{matrix}\middle|1\right],$$

we establish the following transformation formula on quadratic series.

Theorem 2.2 (Reciprocal relation on quadratic series). *For five indeterminate a, b, c, d, e, there holds*

$$Q_n(a,b,c,d,e)=Q_n(m+a,m+b,m+c,d-m,e)\begin{bmatrix}1+a, b, 1+a-b-d, 2c-a, c, c+e+\frac{1-a-d}{2}, 1-e+\frac{a-d}{2}\\ a, c+\frac{1-b-d}{2}, c+\frac{b-a}{2}, 1+\frac{a-d}{2}, 1-d, 1+a-2e, 2c+2e-a\end{bmatrix}_m$$
$$+\frac{(b-a)(b+d-1)(2c-a)(1+2c-d)}{a(2c+b-a)(1+2c-b-d)(1-d)}\Big\{Q'_m(a,b,c,d,e)-Q'_m(3n+a,n+b,n+c,n+d,n+e)$$
$$\times\begin{bmatrix}b, & d-1, & 1+a-b-d, & c, & e, & \frac{1}{2}+a-c-e, & 2+2c-d\\ \frac{a-b}{2}, & 1+\frac{a-d}{2}, & \frac{b+d-1}{2}, & a-2c, & 1+a-2e, & 2c+2e-a, & 1+2c-d\end{bmatrix}_n\Big\}.$$

This relation is said to be *reciprocal* because the right member in braces $\{\cdots\}$ can be obtained from the left member under the exchange $m\rightleftharpoons n$ and the parameter substitutions $a\to\lambda+a$, $d\to\lambda+d$ and $e\to\lambda+e$ with $\lambda=1+2c-a-d$. If applying again this relation to that member in braces $\{\cdots\}$, then we shall get back to the difference on the left hand side. Some consequences of this theorem are exhibited as follows.

Firstly, letting $m=n-1$, $b\to a$, $c=1-n$ in Theorem 2.2 and shifting n to $n+1$ in succession, we see that the last two lines are annihilated and the Q_n-sum on the right hand side of the first line reduces to one. We recover the following known ${}_7F_6$-series identity.

Corollary 2.3 ([5, Eq. (1.7)]).

$${}_7F_6\left[\begin{matrix}a, 1+\frac{a}{3}, & d, & 1-d, & e, & \frac{1}{2}+a-e+n, & -n\\ \frac{a}{3}, & 1+\frac{a-d}{2}, & \frac{1+a+d}{2}, & 1+a-2e, & 2e-a-2n, & 1+a+2n\end{matrix}\middle|1\right]=\begin{bmatrix}\frac{1+a}{2}, 1+\frac{a}{2}, \frac{1+a+d}{2}-e, 1-e+\frac{a-d}{2}\\ \frac{1+a+d}{2}, 1+\frac{a-d}{2}, \frac{1+a}{2}-e, 1-e+\frac{a}{2}\end{bmatrix}_n.$$

Secondly, letting $m=n-1$, $d\to 1-b$, $b=1-n$ in Theorem 2.2 and then shifting n to $n+1$, we derive the summation formula.

Corollary 2.4 ([3, Cor. 16]).

$${}_7F_6\left[\begin{matrix}a, 1+\frac{a}{3}, & c, & e, & \frac{1}{2}+a-c-e, & 1+n, & -n\\ \frac{a}{3}, & 1+a-2c, & 1+a-2e, & 2c+2e-a, & \frac{1+a-n}{2}, & 1+\frac{a-n}{2}\end{matrix}\middle|1\right]=\begin{bmatrix}1+a, 2c-a, c+e-\frac{a+n}{2}, \frac{1+a-n}{2}-e\\ 2c+2e-a, 1+a-2e, c-\frac{a+n}{2}, \frac{1+a-n}{2}\end{bmatrix}_n.$$

Finally, letting $m=n$ and $c=1-n$ in Theorem 2.2 leads directly to the following transformation formula.

Corollary 2.5 (Transformation formula on quadratic series: $\lambda=3-2n-a-d$).

$$Q_n(a,b,1-n,d,e)=\frac{(b-a)(b+d-1)(2-2n-a)(3-2n-d)}{a(2-2n+b-a)(3-2n-b-d)(1-d)}Q_n(\lambda+a,b,1-n,\lambda+d,\lambda+e).$$

In view of the limiting relations

$$\lim_{m\to\infty} Q_n(m+a;m+b,m+c,d-m,e)=\sum_{k=0}^{n-1}\left[\begin{matrix} e, & \frac{1}{2}+a-c-e \\ 1+\frac{a-b}{2}, & \frac{1+b+d}{2}\end{matrix}\right]_k,$$

$$\lim_{n\to\infty} Q'_m(3n+a,n+b,n+c,n+d,n+e)=\sum_{k=0}^{m-1}\left[\begin{matrix} c+e+\frac{1-a-d}{2}, & 1-e+\frac{a-d}{2} \\ 1+c+\frac{b-a}{2}, & c+\frac{3-b-d}{2}\end{matrix}\right]_k,$$

we derive, by letting $m, n \to \infty$ in Theorem 2.2, the nonterminating transformation formula.

Proposition 2.6 (Nonterminating quadratic series transformation: $\Re(2c+d-a)>0$).

$$\begin{aligned} &Q(a,b,c,d,e)+\frac{(a-b)(b+d-1)(2c-a)(1+2c-d)}{a(2c+b-a)(1+2c-b-d)(1-d)}Q'(a,b,c,d,e)\\ =&\Gamma\left[\begin{matrix} a,1-d,1+a-2e,2c+2e-a,c+\frac{1-b-d}{2},c+\frac{b-a}{2},1+\frac{a-d}{2} \\ 1+a,\ c,\ c+e+\frac{1-a-d}{2},1-e+\frac{a-d}{2},\ b\ ,1+a-b-d,2c-a\end{matrix}\right] {}_2H_2^{+}\left[\begin{matrix} e,\frac{1}{2}+a-c-e \\ 1+\frac{a-b}{2},\ \frac{1+b+d}{2}\end{matrix}\middle| 1\right]\\ +&\frac{\Gamma\left[\begin{matrix} a,1+\frac{a-b}{2},1+\frac{a-d}{2},\frac{1+b+d}{2},1+a-2c,1+a-2e,2c+2e-a \\ 1+a,\ b,\ d,\ 1+a-b-d,\ c,\ e,\ \frac{1}{2}+a-c-e\end{matrix}\right]}{(c+\frac{b-a}{2})(c+\frac{1-b-d}{2})} {}_2H_2^{+}\left[\begin{matrix} c+e+\frac{1-a-d}{2},1-e+\frac{a-d}{2} \\ 1+c+\frac{b-a}{2},\ c+\frac{3-b-d}{2}\end{matrix}\middle| 1\right]. \end{aligned}$$

Two nonterminating ${}_7F_6$-series transformations can be deduced by this proposition.

Firstly, letting $b \to a$ in Proposition 2.6 and then recalling Gauss's theorem (cf. [1, §1.2])

$${}_2F_1\left[\begin{matrix} a,b \\ c\end{matrix}\middle| 1\right]=\Gamma\left[\begin{matrix} c,c-a-b \\ c-a,c-b\end{matrix}\right], \qquad \Re(c-a-b)>0,$$

we get the following three-term transform formula.

Corollary 2.7 (Nonterminating transformation between quadratic ${}_7F_6$-series and ${}_2H_2^{+}$-series: $\Re(2c+d-a)>0$).

$$\begin{aligned} &{}_7F_6\left[\begin{matrix} a,1+\frac{a}{3}, & c, & e, & d, & 1-d, & \frac{1}{2}+a-c-e \\ \frac{a}{3}, & 1+a-2c, & 1+a-2e, & 1+\frac{a-d}{2}, & \frac{1+a+d}{2}, & 2c+2e-a\end{matrix}\middle| 1\right]\\ =&\ \Gamma\left[\begin{matrix} 1+a-2e,\ 2c+2e-a,\ c+\frac{1-a-d}{2},\ 1+\frac{a-d}{2},\ \frac{1+a+d}{2},\ c+\frac{d-a}{2} \\ 1+a,c+e+\frac{1-a-d}{2},1-e+\frac{a-d}{2},2c-a,\frac{1+a+d}{2}-e,c+e+\frac{d-a}{2}\end{matrix}\right]\\ +&\ \Gamma\left[\begin{matrix} 1+\frac{a-d}{2},\frac{1+a+d}{2},1+a-2c,1+a-2e,2c+2e-a,c+\frac{1-a-d}{2} \\ 1+a,\ d,\ 1-d,\ 1+c,\ e,\ \frac{1}{2}+a-c-e,c+\frac{3-a-d}{2}\end{matrix}\right] {}_2H_2^{+}\left[\begin{matrix} c+e+\frac{1-a-d}{2},1-e+\frac{a-d}{2} \\ 1+c,\ c+\frac{3-a-d}{2}\end{matrix}\middle| 1\right]. \end{aligned}$$

Instead, letting $c \to a/2$ in Proposition 2.6, we get another transformation.

Corollary 2.8 (Nonterminating transformation between quadratic ${}_7F_6$-series and ${}_2H_2^{+}$-series: $\Re(d)>0$).

$$\begin{aligned} &{}_7F_6\left[\begin{matrix} \frac{a}{2},1+\frac{a}{3}, & b, & d, & 1+a-b-d, & e, & \frac{1+a}{2}-e \\ \frac{a}{3}, & 1+\frac{a-b}{2}, & 1+\frac{a-d}{2}, & \frac{1+b+d}{2}, & 1+a-2e, & 2e\end{matrix}\middle| 1\right]\\ =&\ 2\Gamma\left[\begin{matrix} 1+\frac{a-b}{2},\ 1+\frac{a-d}{2},\ \frac{1+b+d}{2},\ 1+a-2e,\ 2e \\ 1+\frac{a}{2},\ 1+b,\ d,2+a-b-d,\ e,\ \frac{1+a}{2}-e\end{matrix}\right] {}_2H_2^{+}\left[\begin{matrix} e+\frac{1-d}{2},1-e+\frac{a-d}{2} \\ 1+\frac{b}{2},\ \frac{3+a-b-d}{2}\end{matrix}\middle| 1\right]. \end{aligned}$$

3. Transformation from Q-Sum to Very Well-poised Series

Alternatively, define the two sequences by

$$\mathcal{A}_k = \begin{bmatrix} 1+b, \ 1+2a-b-2c-2e, \ 1+c, \ 1+e \\ 1+\frac{a-b}{2}, 1+c+e+\frac{b-a}{2}, 1+a-2c, 1+a-2e \end{bmatrix}_k, \mathcal{B}_k = \begin{bmatrix} d, 1+a-b-d, \frac{1}{2}+a-c-e, 1+c+e+\frac{b-a}{2} \\ 1+\frac{a-d}{2}, \frac{1+b+d}{2}, 1+2c+2e-a, 2a-b-2c-2e \end{bmatrix}_k.$$

We have no difficulty to check the relations

$$\varpi := \mathcal{A}_{-1}\mathcal{B}_0 = \frac{(b-a)(a-2c)(a-2e)(c+e+\frac{b-a}{2})}{2bce(b+2c+2e-2a)},$$

$$\mathcal{R}_2 := \frac{\mathcal{A}_{n-1}\mathcal{B}_n}{\mathcal{A}_{-1}\mathcal{B}_0} = \frac{c+e+\frac{b-a}{2}+n}{c+e+\frac{b-a}{2}} \begin{bmatrix} b, & d, & 1+a-b-d, & c, & e, & \frac{1}{2}+a-c-e \\ \frac{a-b}{2}, 1+\frac{a-d}{2}, \frac{1+b+d}{2}, a-2c, a-2e, 1+2c+2e-a \end{bmatrix}_n;$$

and calculate the finite differences

$$\nabla\mathcal{A}_k = \frac{(a+3k)(2c+2e-a+k)(2c-a+b)(b-a+2e)}{4bce(b+2c+2e-2a)} \times \begin{bmatrix} b, & 2a-b-2c-2e, & c, & e \\ 1+\frac{a-b}{2}, & 1+c+e+\frac{b-a}{2}, & 1+a-2c, & 1+a-2e \end{bmatrix}_k,$$

$$\vartriangle\mathcal{B}_k = \frac{(a+3k+2)(1+b+k)(1-a-d+2c+2e)(b+d+2c+2e-2a)}{(2+a-d)(1+b+d)(1+2c+2e-a)(b+2c+2e-2a)} \times \begin{bmatrix} d, & 1+a-b-d, & \frac{1}{2}+a-c-e, & 1+c+e+\frac{b-a}{2} \\ 2+\frac{a-d}{2}, & \frac{3+b+d}{2}, & 2+2c+2e-a, & 1+2a-b-2c-2e \end{bmatrix}_k.$$

Then by means of the modified Abel lemma on summation by parts, the Q_n-sum can be manipulated as follows:

$$Q_n(a,b,c,d,e) \times \frac{a(b-a+2c)(b-a+2e)(2c+2e-a)}{4bce(b+2c+2e-2a)} = \sum_{k=0}^{n-1} \mathcal{B}_k \nabla \mathcal{A}_k = \varpi\Big\{\mathcal{R}_2 - 1\Big\} + \sum_{k=0}^{n-1} \mathcal{A}_k \vartriangle \mathcal{B}_k$$

$$= \varpi\Big\{\mathcal{R}_2 - 1\Big\} + Q_n(2+a, 2+b, 1+c, d, 1+e)\frac{(2+a)(1+b)(1+2c+2e-a-d)(b+d+2c+2e-2a)}{(1+b+d)(2+a-d)(1+2c+2e-a)(b+2c+2e-2a)}.$$

Rewriting the last equation, we get another recurrence relation for Q_n-sum.

Lemma 3.1 (Recurrence relation).

$$Q_n(a,b,c,d,e) = Q_n(2+a, 2+b, 1+c, d, 1+e)\frac{(2+a)ce(c+e+\frac{1-a-d}{2})(c+e-a+\frac{b+d}{2})(b)_2}{a(c+\frac{b-a}{2})(e+\frac{b-a}{2})(\frac{1+b+d}{2})(1+\frac{a-d}{2})(2c+2e-a)_2}$$
$$+ \frac{(a-b)(a-2c)(a-2e)(b+2c+2e-a)}{a(b+2c-a)(b+2e-a)(2c+2e-a)}\{1-\mathcal{R}_2(a,b,c,d,e)\}.$$

Iterating the last relation m-times, we get

$$Q_n(a,b,c,d,e) = Q_n(2m+a, 2m+b, m+c, d, m+e) \begin{bmatrix} 1+\frac{a}{2}, \frac{b}{2}, \frac{1+b}{2}, c, e, c+e+\frac{1-a-d}{2}, c+e-a+\frac{b+d}{2} \\ \frac{a}{2}, c+e+\frac{1-a}{2}, c+e-\frac{a}{2}, e+\frac{b-a}{2}, c+\frac{b-a}{2}, \frac{1+b+d}{2}, 1+\frac{a-d}{2} \end{bmatrix}_m$$
$$+ \frac{(a-b)(a-2c)(a-2e)(b+2c+2e-a)}{a(2c+b-a)(2e+b-a)(2c+2e-a)} \sum_{k=0}^{m-1} \frac{c+e+\frac{b-a}{2}+2k}{c+e+\frac{b-a}{2}} \Big\{1 - \mathcal{R}_2(2k+a, 2k+b, k+c, d, k+e)\Big\}$$
$$\times \begin{bmatrix} \frac{b}{2}, & \frac{1+b}{2}, & c, & e, & c+e+\frac{1-a-d}{2}, & c+e-a+\frac{b+d}{2} \\ 1+c+e-\frac{a}{2}, & c+e+\frac{1-a}{2}, & 1+e+\frac{b-a}{2}, & 1+c+\frac{b-a}{2}, & \frac{1+b+d}{2}, & 1+\frac{a-d}{2} \end{bmatrix}_k.$$

Denoting by $W_m(a,b,c,d,e)$ the very well-poised partial sum

$$ {}_7H_7{}^m\left[\begin{matrix}1+\frac{2c+2e+b-a}{4}, & \frac{b}{2}, & \frac{1+b}{2}, & c, & e, & c+e+\frac{1-a-d}{2}, & c+e-a+\frac{b+d}{2}\\ \frac{2c+2e+b-a}{4}, & 1+c+e-\frac{a}{2}, & c+e+\frac{1-a}{2}, & 1+e+\frac{b-a}{2}, & 1+c+\frac{b-a}{2}, & \frac{1+b+d}{2}, & 1+\frac{a-d}{2}\end{matrix}\middle|1\right]$$

and then reformulating the $\mathcal{R}_2$-function by singling out k-factorials

$$\mathcal{R}_2(2k+a,2k+b,k+c,d,k+e)=\frac{c+e+\frac{b-a}{2}+n+2k}{c+e+\frac{b-a}{2}+2k}\left[\begin{matrix}b, & d, & 1+a-b-d, & c, & e, & \frac{1}{2}+a-c-e\\ \frac{a-b}{2}, & 1+\frac{a-d}{2}, & \frac{1+b+d}{2}, & a-2c, & a-2e, & 1+2c+2e-a\end{matrix}\right]_n$$
$$\times\left[\begin{matrix}\frac{b+n}{2}, \frac{1+b+n}{2}, c+n, e+n, 1+c+e-\frac{a}{2}, c+e+\frac{1-a}{2}, 1+\frac{a-d}{2}, \frac{1+b+d}{2}\\ \frac{b}{2}, \frac{1+b}{2}, c, e, 1+c+e+\frac{n-a}{2}, c+e+\frac{1+n-a}{2}, 1+\frac{a-d}{2}+n, \frac{1+b+d}{2}+n\end{matrix}\right]_k,$$

we establish another transformation formula for Q_n-sum.

Theorem 3.2 (Transformation from quadratic to very well-poised series).

$$Q_n(a,b,c,d,e)=Q_n(2m+a,2m+b,m+c,d,m+e)\left[\begin{matrix}1+\frac{a}{2}, \frac{b}{2}, \frac{1+b}{2}, c, e, c+e+\frac{1-a-d}{2}, c+e-a+\frac{b+d}{2}\\ \frac{a}{2}, c+e+\frac{1-a}{2}, c+e-\frac{a}{2}, e+\frac{b-a}{2}, c+\frac{b-a}{2}, \frac{1+b+d}{2}, 1+\frac{a-d}{2}\end{matrix}\right]_m$$
$$+\frac{(a-b)(a-2c)(a-2e)(2c+2e+b-a)}{a(2c+b-a)(2e+b-a)(2c+2e-a)}\Big\{W_m(a,b,c,d,e)-W_m(3n+a,n+b,n+c,n+d,n+e)$$
$$\times\left[\begin{matrix}b, & d, & 1+a-b-d, & c, & e, & \frac{1}{2}+a-c-e, & 1+c+e+\frac{b-a}{2}\\ \frac{a-b}{2}, & 1+\frac{a-d}{2}, & \frac{1+b+d}{2}, & a-2c, & a-2e, & 1+2c+2e-a, & c+e+\frac{b-a}{2}\end{matrix}\right]_n\Big\}.$$

Firstly, letting $m=n-1$, $e\to a/2$, $c=1-n$ in Theorem 3.2 and then shifting n to $n+1$, we derive the summation formula.

Corollary 3.3 ([3, Cor.13]).

$${}_7F_6\left[\begin{matrix}\frac{a}{2}, & 1+\frac{a}{3}, & b, & d, & 1+a-b-d, & \frac{1+a}{2}+n, & -n\\ & \frac{a}{3}, & 1+\frac{a-b}{2}, & 1+\frac{a-d}{2}, & \frac{1+b+d}{2}, & -2n, & 1+a+2n\end{matrix}\middle|1\right]=\left[\begin{matrix}1+\frac{a}{2}, \frac{1+b}{2}, \frac{1+d}{2}, 1+\frac{a-b-d}{2}\\ \frac{1}{2}, 1+\frac{a-b}{2}, 1+\frac{a-d}{2}, \frac{1+b+d}{2}\end{matrix}\right]_n.$$

Secondly, letting $n=1+\delta+2m$, $c\to a/2$, $b=1-n$ with $\delta=0,1$ in Theorem 3.2 and noting that $Q_n(a,-\delta,a/2,d,e)=1-\delta$, we find another summation formula.

Corollary 3.4 ([5, Eq. (1.8)]).

$${}_7F_6\left[\begin{matrix}\frac{a}{2}, & 1+\frac{a}{3}, & d, & e, & \frac{1+a}{2}-e, & 1+a-d+n, & -n\\ & \frac{a}{3}, & 1+\frac{a-d}{2}, & 1+a-2e, & 2e, & \frac{1+d-n}{2}, & 1+\frac{a+n}{2}\end{matrix}\middle|1\right]=\begin{cases}\left[\begin{matrix}\frac{1}{2}, 1+\frac{a}{2}, e+\frac{1-d}{2}, 1-e+\frac{a-d}{2}\\ 1+\frac{a}{2}-e, 1+\frac{a-d}{2}, \frac{1-d}{2}, \frac{1}{2}+e\end{matrix}\right]_m, & n=2m;\\ 0, & n=2m+1.\end{cases}$$

Finally, letting $m=n$ and $e=1-n$, we get from Theorem 3.2 the following result.

Corollary 3.5 (Transformation between quadratic and very well-poised series).

$$Q_n(a,b,c,d,1-n)=\frac{(a-b)(a-2c)(a+2n-2)(2+b-a+2c-2n)}{a(2c+b-a)(2+b-a-2n)(2+2c-a-2n)}W_n(a,b,c,d,1-n).$$

We remark that the summation formula displayed in Corollary 2.4 can also be derived by applying Dougall's identity [4] (cf. [1, §5.1]) to the last transformation.

In view of the following limiting relations

$$\lim_{m\to\infty}Q_n(2m+a,2m+b,m+c,d,m+e)=\sum_{k=0}^{n-1}\left[\begin{matrix}d, 1+a-b-d, \frac{1}{2}+a-c-e\\ 1+\frac{a-b}{2}, 1+a-2c, 1+a-2e\end{matrix}\right]_k,$$

$$\lim_{n\to\infty}W_m(3n+a,n+b,n+c,n+d,n+e)=\sum_{k=0}^{m-1}\left[\begin{matrix}c+e+\frac{1-a-d}{2}, c+e-a+\frac{b+d}{2}\\ 1+c+\frac{b-a}{2}, 1+e+\frac{b-a}{2}\end{matrix}\right]_k,$$

we derive, by letting $m,n\to\infty$ in Theorem 3.2, the nonterminating series relation.

Proposition 3.6 (Nonterminating transformation formula on quadratic series:$\Re(1+a+b-2c-2e)>0$).

$$Q(a,b,c,d,e)-\frac{(a-b)(a-2c)(a-2e)(2c+2e+b-a)}{a(2c+b-a)(2e+b-a)(2c+2e-a)}W(a,b,c,d,e)$$
$$=\Gamma\begin{bmatrix}\frac{a}{2},c+e+\frac{1-a}{2},c+e-\frac{a}{2},e+\frac{b-a}{2},c+\frac{b-a}{2},1+\frac{a-d}{2},\frac{1+b+d}{2}\\ 1+\frac{a}{2},\ \frac{b}{2},\ \frac{1+b}{2},\ c,\ e,\ c+e-a+\frac{b+d}{2},\ c+e+\frac{1-a-d}{2}\end{bmatrix}{}_3H_3{}^{+}\begin{bmatrix}d,1+a-b-d,\frac{1}{2}+a-c-e\\ 1+\frac{a-b}{2},1+a-2c,1+a-2e\end{bmatrix}1\end{bmatrix}$$
$$-\frac{\Gamma\begin{bmatrix}a,1+\frac{a-b}{2},1+\frac{a-d}{2},\frac{1+b+d}{2},1+a-2c,1+a-2e,2c+2e-a\\ 1+a,\ b,\ d,\ 1+a-b-d,\ c,\ e,\ \frac{1}{2}+a-c-e\end{bmatrix}}{(c+\frac{b-a}{2})(e+\frac{b-a}{2})}{}_2H_2{}^{+}\begin{bmatrix}c+e+\frac{1-a-d}{2},c+e-a+\frac{b+d}{2}\\ 1+c+\frac{b-a}{2},\ 1+e+\frac{b-a}{2}\end{bmatrix}1\end{bmatrix}.$$

Letting $c\to a/2$ in Proposition 3.6 and noticing the ${}_3H_3{}^{+}$-series can be summed by Watson's sum [7](cf. [1, §3.3])

$$ {}_3F_2\begin{bmatrix}a,\ b,\ c\\ \frac{1+a+b}{2},2c\end{bmatrix}1\end{bmatrix}=\Gamma\begin{bmatrix}\frac{1}{2},\frac{1}{2}+c,\frac{1+a+b}{2},\frac{1-a-b}{2}+c\\ \frac{1+a}{2},\frac{1+b}{2},c+\frac{1-a}{2},c+\frac{1-b}{2}\end{bmatrix},\quad \Re(a+b-2c)<1, \tag{4}$$

we get the following transformation formula.

Corollary 3.7 (Nonterminating transformation from quadratic ${}_7F_6$-series to ${}_2H_2{}^{+}$-series: $\Re(1+b-2e)>0$).

$$ {}_7F_6\begin{bmatrix}\frac{a}{2},1+\frac{a}{3},\ b,\ d,\ 1+a-b-d,\ e,\ \frac{1+a}{2}-e\\ \frac{a}{3},\ 1+\frac{a-b}{2},1+\frac{a-d}{2},\ \frac{1+b+d}{2},\ 1+a-2e,\ 2e\end{bmatrix}1\end{bmatrix}$$
$$=\Gamma\begin{bmatrix}\frac{1}{2},\ \frac{1}{2}+e,\ \frac{1+b}{2}-e,\ 1+\frac{a}{2}-e,\ 1+\frac{a-b}{2},\ e+\frac{b-a}{2},\ 1+\frac{a-d}{2},\ \frac{1+b+d}{2}\\ \frac{1+b}{2},\frac{1+d}{2},,\frac{1+b+d}{2}-e,1+\frac{a}{2},e+\frac{b+d-a}{2},e+\frac{1-d}{2},1+\frac{a-b-d}{2},1+\frac{a-d}{2}-e\end{bmatrix}$$
$$-\Gamma\begin{bmatrix}1+\frac{a-b}{2},1+\frac{a-d}{2},1+a-2e,\frac{1+b+d}{2},2e,e+\frac{b-a}{2}\\ 1+\frac{a}{2},1+b,d,e,1+a-b-d,\frac{1+a}{2}-e,1+e+\frac{b-a}{2}\end{bmatrix}{}_2H_2{}^{+}\begin{bmatrix}e+\frac{1-d}{2},e+\frac{b+d-a}{2}\\ 1+\frac{b}{2},1+e+\frac{b-a}{2}\end{bmatrix}1\end{bmatrix}.$$

Instead, letting $b\to a$ in Proposition 3.6 and recalling Whipple's theorem [8](cf. [1, §3.4])

$$ {}_3F_2\begin{bmatrix}a,\ 1-a,\ c\\ b,1+2c-b\end{bmatrix}1\end{bmatrix}=\Gamma\begin{bmatrix}\frac{b}{2},\ \frac{1+b}{2},\ c+\frac{1-b}{2},\ 1+c-\frac{b}{2}\\ \frac{a+b}{2},c+\frac{1+a-b}{2},\frac{1-a+b}{2},1+c-\frac{a+b}{2}\end{bmatrix},\quad \Re(c)>0, \tag{5}$$

we have another nonterminating transformation.

Corollary 3.8 (Nonterminating transformation from quadratic ${}_7F_6$-series to ${}_2H_2{}^{+}$-series: $\Re(1+2a-2c-2e)>0$).

$$ {}_7F_6\begin{bmatrix}a,1+\frac{a}{3},\ d,\ 1-d,\ c,\ e,\ \frac{1}{2}+a-c-e\\ \frac{a}{3},\ 1+\frac{a-d}{2},\ \frac{1+a+d}{2},1+a-2c,1+a-2e,\ 2c+2e-a\end{bmatrix}1\end{bmatrix}$$
$$=\Gamma\begin{bmatrix}c+e+\frac{1-a}{2},\ c+e-\frac{a}{2},\ 1+\frac{a-d}{2},\ \frac{1+a+d}{2},\ \frac{1+a}{2}-c,\ 1+\frac{a}{2}-c,\ \frac{1+a}{2}-e,\ 1+\frac{a}{2}-e\\ \frac{1+a}{2},1+\frac{a}{2},c+e+\frac{d-a}{2},c+e+\frac{1-a-d}{2},\frac{1+a+d}{2}-c,1+\frac{a-d}{2}-c,\frac{1+a+d}{2}-e,1+\frac{a-d}{2}-e\end{bmatrix}$$
$$-\Gamma\begin{bmatrix}1+a-2c,1+a-2e,2c+2e-a,1+\frac{a-d}{2},\frac{1+a+d}{2}\\ 1+a,\ 1+c,\ 1+e,\ \frac{1}{2}+a-c-e,\ d,\ 1-d\end{bmatrix}{}_2H_2{}^{+}\begin{bmatrix}c+e+\frac{1-a-d}{2},c+e+\frac{d-a}{2}\\ 1+c,\ \ 1+e\end{bmatrix}1\end{bmatrix}.$$

Remark. We must point out that Watson's theorem (4) and Whipple's theorem (5) are indeed the limiting cases of Corollary 3.7 and Corollary 3.8 respectively. If we substitute b by $a-b$, then let $a\to\infty$ in Corollary 3.7, we obtain Watson's theorem (4). If we substitute e by $a/2-e$, and then let $a\to\infty$ in Corollary 3.8, we have Whipple's theorem (5).

4. A Strange Transformation on Terminating ${}_{11}F_{10}$-Series

Noting $Q_n(a,b,c,d,e) = Q_n(a,d,c,b,e)$ in Lemma 2.1, we have

$$Q_n(a,b,c,d,e) = Q_n(1+a,b-1,1+c,1+d,e)\frac{(1+a)cd(2c-a)(1+a-b-d)(c+e+\frac{1-a-b}{2})(1-e+\frac{a-b}{2})}{a(1-b)(1+a-2e)(2c+2e-a)(1+\frac{a-b}{2})(c+\frac{d-a}{2})(c+\frac{1-b-d}{2})}$$
$$+\frac{(d-a)(2c-a)(b+d-1)(1+2c-b)}{a(2c+d-a)(1+2c-b-d)(b-1)}\Big\{1-\mathcal{R}_1(a,d,c,b,e)\Big\}.$$

Substituting the last expression into Lemma 3.1 and then simplifying the result, we get the following recurrence relation for Q_n-sum

$$Q_n(a,b,c,d,e) = Q_n(3+a,1+b,2+c,1+d,1+e)\frac{bde(c)_2}{(2c+2e-a)_3}$$
$$\times\frac{(3+a)(2c-a)(1+a-b-d)(c+e+\frac{1-a-b}{2})(c+e+\frac{1-a-d}{2})(c+e-a+\frac{b+d}{2})}{a(1+a-2e)(1+\frac{a-b}{2})(1+\frac{a-d}{2})(c+\frac{b-a}{2})(c+\frac{d-a}{2})(\frac{1+b+d}{2})(c+\frac{1-b-d}{2})}$$
$$+\frac{(a-d)(2c-a)(b+d-1)(1+2c-b)}{a(2c+d-a)(1+2c-b-d)(b-1)}\Big\{1-\mathcal{R}_1(a,d,c,b,e)\Big\}$$
$$-\frac{cd(1+a-b-d)(c+e+\frac{b-a}{2})(c+e+\frac{1-a-b}{2})(2c-a)_2}{a(b-1)(c+\frac{b-a}{2})(c+\frac{d-a}{2})(c+\frac{1-b-d}{2})(2c+2e-a)_2}\Big\{1-\mathcal{R}_2(1+a,b-1,1+c,1+d,e)\Big\}.$$

Iterating the last relation m-times, we get the expression

$$Q_n(a,b,c,d,e) = Q_n(3m+a,m+b,2m+c,m+d,m+e)\frac{(c)_{2m}}{(2c+2e-a)_{3m}}$$
$$\times\begin{bmatrix}1+\frac{a}{3}, b, d, 1+a-b-d, e, c+e+\frac{1-a-b}{2}, c+e+\frac{1-a-d}{2}, c+e-a+\frac{b+d}{2}, 2c-a\\ \frac{a}{3}, \quad 1+\frac{a-b}{2}, \quad 1+\frac{a-d}{2}, \quad \frac{1+b+d}{2}, \quad 1+a-2e, \quad c+\frac{b-a}{2}, \quad c+\frac{d-a}{2}, \quad c+\frac{1-b-d}{2}\end{bmatrix}_m$$
$$+\frac{(a-d)\,(2c-a)\,(b+d-1)(1-b+2c)}{a(b-1)(2c+d-a)(1+2c-b-d)}$$
$$\times\sum_{k=0}^{m-1}\frac{1-b+2c+3k}{1-b+2c}\Big\{1-\mathcal{R}_1(3k+a,k+d,2k+c,k+b,k+e)\Big\}\frac{(c)_{2k}}{(2c+2e-a)_{3k}}$$
$$\times\begin{bmatrix}b-1, d, 1+a-b-d, e, c+e+\frac{1-a-b}{2}, c+e+\frac{1-a-d}{2}, c+e-a+\frac{b+d}{2}, 1+2c-a\\ 1+\frac{a-b}{2}, \quad \frac{a-d}{2}, \quad \frac{b+d-1}{2}, \quad 1+a-2e, \quad c+\frac{b-a}{2}, \quad 1+c+\frac{d-a}{2}, \quad c+\frac{3-b-d}{2}\end{bmatrix}_k$$
$$-\frac{(c+e+\frac{b-a}{2})cd(1+a-b-d)(c+e+\frac{1-a-b}{2})(2c-a)_2}{a(b-1)(c+\frac{b-a}{2})(c+\frac{d-a}{2})(c+\frac{1-b-d}{2})(2c+2e-a)_2}$$
$$\times\sum_{k=0}^{m-1}\frac{c+e+\frac{b-a}{2}+2k}{c+e+\frac{b-a}{2}}\Big\{1-\mathcal{R}_2(3k+1+a,k+b-1,2k+1+c,k+1+d,k+e)\Big\}\frac{(1+c)_{2k}}{(2+2c+2e-a)_{3k}}$$
$$\times\begin{bmatrix}b-1, 1+d, 2+a-b-d, e, c+e+\frac{3-a-b}{2}, c+e+\frac{1-a-d}{2}, c+e-a+\frac{b+d}{2}, 2+2c-a\\ 1+\frac{a-b}{2}, \quad 1+\frac{a-d}{2}, \quad \frac{1+b+d}{2}, \quad 1+a-2e, \quad 1+c+\frac{b-a}{2}, \quad 1+c+\frac{d-a}{2}, \quad c+\frac{3-b-d}{2}\end{bmatrix}_k.$$

Reformulating the both $\mathcal{R}$-functions by singling out k-factorials

$$\begin{aligned}&\mathcal{R}_1(3k+a,k+d,2k+c,k+b,k+e)\\&=\frac{1+2c-b+3k+n}{1+2c-b+3k}\begin{bmatrix}b-1, & d, & 1+a-b-d, & c, & e, & \frac{1}{2}+a-c-e\\ 1+\frac{a-b}{2}, & \frac{a-d}{2}, & \frac{b+d-1}{2}, & a-2c, & 1+a-2e, & 2c+2e-a\end{bmatrix}_n\frac{(c+n)_{2k}}{(2c+2e-a+n)_{3k}}\\&\times\begin{bmatrix}b-1+n,d+n,e+n,1+a-2e,1+a-b-d+n,1+2c-a-n,1+\frac{a-b}{2},\frac{a-d}{2},\frac{b+d-1}{2}\\ b-1,\ d,\ e,1+a-2e+n,1+a-b-d,1+2c-a,1+\frac{a-b}{2}+n,\frac{a-d}{2}+n,\frac{b+d-1}{2}+n\end{bmatrix}_k,\end{aligned}$$

$$\begin{aligned}&\mathcal{R}_2(3k+1+a,k+b-1,2k+1+c,k+1+d,k+e)\\&=\frac{c+e+\frac{b-a}{2}+2k+n}{c+e+\frac{b-a}{2}+2k}\begin{bmatrix}b-1, & 1+d, & 2+a-b-d, & 1+c, & e, & \frac{1}{2}+a-c-e\\ 1+\frac{a-b}{2}, & 1+\frac{a-d}{2}, & \frac{1+b+d}{2}, & a-2c-1, & 1+a-2e, & 2+2c+2e-a\end{bmatrix}_n\frac{(1+c+n)_{2k}}{(2+2c+2e-a+n)_{3k}}\\&\times\begin{bmatrix}b-1+n,1+d+n,e+n,1+a-2e,2+a-b-d+n,2+2c-a-n,1+\frac{a-b}{2},1+\frac{a-d}{2},\frac{1+b+d}{2}\\ b-1,1+d,e,1+a-2e+n,2+a-b-d,2+2c-a,1+\frac{a-b}{2}+n,1+\frac{a-d}{2}+n,\frac{1+b+d}{2}+n\end{bmatrix}_k;\end{aligned}$$

and then denoting by $U_m(a,b,c,d,e)$, $V_m(a,b,c,d,e)$ the following two partial series respectively

$${}_{11}H_{11}{}^{m}\left[\begin{matrix}1+\frac{1+2c-b}{3},\ b-1,\ d,\ e,\ 1+a-b-d,1+2c-a,c+e+\frac{1-a-b}{2},c+e+\frac{1-a-d}{2},c+e-a+\frac{b+d}{2},\frac{c}{2},\frac{1+c}{2}\\ \frac{1+2c-b}{3},1+\frac{a-b}{2},\frac{a-d}{2},1+a-2e,\frac{b+d-1}{2},c+\frac{b-a}{2},1+c+\frac{d-a}{2},c+\frac{3-b-d}{2},\frac{2c+2e-a}{3},\frac{1+2c+2e-a}{3},\frac{2+2c+2e-a}{3}\end{matrix}\middle|\frac{4}{27}\right],$$

$${}_{11}H_{11}{}^{m}\left[\begin{matrix}1+\frac{2c+2e+b-a}{4},b-1,1+d,e,2+a-b-d,2+2c-a,c+e+\frac{3-a-b}{2},c+e+\frac{1-a-d}{2},c+e-a+\frac{b+d}{2},\frac{1+c}{2},1+\frac{c}{2}\\ \frac{2c+2e+b-a}{4},1+\frac{a-b}{2},1+\frac{a-d}{2},1+a-2e,\frac{1+b+d}{2},1+c+\frac{b-a}{2},1+c+\frac{d-a}{2},c+\frac{3-b-d}{2},\frac{2+2c+2e-a}{3},\frac{3+2c+2e-a}{3},\frac{4+2c+2e-a}{3}\end{matrix}\middle|\frac{4}{27}\right],$$

we establish the following six-term transformation formula.

Theorem 4.1. *For five indeterminate a, b, c, d, e, there holds*

$$\begin{aligned}Q_n(a,b,c,d,e)&=Q_n(3m+a,m+b,2m+c,m+d,m+e)\frac{(c)_{2m}}{(2c+2e-a)_{3m}}\\&\times\begin{bmatrix}1+\frac{a}{3},b,d,1+a-b-d,e,c+e+\frac{1-a-b}{2},c+e+\frac{1-a-d}{2},c+e-a+\frac{b+d}{2},2c-a\\ \frac{a}{3},\ 1+\frac{a-b}{2},\ 1+\frac{a-d}{2},\ \frac{1+b+d}{2},\ 1+a-2e,\ c+\frac{b-a}{2},\ c+\frac{d-a}{2},\ c+\frac{1-b-d}{2}\end{bmatrix}_m\\&+\frac{(a-d)(2c-a)(b+d-1)(1+2c-b)}{a(b-1)(2c+d-a)(1+2c-b-d)}\Big\{U_m(a,b,c,d,e)-U_m(3n+a,n+b,n+c,n+d,n+e)\\&\qquad\times\begin{bmatrix}b-1, & d, & 1+a-b-d, & c, & e, & \frac{1}{2}+a-c-e, & 2+2c-b\\ 1+\frac{a-b}{2}, & \frac{a-d}{2}, & \frac{b+d-1}{2}, & a-2c, & 1+a-2e, & 2c+2e-a, & 1+2c-b\end{bmatrix}_n\Big\}\\&-\frac{cd(c+e+\frac{b-a}{2})(1+a-b-d)(c+e+\frac{1-a-b}{2})(2c-a)_2}{a(b-1)(c+\frac{b-a}{2})(c+\frac{d-a}{2})(c+\frac{1-b-d}{2})(2c+2e-a)_2}\Big\{V_m(a,b,c,d,e)-V_m(3n+a,n+b,n+c,n+d,n+e)\\&\qquad\times\begin{bmatrix}b-1, & 1+d, & 2+a-b-d, & 1+c, & e, & \frac{1}{2}+a-c-e, & 1+c+e+\frac{b-a}{2}\\ 1+\frac{a-b}{2}, & 1+\frac{a-d}{2}, & \frac{1+b+d}{2}, & a-2c-1, & 1+a-2e, & 2+2c+2e-a, & c+e+\frac{b-a}{2}\end{bmatrix}_n\Big\}.\end{aligned}$$

In this theorem, letting $m=n$, $b\to a$, $a\to 1+a$, $e=1-n$ and shifting n to $n+1$ in succession, then recalling the summation formula displayed in Corollary 2.3, we find the following strange transformation on terminating ${}_{11}F_{10}$-series.

Proposition 4.2 (Terminating transformation on ${}_{11}F_{10}$-series).

$$\begin{aligned}&{}_{11}F_{10}\left[\begin{matrix}a,\ 1+\frac{2c-a}{3},\ \frac{c}{2},\ \frac{1+c}{2},\ d,\ 1-d,\ 2c-a,\ c-a-\frac{1}{2}-n,\ c-\frac{a+d}{2}-n,\ c+\frac{d-a-1}{2}-n,\ -n\\ \frac{2c-a}{3},\ c,\frac{1+a-d}{2},\frac{a+d}{2},1+c-\frac{a+d}{2},c+\frac{1+d-a}{2},\frac{2c-a-1-2n}{3},\frac{2c-a-2n}{3},\frac{1+2c-a-2n}{3},2+a+2n\end{matrix}\middle|\frac{4}{27}\right]\\&=\begin{bmatrix}\frac{a}{2},\frac{1+a}{2},\frac{1+a-d}{2}-c,\frac{a+d}{2}-c\\ \frac{a}{2}-c,\frac{1+a}{2}-c,\frac{1+a-d}{2},\frac{a+d}{2}\end{bmatrix}_{n+1}+\frac{2d(1-d)(c-n)(2c-2a-1-2n)}{(1+a-d)(a+d)(2c-a-1-2n)_2}\\&\times{}_{11}F_{10}\left[\begin{matrix}a,\ 1+\frac{c-n}{2},\ \frac{1+c}{2},\ 1+\frac{c}{2},\ 1+d,\ 2-d,\ 1+2c-a,\ \frac{1}{2}+c-a-n,\ c-\frac{a+d}{2}-n,\ c+\frac{d-a-1}{2}-n,\ -n\\ \frac{c-n}{2},\ 1+c,\frac{3+a-d}{2},1+\frac{a+d}{2},1+c-\frac{a+d}{2},c+\frac{1+d-a}{2},\frac{1+2c-a-2n}{3},\frac{2+2c-a-2n}{3},\frac{3+2c-a-2n}{3},2+a+2n\end{matrix}\middle|\frac{4}{27}\right].\end{aligned}$$

Letting $n \to \infty$ in the last proposition and then rearranging the parameters, we get the following interesting transformation.

Corollary 4.3 (Nonterminating transformation between ${}_7F_6$-series and ${}_6F_5$-series).

$$
{}_7F_6\left[\begin{matrix}2a, 1+\frac{2a}{3}, & 2b, & 2d, & 1-2d, & \frac{a+b}{2}, & \frac{1+a+b}{2} \\ \frac{2a}{3}, & a+b, & 1+a-d, & a+d, & b+d, & \frac{1}{2}+b-d\end{matrix}\middle| -\frac{1}{4}\right] - \frac{d(1-2d)}{(1+2b-2d)(b+d)}
$$
$$
\times {}_6F_5\left[\begin{matrix}1+2a, & 2b, & 1+2d, & 2-2d, & \frac{1+a+b}{2}, & 1+\frac{a+b}{2} \\ & 1+a+b, & 1+a-d, & \frac{1}{2}+a+d, & 1+b+d, & \frac{3}{2}+b-d\end{matrix}\middle| -\frac{1}{4}\right]
$$
$$
= \Gamma\left[\begin{matrix}-2a, & \frac{1}{2}-2a, & \frac{1}{2}+b-d, & b+d \\ b, & \frac{1}{2}+b, & \frac{1}{2}-d-2a, & d-2a\end{matrix}\right].
$$

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