Filomat 28:2 (2014), 373–381
DOI 10.2298/FIL1402373M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

AN ANALOGUE OF THE OSTROWSKI INEQUALITY AND APPLICATIONS

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Abstract. A new analogue of the Ostrowski inequality is introduced in three different cases for functions in $L^1[a,b]$ and $L^\infty[a,b]$ spaces and its application is given for deriving error bounds of some quadrature rules.

1. Introduction

Let $L^p[a,b]$ $(1 \le p < \infty)$ denote the space of p-power integrable functions on the interval $[a,b]$ with the standard norm

$$\|f\|_p = \left(\int_a^b \| f(t) \|^p dt \right)^{1/p},$$

and $L^\infty[a,b]$ the space of all essentially bounded functions on $[a,b]$ with the norm

$$\|f\|_\infty = ess \sup_{x\in[a,b]} \left| f(x) \right|.$$

If $h \in L^1[a,b]$ and $g \in L^\infty[a,b]$, then the following inequality holds

$$\left| \int_a^b h(x)\, g(x)\, dx \right| \le \| h \|_1 \| g \|_\infty .$$

For two absolutely continuous functions $f, g : [a,b] \to \mathbf{R}$ such that $f, g, fg \in L^1[a,b]$, the Chebyshev functional is defined by

$$\begin{aligned}\mathbf{T}(f,g) &= \tfrac{1}{b-a} \int_a^b \left(f(x) - \tfrac{1}{b-a} \int_a^b f(x)\, dx \right) \left(g(x) - \tfrac{1}{b-a} \int_a^b g(x)\, dx \right) dx \\ &= \tfrac{1}{b-a} \int_a^b f(x)\, g(x)\, dx - \tfrac{1}{(b-a)^2} \left(\int_a^b f(x)\, dx \right) \left(\int_a^b g(x)\, dx \right).\end{aligned}$$

2010 *Mathematics Subject Classification*. 26D10, 26D20, 41A35, 65D30.
Keywords. Ostrowski inequality, Numerical quadrature rules, Error bounds, Kernel function, Normed spaces.
Received: 7 Jan 2013; Accepted: 7 Feb 2013
Communicated by Vladimir Rakocevic
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A well-known inequality in the literature, which is related to the Chebyshev functional, is the Ostrowski inequality [15]. If $f, g : [a, b] \to \mathbf{R}$ is a differentiable function with bounded derivative, then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left(\frac{(x-a)^2 + (b-x)^2}{2\,(b-a)} \right) \| f' \|_\infty, \tag{1}$$

for all $x \in [a, b]$. The above result has been extended for absolutely continuous functions and Lebesgue p-norms of the derivative f' in [3,4] as follows: Let $f : [a, b] \to \mathbf{R}$ be absolutely continuous on $[a, b]$. Then for all $x \in [a, b]$ and $\frac{1}{p} + \frac{1}{q} = 1$ $(p > 1)$ we have

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \begin{cases} \frac{1}{(p+1)^{\frac{1}{p}}} \left(\left(\frac{x-a}{b-a}\right)^{p+1} + \left(\frac{b-x}{b-a}\right)^{p+1} \right) (b-a)^{\frac{1}{q}} \| f' \|_q, \\ \left(\frac{1}{2} + \frac{1}{b-a} \left| x - \frac{a+b}{2} \right| \right) \| f' \|_1 . \end{cases} \tag{2}$$

The constants $1/(p+1)^{1/p}$ and $1/2$ in (2) are respectively sharp in the sense that they cannot be replaced by a smaller constant.

The inequalities (1), (2) can also be obtained, in a slightly different form, as particular cases of some results established by Fink in [10] for n-time differentiable functions. For other Ostrowski type inequalities concerning Lipschitzian type functions, see [5]. The cases of bounded variation functions and monotonic functions have been studied in [6]. For various generalizations, refinements and related Ostrowski type inequalities for functions of one or several variables one may refer to the monograph [7] and the references therein. See also [1,8,11,15] and [13,14,16,19] in this regard. Moreover, the Ostrowski inequality has an important role in numerical quadrature rules [9,12].

In this paper, we introduce a new analogue of the Ostrowski inequality in three different cases and apply them for some quadrature rules. First of all, let us consider the following well-known kernel on $[a, b]$

$$K(x; t) = \begin{cases} t - a & t \in [a, x], \\ t - b & t \in (x, b]. \end{cases} \tag{3}$$

After some computations, it can directly be concluded that

$$\int_a^b |K(x; t)|\, dt = \int_a^x (t-a)\, dt - \int_x^b (t-b)\, dt = \frac{1}{2} \left((x-a)^2 + (b-x)^2 \right),$$

and

$$\int_a^b f'(t)\, K(x; t)\, dt = (b-a) f(x) - \int_a^b f(x)\, dx. \tag{4}$$

2. Main Results

Theorem 1. *Let $f : \mathbf{I} \to \mathbf{R}$, where $\mathbf{I}$ is an interval, be a function differentiable in the interior $\mathbf{I}^0$ of $\mathbf{I}$, and let $[a, b] \subset \mathbf{I}^0$. If $\alpha(x) \leq f'(x) \leq \beta(x)$ for any $\alpha, \beta \in C[a, b]$ and $x \in [a, b]$ then the following inequality holds*

$$\begin{aligned} \tfrac{1}{b-a} \left(\int_a^x (t-a)\, \alpha(t)\, dt + \int_x^b (t-b)\, \beta(t)\, dt \right) &\leq f(x) - \tfrac{1}{b-a} \int_a^b f(t)\, dt \\ &\leq \tfrac{1}{b-a} \left(\int_a^x (t-a)\, \beta(t)\, dt + \int_x^b (t-b)\, \alpha(t)\, dt \right). \end{aligned} \tag{5}$$

Proof. By referring to the kernel (3) and identity (4) first we have

$$\begin{aligned}&\int_a^b K(x;t)\left(f'(t)-\tfrac{\alpha(t)+\beta(t)}{2}\right)dt\\ &=(b-a)f(x)-\int_a^b f(t)\,dt-\tfrac{1}{2}\left(\int_a^b K(x;t)\,(\alpha(t)+\beta(t))\,dt\right)\\ &=(b-a)f(x)-\int_a^b f(t)\,dt-\tfrac{1}{2}\left(\int_a^x (t-a)(\alpha(t)+\beta(t))\,dt+\int_x^b (t-b)(\alpha(t)+\beta(t))\,dt\right).\end{aligned} \tag{6}$$

Also, the given assumption $\alpha(x)\le f'(x)\le\beta(x)$ implies that

$$\left|f'(t)-\frac{\alpha(t)+\beta(t)}{2}\right|\le\frac{\beta(t)-\alpha(t)}{2}. \tag{7}$$

Therefore, from (6) and (7) one can conclude that

$$\begin{aligned}&\left|(b-a)f(x)-\int_a^b f(t)\,dt-\tfrac{1}{2}\left(\int_a^x (t-a)(\alpha(t)+\beta(t))\,dt+\int_x^b (t-b)(\alpha(t)+\beta(t))\,dt\right)\right|\\ &=\left|\int_a^b K(x;t)\left(f'(t)-\tfrac{\alpha(t)+\beta(t)}{2}\right)dt\right|\le\int_a^b |K(x;t)|\,\tfrac{\beta(t)-\alpha(t)}{2}\,dt\\ &=\tfrac{1}{2}\left(\int_a^x (t-a)(\beta(t)-\alpha(t))\,dt-\int_x^b (t-b)(\beta(t)-\alpha(t))\,dt\right).\end{aligned} \tag{8}$$

By re-arranging (8), the main inequality (5) will be derived.

Theorem 1 is actually remarkable as it improves all previous results which made use of the Lebesgue norms of $f'(x)$ in (1) and (2). Moreover, a further advantage of this theorem is that necessary computations in bounds (5) are just in terms of the pre-assigned functions $\alpha(t),\beta(t)$ (not f').

Special case 1. Suppose that $f'(x)$ is bounded at two arbitrary linear functions, e.g. $\alpha(x)=\alpha_1x+\alpha_0\neq 0$ and $\beta(x)=\beta_1x+\beta_0\neq 0$. In this case, the main inequality (5) takes the form

$$\begin{aligned}&\tfrac{(x-a)^2}{b-a}\left(\tfrac{\alpha_1}{3}(x-a)+\tfrac{\alpha_0+a\alpha_1}{2}\right)-\tfrac{(x-b)^2}{b-a}\left(\tfrac{\beta_1}{3}(x-b)+\tfrac{\beta_0+b\beta_1}{2}\right)\\ &\le f(x)-\tfrac{1}{b-a}\int_a^b f(t)\,dt\le\\ &\tfrac{(x-a)^2}{b-a}\left(\tfrac{\beta_1}{3}(x-a)+\tfrac{\beta_0+a\beta_1}{2}\right)-\tfrac{(x-b)^2}{b-a}\left(\tfrac{\alpha_1}{3}(x-b)+\tfrac{\alpha_0+b\alpha_1}{2}\right).\end{aligned} \tag{9}$$

In [2], Dragomir obtained a special case of (5) for $\alpha(x)=\alpha_0\neq 0$ and $\beta(x)=\beta_0\neq 0$ as follows

$$\frac{\alpha_0(x-a)^2-\beta_0(b-x)^2}{2(b-a)}\le f(x)-\frac{1}{b-a}\int_a^b f(t)dt\le\frac{\beta_0(x-a)^2-\alpha_0(b-x)^2}{2(b-a)},$$

which is exactly a special case of (9) for $\alpha_1=\beta_1=0$.

Remark 1. Although $\alpha(x)\le f'(x)\le\beta(x)$ is a straightforward condition in theorem 1, sometimes one might not be able to easily obtain both bounds of $\alpha(x)$ and $\beta(x)$ for $f'(x)$. In this case, we can make use of two analogue theorems. The first one would be helpful when f' is unbounded from above and the second one would be helpful when f' is unbounded from below.

Theorem 2. *Let $f:\mathbf{I}\to\mathbf{R}$, where $\mathbf{I}$ is an interval, be a function differentiable in the interior $\mathbf{I}^0$ of $\mathbf{I}$, and let $[a,b]\subset\mathbf{I}^0$.*

If $\alpha(x) \leq f'(x)$ *for any* $\alpha, \beta \in C[a,b]$ *and* $x \in [a,b]$ *then*

$$\frac{1}{b-a}\left(\int_a^x (t-a)\,\alpha(t)\,dt + \int_x^b (t-b)\,\alpha(t)\,dt - \max\{x-a, b-x\}\left(f(b)-f(a)-\int_a^b \alpha(t)\,dt\right)\right)$$

$$\leq f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \leq \tag{10}$$

$$\frac{1}{b-a}\left(\int_a^x (t-a)\,\alpha(t)\,dt + \int_x^b (t-b)\,\alpha(t)\,dt + \max\{x-a, b-x\}\left(f(b)-f(a)-\int_a^b \alpha(t)\,dt\right)\right).$$

Proof. Since

$$\int_a^b K(x;t)\,\left(f'(t)-\alpha(t)\right)\,dt$$

$$= (b-a)\,f(x) - \int_a^b f(t)\,dt - \left(\int_a^b K(x;t)\;\alpha(t)\,dt\right)$$

$$= (b-a)\,f(x) - \int_a^b f(t)\,dt - \left(\int_a^x (t-a)\,\alpha(t)\,dt + \int_x^b (t-b)\,\alpha(t)\,dt\right),$$

so

$$\left|(b-a)\,f(x) - \int_a^b f(t)\,dt - \left(\int_a^x (t-a)\,\alpha(t)\,dt + \int_x^b (t-b)\,\alpha(t)\,dt\right)\right|$$

$$= \left|\int_a^b K(x;t)\,\left(f'(t)-\alpha(t)\right)\,dt\right| \leq \int_a^b |K(x;t)|\,\left(f'(t)-\alpha(t)\right)\,dt \tag{11}$$

$$\leq \max_{t\in[a,b]} |K(x;t)|\,\int_a^b \left(f'(t)-\alpha(t)\right)\,dt = \max\{x-a, b-x\}\left(f(b)-f(a)-\int_a^b \alpha(t)\,dt\right).$$

By re-arranging (11), inequality (10) will be derived.

Special case 2. If $\alpha(x) = \alpha_0 \neq 0$ then (10) becomes

$$\alpha_0\left(x-\frac{a+b}{2}\right) - \max\{x-a, b-x\}\left(\frac{f(b)-f(a)}{b-a} - \alpha_0\right) \leq f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt$$

$$\leq \alpha_0\left(x-\frac{a+b}{2}\right) + \max\{x-a, b-x\}\left(\frac{f(b)-f(a)}{b-a} - \alpha_0\right).$$

Theorem 3. *Let* $f : \mathbf{I} \to \mathbf{R}$*, where* $\mathbf{I}$ *is an interval, be a function differentiable in the interior* $\mathbf{I}^0$ *of* $\mathbf{I}$*, and let* $[a,b] \subset \mathbf{I}^0$*. If* $f'(x) \leq \beta(x)$ *for any* $\alpha, \beta \in C[a,b]$ *and* $x \in [a,b]$ *then*

$$\frac{1}{b-a}\left(\int_a^x (t-a)\,\beta(t)\,dt + \int_x^b (t-b)\,\beta(t)\,dt - \max\{x-a, b-x\}\left(\int_a^b \beta(t)\,dt - f(b) + f(a)\right)\right)$$

$$\leq f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \leq \tag{12}$$

$$\frac{1}{b-a}\left(\int_a^x (t-a)\,\beta(t)\,dt + \int_x^b (t-b)\,\beta(t)\,dt + \max\{x-a, b-x\}\left(\int_a^b \beta(t)\,dt - f(b) + f(a)\right)\right).$$

Proof. Since

$$\int_a^b K(x;t)\,\left(f'(t)-\beta(t)\right)\,dt$$

$$= (b-a)\,f(x) - \int_a^b f(t)\,dt - \left(\int_a^b K(x;t)\;\beta(t)\,dt\right)$$

$$= (b-a)\,f(x) - \int_a^b f(t)\,dt - \left(\int_a^x (t-a)\,\beta(t)\,dt + \int_x^b (t-b)\,\beta(t)\,dt\right),$$

so

$$\left|(b-a)\,f(x)-\int_a^b f(t)\,dt-\left(\int_a^x (t-a)\,\beta(t)\,dt+\int_x^b (t-b)\,\beta(t)\,dt\right)\right|$$

$$=\left|\int_a^b K(x;t)\,\left(f'(t)-\beta(t)\right)\,dt\right| \le \int_a^b |K(x;t)|\,\left(\beta(t)-f'(t)\right)\,dt \tag{13}$$

$$\le \max_{t\in[a,b]} |K(x;t)|\,\int_a^b \left(\beta(t)-f'(t)\right)\,dt \;=\max\{x-a,b-x\}\left(\int_a^b \beta(t)\,dt-f(b)+f(a)\right).$$

By re-arranging (13), inequality (12) will be derived.
Special case 3. If $\beta(x)=\beta_0\neq 0$ then (12) becomes

$$\beta_0\left(x-\tfrac{a+b}{2}\right)-\max\{x-a,b-x\}\left(\beta_0-\tfrac{f(b)-f(a)}{b-a}\right)\le f(x)-\tfrac{1}{b-a}\int_a^b f(t)\,dt$$

$$\le \beta_0\left(x-\tfrac{a+b}{2}\right)+\max\{x-a,b-x\}\left(\beta_0-\tfrac{f(b)-f(a)}{b-a}\right).$$

3. Applications in numerical quadrature rules

A general $(n+1)$-point weighted quadrature formula is denoted by

$$\int_a^b w(x)\,f(x)\,dx=\sum_{k=0}^{n} w_k\,f(x_k)+R_{n+1}(f), \tag{14}$$

where $w(x)$ is a positive function on $[a,b]$, $\{x_k\}_{k=0}^n$ and $\{w_k\}_{k=0}^n$ are respectively nodes and weight coefficients and $R_{n+1}(f)$ is the corresponding error [18].
Let $\mathbf{\Pi}_d$ be the set of algebraic polynomials of degree at most d. The quadrature formula (14) has degree of exactness d if for every $p\in\mathbf{\Pi}_d$ we have $R_{n+1}(p)=0$. In addition, if $R_{n+1}(p)\neq 0$ for some $\mathbf{\Pi}_{d+1}$, formula (14) has precise degree of exactness d. The convergence order of quadrature formula (14) depends on the smoothness of the function f as well as on its degree of exactness. It is well known that for given $n+1$ mutually different nodes $\{x_k\}_{k=0}^n$ we can always achieve a degree of exactness $d=n$ by interpolating at these nodes and integrating the interpolated polynomial instead of f. Namely, taking the node polynomial

$$\Psi_{n+1}(x)=\prod_{k=0}^{n}(x-x_k),$$

by integrating the Lagrange interpolation formula

$$f(x)=\sum_{k=0}^{n} f(x_k)\,L(x\,;\,x_k)+r_{n+1}(f;x),$$

where

$$L(x\,;\,x_k)=\frac{\Psi_{n+1}(x)}{\Psi'_{n+1}(x_k)(x-x_k)}\qquad (k=0,1,...,n)\,,$$

we obtain (14), with

$$w_k=\frac{1}{\Psi'_{n+1}(x_k)}\int_a^b \frac{\Psi_{n+1}(x)\,w(x)}{x-x_k}\,dx\quad (k=0,1,...,n)\,,$$

and

$$R_{n+1}(f) = \int_a^b r_{n+1}(f;x)\, w(x)\, dx.$$

Note that for each $f \in \mathbf{\Pi}_n$ we have $r_{n+1}(f;x) = 0$ and therefore $R_{n+1}(f) = 0$.

Quadrature formulae obtained in this way are known as interpolatory. If a quadrature is not of the interpolatory type, i.e. if it does not follow the concept of the degree of exactness, then it would be a nonstandard quadrature rule.

Usually the simplest interpolatory quadrature formula of type (14) with predetermined nodes $\{x_k\}_{k=0}^n \in [a,b]$ is called a weighted Newton-Cotes formula. For $w(x) = 1$ and the equidistant nodes $\{x_k\}_{k=0}^n = \{a+kh\}_{k=0}^n$ with $h = (b-a)/n$, the classical Newton-Cotes formula including the midpoint rule for $n = 0$ and $w(x) = 1$, the trapezoidal rule for $n = 1$ and $w(x) = 1$ and so on are derived. In this section, we use theorems 1, 2 and 3 to obtain new error bounds for midpoint rule and six further nonstandard quadratures as follows

$$I_1(f): \quad \int_a^b f(x)\,dx \cong (b-a)\, f(\frac{a+b}{2}),$$

$$I_2(f): \quad \int_a^b f(x)\,dx \cong (b-a)\, f(a),$$

$$I_3(f): \quad \int_a^b f(x)\,dx \cong (b-a)\, f(b),$$

$$I_4(f): \quad \int_a^b f(t)\,dt \cong \frac{b-a}{2}\left(-f(a) + 2f(\frac{a+b}{2}) + f(b)\right),$$

$$I_5(f): \quad \int_a^b f(t)\,dt \cong \frac{b-a}{2}\left(f(a) + 2f(\frac{a+b}{2}) - f(b)\right),$$

$$I_6(f): \quad \int_a^b f(x)\,dx \cong (b-a)\, \left(2f(a) - f(b)\right),$$

$$I_7(f): \quad \int_a^b f(x)\,dx \cong (b-a)\, \left(-f(a) + 2f(b)\right).$$

Corollary 1. If $\alpha(x) \le f'(x) \le \beta(x)$ for any $x \in [a,b]$ and $\alpha, \beta \in C[a,b]$ then by replacing $x = \frac{a+b}{2} \in [a,b]$ in (5), the error of midpoint rule $I_1(f)$ can be bounded as

$$\begin{aligned} \int_a^{\frac{a+b}{2}} (t-a)\,\alpha(t)\,dt + \int_{\frac{a+b}{2}}^b (t-b)\,\beta(t)\,dt \le\ & (b-a)f(\tfrac{a+b}{2}) - \int_a^b f(t)\,dt \\ & \le \int_a^{\frac{a+b}{2}} (t-a)\,\beta(t)\,dt + \int_{\frac{a+b}{2}}^b (t-b)\,\alpha(t)\,dt. \end{aligned} \tag{15}$$

For instance, if $\alpha(x) = \alpha_1 x + \alpha_0 \neq 0$ and $\beta(x) = \beta_1 x + \beta_0 \neq 0$ in (15) then

$$\begin{aligned} \tfrac{(b-a)^2}{4}\left(\tfrac{b-a}{6}(\alpha_1+\beta_1) + \tfrac{\alpha_0 + a\,\alpha_1 - (\beta_0 + b\,\beta_1)}{2}\right) \le (b-a)\, f(\tfrac{a+b}{2}) - \int_a^b f(t)\,dt \\ \le \tfrac{(b-a)^2}{4}\left(\tfrac{b-a}{6}(\alpha_1+\beta_1) + \tfrac{\beta_0 + a\,\beta_1 - (\alpha_0 + b\,\alpha_1)}{2}\right), \end{aligned}$$

provided that $\alpha_1\, t + \alpha_0 \le f'(t) \le \beta_1\, t + \beta_0 \quad \forall t \in [a,b]$.

Corollary 2. If $\alpha(x) \le f'(x) \le \beta(x)$ for any $x \in [a,b]$ and $\alpha, \beta \in C[a,b]$ then by replacing $x = a \in [a,b]$ in (5), the error of nonstandard quadrature $I_2(f)$ can be bounded as

$$\int_a^b (t-b)\,\beta(t)\,dt \le (b-a)f(a) - \int_a^b f(t)\,dt \le \int_a^b (t-b)\,\alpha(t)\,dt. \tag{16}$$

For instance, if $\alpha(x) = \alpha_1 x + \alpha_0 \neq 0$ and $\beta(x) = \beta_1 x + \beta_0 \neq 0$ then

$$(b-a)^2\left(\frac{\beta_1}{3}(b-a) - \frac{\beta_0 + b\beta_1}{2}\right) \leq (b-a)f(a) - \int_a^b f(t)\,dt \leq (b-a)^2\left(\frac{\alpha_1}{3}(b-a) - \frac{\alpha_0 + b\alpha_1}{2}\right).$$

provided that $\alpha_1\, t + \alpha_0 \leq f'(t) \leq \beta_1\, t + \beta_0 \quad \forall t \in [a,b]$.

Corollary 3. If $\alpha(x) \leq f'(x) \leq \beta(x)$ for any $x \in [a,b]$ and $\alpha, \beta \in C[a,b]$ then by replacing $x = b \in [a,b]$ in (5), the error of nonstandard quadrature $I_3(f)$ can be bounded as

$$\int_a^b (t-a)\,\alpha(t)\,dt \leq (b-a)f(b) - \int_a^b f(t)\,dt \leq \int_a^b (t-a)\,\beta(t)\,dt. \tag{17}$$

For instance, if $\alpha(x) = \alpha_1 x + \alpha_0 \neq 0$ and $\beta(x) = \beta_1 x + \beta_0 \neq 0$ then

$$(b-a)^2\left(\frac{\alpha_1}{3}(b-a) + \frac{\alpha_0 + a\alpha_1}{2}\right) \leq (b-a)f(b) - \int_a^b f(t)\,dt \leq (b-a)^2\left(\frac{\beta_1}{3}(b-a) + \frac{\beta_0 + a\beta_1}{2}\right).$$

provided that $\alpha_1\, t + \alpha_0 \leq f'(t) \leq \beta_1\, t + \beta_0 \quad \forall t \in [a,b]$.

Corollary 4. If $\alpha(x) \leq f'(x) \leq \beta(x)$ for any $x \in [a,b]$ and $\alpha, \beta \in C[a,b]$ then by replacing $x = b \in [a,b]$ in (5), the the error of nonstandard quadrature $I_4(f)$ can be bounded as

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}} (t-a)\,\alpha(t)\,dt + \int_{\frac{a+b}{2}}^b (t-b)\,\alpha(t)\,dt + \tfrac{b-a}{2}\int_a^b \alpha(t)\,dt \\
&\leq \tfrac{b-a}{2}\left(-f(a) + 2f(\tfrac{a+b}{2}) + f(b)\right) - \int_a^b f(t)\,dt \leq \\
&\int_a^{\frac{a+b}{2}} (t-a)\,\beta(t)\,dt + \int_{\frac{a+b}{2}}^b (t-b)\,\beta(t)\,dt + \tfrac{b-a}{2}\int_a^b \beta(t)\,dt\,.
\end{aligned} \tag{18}$$

Proof. To prove (18) we need to use the results of both theorems 2 and 3 simultaneously such that by replacing $x = (a+b)/2$ in (10) we first obtain

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}} (t-a)\,\alpha(t)\,dt + \int_{\frac{a+b}{2}}^b (t-b)\,\alpha(t)\,dt + \tfrac{b-a}{2}\int_a^b \alpha(t)\,dt \\
&\leq \tfrac{b-a}{2}\left(-f(a) + 2f(\tfrac{a+b}{2}) + f(b)\right) - \int_a^b f(t)\,dt\,,
\end{aligned} \tag{19}$$

provided that $\alpha(t) \leq f'(t) \quad \forall t \in [a,b]$. On the other hand, replacing $x = (a+b)/2$ in (12) gives

$$\begin{aligned}
&\tfrac{b-a}{2}\left(-f(a) + 2f(\tfrac{a+b}{2}) + f(b)\right) - \int_a^b f(t)\,dt \leq \\
&\int_a^{\frac{a+b}{2}} (t-a)\,\beta(t)\,dt + \int_{\frac{a+b}{2}}^b (t-b)\,\beta(t)\,dt + \tfrac{b-a}{2}\int_a^b \beta(t)\,dt\,,
\end{aligned} \tag{20}$$

provided that $f'(t) \leq \beta(t) \quad \forall t \in [a,b]$. Now, combining two latter results (19) and (20) leads us to inequality (18).

Corollary 5. If $\alpha(x) \leq f'(x) \leq \beta(x)$ for any $x \in [a,b]$ and $\alpha, \beta \in C[a,b]$ then by replacing $x = b \in [a,b]$ in (5), the the error of nonstandard quadrature $I_5(f)$ can be bounded as

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}} (t-a)\,\beta(t)\,dt + \int_{\frac{a+b}{2}}^b (t-b)\,\beta(t)\,dt - \tfrac{b-a}{2}\int_a^b \beta(t)\,dt \\
&\leq \tfrac{b-a}{2}\left(f(a) + 2f(\tfrac{a+b}{2}) - f(b)\right) - \int_a^b f(t)\,dt \leq \\
&\int_a^{\frac{a+b}{2}} (t-a)\,\alpha(t)\,dt + \int_{\frac{a+b}{2}}^b (t-b)\,\alpha(t)\,dt - \tfrac{b-a}{2}\int_a^b \alpha(t)\,dt\,.
\end{aligned} \tag{21}$$

Proof. The proof of (21) is similar to that of corollary 4 if one replaces $x = (a + b)/2$ in respectively (10) and (12) and then combines them together.

Corollary 6. If $\alpha(x) \le f'(x) \le \beta(x)$ for any $x \in [a, b]$ and $\alpha, \beta \in C[a, b]$ then by replacing $x = b \in [a, b]$ in (5), the the error of nonstandard quadrature $I_6(f)$ can be bounded as

$$\int_a^b (t + a - 2b)\,\beta(t)\,dt \le (b - a)\,(2f(a) - f(b)) - \int_a^b f(t)\,dt \le \int_a^b (t + a - 2b)\,\alpha(t)\,dt\,. \quad (22)$$

Proof. Again, to prove (22) we need to use the results of both theorems 2 and 3 simultaneously such that by replacing $x = a$ in (10) we first obtain

$$(b - a)\,(2f(a) - f(b)) - \int_a^b f(t)\,dt \le \int_a^b (t + a - 2b)\,\alpha(t)\,dt\,, \quad (23)$$

provided that $\alpha(t) \le f'(t) \quad \forall t \in [a, b]$. On the other hand, replacing $x = a$ in (12) gives

$$\int_a^b (t + a - 2b)\,\beta(t)\,dt \le (b - a)\,(2f(a) - f(b)) - \int_a^b f(t)\,dt\,, \quad (24)$$

provided that $f'(t) \le \beta(t) \quad \forall t \in [a, b]$. Therefore, combining two latter results (23) and (24) yields (22).

Corollary 7. If $\alpha(x) \le f'(x) \le \beta(x)$ for any $x \in [a, b]$ and $\alpha, \beta \in C[a, b]$ then by replacing $x = b \in [a, b]$ in (5), the the error of nonstandard quadrature $I_7(f)$ can be bounded as

$$\int_a^b (t - 2a + b)\,\alpha(t)\,dt \le (b - a)\,(-f(a) + 2f(b)) - \int_a^b f(t)\,dt \le \int_a^b (t - 2a + b)\,\beta(t)\,dt\,. \quad (25)$$

Proof. The proof of (25) is similar to that of corollary 6 if one replaces $x = b$ in respectively (10) and (12) and then combines them together.

References

[1] P. Cerone, S.S. Dragomir, Improvement of some Ostrowski-Gruss type inequalities, New bounds for the three-point rule involving the Riemann-Stieltjes integrals in: C. Gulati, et al. (Eds.), Advances in Statistics Combinatorics and Related Areas, World Science Publishing, 2002, pp. 53-62.

[2] S.S. Dragomir, Ostrowski's type inequalities for some classes of continuous functions of selfadjoint operators in Hilbert spaces, Comput. Math. Appl., 62 (2011) 4439-4448.

[3] S.S. Dragomir, S. Wang, A new inequality of Ostrowski's type in Lp-norm and applications to some special means and to some numerical quadrature rules, Indian J. Math. 40 (1998) 245-304.

[4] S.S. Dragomir, S. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, Appl. Math. Lett. 11 (1998) 105-109.

[5] S.S. Dragomir, The Ostrowski's integral inequality for Lipschitzian mappings and applications, Comput. Math. Appl. 38 (1999) 33-37.

[6] S.S. Dragomir, Ostrowski's inequality for monotonous mappings and applications, J. Korean Soc. Industr. Appl. Math. 3 (1999) 127-135.

[7] S.S. Dragomir, Th. M. Rassias (Eds.), Ostrowski Type Inequalities and Applications in Numerical Integration , Kluwer Academic Publisher, 2002.

[8] S.S. Dragomir, On the Ostrowski's inequality for Riemann-Stieltjes integral, Korean J. Appl. Math. 7 (2000) 477-485.

[9] I. Fedotov, S.S. Dragomir, An inequality of Ostrowski type and its applications for Simpson's rule and special means, , Math. Inequal. Appl. 2 (1999) 491-499.

[10] A.M. Fink, Bounds on the deviation of a function from its averages, Czechoslovak Math. J. 42 (1992) 298-310.

[11] P. Kumar, The Ostrowski type moments integral inequalities and moment bounds for continuous random variables, Comput. Math. Appl. 49 (2005) 1929-1940.

[12] M. Masjed-Jamei, S. S. Dragomir, A new generalization of the Ostrowski inequality and applications, Filomat, 25 (2011) 115-123.

[13] M. Masjed-Jamei, A linear constructive approximation for integrable functions and a parametric quadrature model based on a generalization of Ostrowski-Gruss type inequalities, Elect. Trans. Numer. Anal., **38** (2011) 218-232.

[14] M. Masjed-Jamei, S. Dragomir, A generalization of the Ostrowski-Gruss inequality, Anal. Appl. 2013,To appear.

[15] M. Masjed-Jamei, A certain class of weighted approximations for integrable functions and applications, *Numer. Func. Anal. Opt*. 2013, To appear.

[16] Milovanović and Pečarić, On generalization of the inequality of A. Ostrowski and some related applications, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No 544, No 576 (1976) 155-158.

[17] A. Ostrowski, Uber die absolutabweichung einer differentiebaren funktion van ihrem integralmittelwert, Comment Math. Helv 10 (1938) 226-227.

[18] J. Stoer and R. Bulirsch, Introduction to Numerical Analysis, second edition, Springer-Verlag, New York, 1993.

[19] Y.-X. Xiao, Remarks on Ostrowski-like inequalities, Appl. Math. Comput. 2012, To appear.