Minimal properties of the Drazin–inverse solution of a matrix equation

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Abstract. We present the Drazin-inverse solution of the matrix equation $AXB = G$ as a least-squares solution of a specified minimization problem. Some important properties of the Moore-Penrose inverse are extended on the Drazin inverse by exploring the minimal norm properties of the Drazin-inverse solution of the matrix equation $AXB = G$. The least squares properties of the Drazin-inverse solution lead to new representations of the Drazin inverse of a given matrix, which are justified by illustrative examples.

1. Introduction

Penrose in his paper [11] was the first who showed the close correlation between the Moore-Penrose inverse and the least-squares solution of a system of linear equations. Later, based on this result, the least-square properties of the Moore-Penrose inverse matrix have been investigated and many useful results have been established.

Some minimal properties of the Drazin-inverse solution of a given system of linear equations have also presented, recently, in the papers [2, 15, 18]. These properties correspond to analogous properties of the Moore-Penrose inverse solution. Particularly, it is shown that if $b \in \mathcal{R}(A^p)$, where $p = \text{ind}(A)$, then the Drazin-inverse solution is the unique solution of the system $Ax = b$ which belongs to $\mathcal{R}(A^p)$ [2]. Establishing a relation between the Drazin inverse and the solutions of a given system of linear equations, naturally imposed the idea of exploring the minimal properties of the Drazin inverse. Also, Wei et al. in [15, 18] have shown that the Drazin-inverse solution of the system $Ax = b$ is a solution of minimum P-norm, where P is the Jordan matrix obtained from the Jordan decomposition of the matrix A.

We assume that A, B, $G \in \mathbb{C}^{n \times n}$ are given matrices. It is shown that the matrix $X = A^D G B^D$ is the unique solution of the *restricted matrix equation* (see [14])

$$AXB = G, \quad \text{where } \mathcal{R}(G) \subset \mathcal{R}(A^{k_1}), \ \mathcal{N}(B^{k_2}) \subset \mathcal{N}(G), \ k_1 = \text{ind}(A), \ k_2 = \text{ind}(B). \tag{1.1}$$

The solution is obtained by imposing additional restrictions

$$\mathcal{R}(X) \subset \mathcal{R}(A^{k_1}), \quad \mathcal{N}(B^{k_2}) \subset \mathcal{N}(X) \tag{1.2}$$

on the matrix equation (1.1). In the sequel we use the term *the Drazin-inverse solution* of the restricted matrix equation $AXB = G$ to indicate the matrix $A^D G B^D$. As a consequence, it seems reasonable to call the matrix equation (1.1) *the Drazin consistent matrix equation*. Although, in the case when $\mathcal{R}(G) \nsubseteq \mathcal{R}(A^{k_1})$ or $\mathcal{N}(B^{k_2}) \nsubseteq \mathcal{N}(G)$, the matrix $A^D G B^D$ is not a solution of the system $AXB = G$, for convenience we also call it the Drazin–inverse solution of the matrix equation $AXB = G$.

The Drazin inverse has many applications in the theory of finite Markov chains, as well as in the study of differential equations and singular linear difference equations [2], cryptography [7] etc. Taking into account its importance many computational techniques were developed. The Cramer's rules derived for general restricted linear equation [3], as well for restricted matrix equation [4], were generalized for computing the Drazin-inverse solution of a restricted matrix equation in [14]. However, more common problem in the literature is to find a solution of the system

$$Ax = b, \ b \in \mathcal{R}(A^k), \ k = \mathrm{ind}(A), \tag{1.3}$$

i.e., a solution of the form $A^D b$. Many different techniques were developed in order to compute it [13, 16, 19, 20]. An application of gradient iterative methods for computing $A^D b$ is presented in [8], regarding general linear system $Ax = b$. This application is based on the minimal properties of the Drazin-inverse solution. Various representations of the Drazin inverse and corresponding computational procedures are given in [17]. Index splitting methods for computing the Drazin inverse and its relative iterations for the minimal P-norm solution of singular linear system (1.3) are presented in [15, 18]. Main characteristics of the Drazin inverse solution $A^D b$ are derived in [18]. An application of the Drazin inverse A^D in solving singular linear system $Ax = b$ is presented in [10].

The initial idea of the present paper is to reveal the minimal properties of the Drazin-inverse solution $A^D X B^D$ of a given matrix equation

$$AXB = G, \ G \in \mathbb{C}^{n \times n} \text{ is an arbitrary matrix}, \tag{1.4}$$

and consequently the Drazin inverse of a matrix, which are analogous to the minimal properties of the Moore-Penrose inverse. Our goal is achieved by stating the problem of computing the Drazin–inverse solution of the matrix equation (1.4) as a problem of finding a least–squares solution of appropriately defined matrix equation. Later, using the obtained results, we derive new representations of the Drazin inverse of a given square matrix, which involve the Moore-Penrose inverse.

The organization of the remainder of the paper is the following. Preliminary notions for the observed problem as well as auxiliary results are given in Section 2. The third section is devoted to the construction of a minimization problem whose solution is the matrix $A^D G B^D$, in the general case which does not include any restrictions, i.e., A, B, $G \in \mathbb{C}^{n \times n}$ are arbitrary matrices. In this section, we also present a new representation of the Drazin inverse of an arbitrary square matrix A in terms of its Jordan basis, powers of A and the Moore–Penrose inverse. Illustrative examples obtained by testing the results explained in Sections 3 are presented in the last section.

2. Preliminaries

For a given matrix $A = (a_{ij}) \in \mathbb{C}^{m \times n}$, we denote $a = \mathrm{vec}(A) \in \mathbb{C}^{mn}$ to be the vector obtained by stacking the rows of A into a column vector.

The Kronecker product $A \otimes B$ of two matrices $A = (a_{ij}) \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{p \times q}$ is the $mp \times nq$ matrix expressible in partitioned form as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \ldots & a_{1n}B \\ a_{21}B & a_{22}B & \ldots & a_{2n}B \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1}B & a_{m2}B & \ldots & a_{mn}B \end{bmatrix}.$$

Main properties of the Kronecker product are summarized from [1, 6, 9] in the following proposition.

Proposition 2.1. [1] *Let A, B, E, F be matrices of appropriate dimensions. Then the following hold:*
1) $(A \otimes B)(E \otimes F) = AE \otimes BF$,
2) *For any $q \in \mathbb{N}$ it holds* $(A \otimes I)^q = A^q \otimes I$,
3) *If* $\text{ind}(A) = k$, *then* $\text{ind}(A \otimes I) = k$.
4) *If A is a square nonsingular matrix, then the matrix $A \otimes I$ is nonsingular and* $(A \otimes I)^{-1} = A^{-1} \otimes I$,

An important application of the Kronecker product is the possibility to rewrite the matrix equation (1.4) into the equivalent vector equation

$$(A \otimes B^T)\text{vec}(X) = \text{vec}(G).$$

For simplicity, we use the notation $A_B = A \otimes B$.

In order to find a solution of the matrix equation (1.4), the most common approach is to minimize the functional

$$\|AXB - G\|_F^2, \tag{2.1}$$

where $\| \cdot \|_F$ denotes the Frobenious matrix norm. Two important generalized inverses that occur naturally in solving this problem are presented below.

Definition 2.1. [1] *Let $A \in \mathbb{C}^{m \times n}$. The matrix $X \in \mathbb{C}^{n \times m}$ satisfying the equations*

$$(1) \quad AXA = A \qquad (2) \quad XAX = X \qquad (3) \quad (AX)^* = AX \qquad (4) \quad (XA)^* = XA$$

is called the Moore-Penrose inverse of A denoted by $A^\dagger$. The matrix X which satisfies only the first and the third equation is called $\{1,3\}$-inverse of A, denoted by $A^{(1,3)}$.

Despite the Moore-Penrose inverse, it appears that the Drazin inverse also possesses some kind of least-squares and minimal properties. These properties are mainly explored in the articles [15, 18].

Definition 2.2. *Let $A \in \mathbb{C}^{n \times n}$ and $k = \text{ind}(A)$. The matrix $X \in \mathbb{C}^{n \times n}$ satisfying the conditions*

$$(1^k) \ A^k XA = A^k \qquad (2) \ XAX = X \qquad (5) \ AX = XA$$

is called the Drazin inverse A^D of the matrix A.

We use the following usual notations:

$$\mathcal{N}(A) = \{x \in \mathbb{C}^n | Ax = 0\}, \quad \mathcal{R}(A) = \{Ax | x \in \mathbb{C}^n\},$$

$$\widetilde{\mathcal{N}}(A) = \{X \in \mathbb{C}^{n \times n} | AX = 0\}, \quad \widetilde{\mathcal{R}}(A) = \{AX | X \in \mathbb{C}^{n \times n}\}.$$

Let

$$A = PJ_A P^{-1} = P \begin{bmatrix} C_A & 0 \\ 0 & N_A \end{bmatrix} P^{-1}, \ B = QJ_B Q^{-1} = Q \begin{bmatrix} C_B & 0 \\ 0 & N_B \end{bmatrix} Q^{-1}$$

be a Jordan decompositions of matrices $A, B \in \mathbb{C}^{n \times n}$. We explore the following matrix and vector norms

$$\|X\|_{P,Q} = \|P^{-1} XQ\|_F, \ \ \|X\|_P = \|P^{-1} X\|_F \ \text{ and } \ \|x\|_P = \|P^{-1} x\|_2,$$

where $X \in \mathbb{C}^{n \times n}$, $\| \cdot \|_F$ denotes the Frobenius matrix norm, $x \in \mathbb{C}^n$ and $\| \cdot \|_2$ denotes the Euclidean vector norm.

For the sake of completeness we give the proof of the following propositions.

Proposition 2.2. *Let $A \in \mathbb{C}^{n \times n}$, P be a matrix whose columns are a Jordan basis of the matrix A and let J be a Jordan matrix of A. Then $P_I J_I P_I^{-1} = A_I$. Moreover, $\|X\|_P = \|\text{vec}(X)\|_{P_I}$.*

Proof. From the assumptions it follows that $A = PJP^{-1}$. Now from the first statement of Proposition 2.1, we obtain

$$P_I J_I P_I^{-1} = (PJP^{-1})_I = A_I.$$

Finally, using $\text{Tr}(A^*B) = \text{vec}(A)^*\text{vec}(B)$ and elementary properties of the Kronecker product from [6], we get

$$\|X\|_P = \|P^{-1}X\|_F = \|(P^{-1} \otimes I)\text{vec}(X)\|_2 = \|(P \otimes I)^{-1}\text{vec}(X)\|_2$$
$$= \|P_I^{-1}\text{vec}(X)\|_2 = \|\text{vec}(X)\|_{P_I},$$

which completes the proof. $\square$

Proposition 2.3. *Let $A \in \mathbb{C}^{n\times n}$, P be a matrix whose columns are a Jordan basis of the matrix A and $k = \text{ind}(A)$. Then*

$$\widetilde{\mathcal{R}}(A^k) \oplus \widetilde{\mathcal{N}}(A^k) = \mathbb{C}^{n\times n}.$$

Moreover, the spaces are orthogonal with respect to the P-norm.

Proof. Let $X \in \mathbb{C}^{n\times n}$ be an arbitrary matrix. Then $A^k X \in \widetilde{\mathcal{R}}(A^k) = \widetilde{\mathcal{R}}(A^{2k})$. Then there exists a matrix $V \in \mathbb{C}^{n\times n}$ such that $A^{2k}V = A^k X$. Let $U = A^k V \in \widetilde{\mathcal{R}}(A^k)$. Then $A^k U = A^k X$, i.e., $X - U \in \widetilde{\mathcal{N}}(A^k)$. From $X = U + X - U$, $A^k(X - U) = 0$, and $U \in \widetilde{\mathcal{R}}(A^k)$, it follows $\widetilde{\mathcal{R}}(A^k) + \widetilde{\mathcal{N}}(A^k) = \mathbb{C}^{n\times n}$.

Now let us suppose that $X \in \widetilde{\mathcal{R}}(A^k) \cap \widetilde{\mathcal{N}}(A^k)$. Then $X = A^k Y$ for some $Y \in \mathbb{C}^{n\times n}$ and $A^k X = 0$. From here, we obtain $A^{2k}Y = A^k X = 0$. Consequently, $Y \in \widetilde{\mathcal{N}}(A^{2k}) = \widetilde{\mathcal{N}}(A^k)$, from which follows $X = A^k Y = 0$. Therefore, we obtain $\widetilde{\mathcal{R}}(A^k) \oplus \widetilde{\mathcal{N}}(A^k) = \mathbb{C}^{n\times n}$.

Let $X \in \widetilde{\mathcal{R}}(A^k)$ and $Y \in \widetilde{\mathcal{N}}(A^k)$. Then

$$X = A^k Z, \text{ for some } Z \in \mathbb{C}^{n\times n} \text{ and } A^k Y = 0.$$

Using the Kronecker product, the previous equalities can be converted to

$$x = A_I^k z \text{ and } A_I^k y = 0,$$

where $x = \text{vec}(X), y = \text{vec}(Y), z = \text{vec}(Z)$. Consequently $x \in \mathcal{R}(A_I^k)$ and $y \in \mathcal{N}(A_I^k)$. From the second and third property of Proposition 2.1 follows the orthogonality of the spaces $\mathcal{R}(A_I^k)$ and $\mathcal{N}(A_I^k)$. Thus, it follows that

$$\|x + y\|_{P_I}^2 = \|x\|_{P_I}^2 + \|y\|_{P_I}^2.$$

Using Proposition 2.2 we obtain

$$\|X + Y\|_P^2 = \|x + y\|_{P_I}^2 = \|x\|_{P_I}^2 + \|y\|_{P_I}^2 = \|X\|_P^2 + \|Y\|_P^2.$$

which completes the proof. $\square$

3. Drazin-inverse solution of a general matrix equation

As it is well known, the Drazin inverse always exists for a square matrix, although it provides a solution of the matrix equation (1.4) only in the case when $\mathcal{R}(G) \subset \mathcal{R}(A^{k_1})$, $\mathcal{N}(B^{k_2}) \subset \mathcal{N}(G)$ and the restrictions (1.2) are imposed. In this section, our purpose is to develop a methodology for finding the matrix of the form $A^D G B^D$, for arbitrary square matrices A, B and G of appropriate dimensions. Observing the particular cases $B = I, G = I$ and $A = I, G = I$, as a consequence, we obtain two new representations of the Drazin inverse of an arbitrary square matrix. In order to achieve this goal, we introduce the *modified Drazin normal matrix equation* of the form

$$A^{2k_1}XB^{2k_2} = A^{k_1}GB^{k_2}, \quad A, B, G \in \mathbb{C}^{n\times n}, \; k_1 = \text{ind}(A), k_2 = \text{ind}(B). \tag{3.1}$$

Lemma 3.1. *Let $A, B, G \in \mathbb{C}^{n \times n}$. The set of all solutions of the equation (3.1) is given by*

$$X = (A^{k_1})^D G (B^{k_2})^D + Y - A^D A Y B B^D, \quad Y \in \mathbb{C}^{n \times n}. \tag{3.2}$$

Proof. First we show that (3.2) is a solution of the system

$$
\begin{aligned}
A^{2k_1} X B^{2k_2} &= A^{2k_1} (A^{k_1})^D G (B^{k_2})^D B^{2k_2} + A^{2k_1} Y B^{2k_2} - A^{2k_1} A^D A Y B B^D B^{2k_2} \\
&= A^{k_1} A A^D G B^D B B^{k_2} + A^{2k_1} Y B^{2k_2} - A^{2k_1} Y B^{2k_2} \\
&= A^{k_1} G B^{k_2}.
\end{aligned}
$$

Moreover, let $Y \in \mathbb{C}^{n \times n}$ be arbitrary solution of (3.1), i.e., let $A^{2k_1} Y B^{2k_2} = A^{k_1} G B^{k_2}$. We can write

$$Y = (A^{k_1})^D G (B^{k_2})^D + Y - (A^{k_1})^D G (B^{k_2})^D.$$

Since,

$$
\begin{aligned}
(A^{k_1})^D G (B^{k_2})^D &= ((A^{k_1})^D)^2 A^{k_1} G B^{k_2} ((B^{k_2})^D)^2 \\
&= (A^{2k_1})^D A^{2k_1} Y B^{2k_2} (B^{2k_2})^D = A^D A Y B B^D,
\end{aligned}
$$

we complete the proof. $\square$

The following theorem gives the initial idea for finding the Drazin-inverse solution of the matrix equation (1.4), by using the least–squares properties of the Drazin-inverse solution.

Theorem 3.1. *Assume that $A, B, G \in \mathbb{C}^{n \times n}$ and $k_1 = \mathrm{ind}(A)$, $k_2 = \mathrm{ind}(B)$. Let P be a matrix whose columns are a Jordan basis of the matrix A and Q be a matrix whose columns are Jordan basis of B. If the matrix $\hat{X}$ is a PQ-norm least-squares solution of the matrix equation*

$$A^{k_1} X B^{k_2} = G,$$

i.e., it satisfies

$$\| G - A^{k_1} \hat{X} B^{k_2} \|_{P,Q}^2 = \min_X \| G - A^{k_1} X B^{k_2} \|_{P,Q}^2 \tag{3.3}$$

then $\hat{X}$ is a solution of the equation (3.1). Moreover, if $\mathcal{R}(G|_{\mathcal{N}(B^{k_2})}) \subset \mathcal{N}(A^{k_1})$ the converse statement is also valid.

Proof. Let

$$P^{-1} A^{k_1} P = \begin{bmatrix} C_A^{k_1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^{k_1}) \\ \mathcal{N}(A^{k_1}) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^{k_1}) \\ \mathcal{N}(A^{k_1}) \end{bmatrix}; \quad Q^{-1} B^{k_2} Q = \begin{bmatrix} C_B^{k_2} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^{k_2}) \\ \mathcal{N}(B^{k_2}) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(B^{k_2}) \\ \mathcal{N}(B^{k_2}) \end{bmatrix},$$

be Jordan matrices of A^{k_1} and B^{k_2}, respectively. Let

$$P^{-1} G Q = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^{k_2}) \\ \mathcal{N}(B^{k_2}) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^{k_1}) \\ \mathcal{N}(A^{k_1}) \end{bmatrix}$$

and

$$P^{-1} X Q = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^{k_2}) \\ \mathcal{N}(B^{k_2}) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^{k_1}) \\ \mathcal{N}(A^{k_1}) \end{bmatrix}.$$

Since

$$
\begin{aligned}
P^{-1} P_{\mathcal{R}(A^{k_1}), \mathcal{N}(A^{k_1})} G P_{\mathcal{R}(B^{k_2}), \mathcal{N}(B^{k_2})} Q &= P^{-1} A^D A G B B^D Q = P^{-1} A^D A P P^{-1} G Q Q^{-1} B B^D Q \\
&= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^{k_2}) \\ \mathcal{N}(B^{k_2}) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^{k_1}) \\ \mathcal{N}(A^{k_1}) \end{bmatrix},
\end{aligned}
$$

we have that,

$$P^{-1}(G - A^{k_1}XB^{k_2})Q = \begin{bmatrix} G_1 & G_2 \\ 0 & 0 \end{bmatrix} - P^{-1}A^{k_1}XB^{k_2}Q + \begin{bmatrix} 0 & 0 \\ G_3 & G_4 \end{bmatrix}$$

$$= P^{-1}A^DAGBB^DQ - P^{-1}A^{k_1}XB^{k_2}Q + \begin{bmatrix} 0 & G_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ G_3 & G_4 \end{bmatrix}. \tag{3.4}$$

Since $A^{k_1}P\begin{bmatrix} 0 & 0 \\ G_3 & G_4 \end{bmatrix} = P\begin{bmatrix} C_A^{k_1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ G_3 & G_4 \end{bmatrix} = 0$, it follows that

$$\widetilde{\mathcal{R}}\left(P\begin{bmatrix} 0 & 0 \\ G_3 & G_4 \end{bmatrix}\right) \subset \widetilde{\mathcal{N}}(A^{k_1}).$$

Also, we have $A^{k_1}P\begin{bmatrix} C_A^{-k_1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & G_2 \\ 0 & 0 \end{bmatrix} = P\begin{bmatrix} 0 & G_2 \\ 0 & 0 \end{bmatrix}$, which implies $\widetilde{\mathcal{R}}\left(P\begin{bmatrix} 0 & G_2 \\ 0 & 0 \end{bmatrix}\right) \subset \widetilde{\mathcal{R}}(A^{k_1})$ and consequently

$$\widetilde{\mathcal{R}}\left(A^DAGBB^DQ - A^{k_1}XB^{k_2}Q + P\begin{bmatrix} 0 & G_2 \\ 0 & 0 \end{bmatrix}\right) \subset \widetilde{\mathcal{R}}(A^{k_1}).$$

Following the results from Proposition 2.3 and by using (3.4) we get

$$\|G - A^{k_1}XB^{k_2}\|_{P,Q}^2 = \left\| P^{-1}A^DAGBB^DQ - P^{-1}A^{k_1}XB^{k_2}Q + P^{-1}P\begin{bmatrix} 0 & G_2 \\ 0 & 0 \end{bmatrix} + P^{-1}P\begin{bmatrix} 0 & 0 \\ G_3 & G_4 \end{bmatrix} \right\|_F^2,$$

$$= \left\| A^DAGBB^DQ - A^{k_1}XB^{k_2}Q + P\begin{bmatrix} 0 & G_2 \\ 0 & 0 \end{bmatrix} + P\begin{bmatrix} 0 & 0 \\ G_3 & G_4 \end{bmatrix} \right\|_P^2,$$

$$= \left\| A^DAGBB^DQ - A^{k_1}XB^{k_2}Q + P\begin{bmatrix} 0 & G_2 \\ 0 & 0 \end{bmatrix} \right\|_P^2 + \left\| P\begin{bmatrix} 0 & 0 \\ G_3 & G_4 \end{bmatrix} \right\|_P^2.$$

Or equivalently,

$$\|G - A^{k_1}XB^{k_2}\|_{P,Q}^2 = \left\| A^DAGBB^D - A^{k_1}XB^{k_2} + P\begin{bmatrix} 0 & G_2 \\ 0 & 0 \end{bmatrix}Q^{-1} \right\|_{P,Q}^2 + \left\| P\begin{bmatrix} 0 & 0 \\ G_3 & G_4 \end{bmatrix}Q^{-1} \right\|_{P,Q}^2.$$

Evidently $\|G - A^{k_1}XB^{k_2}\|_{P,Q}^2$ attains minimal value in the case

$$A^{k_1}XB^{k_2} = A^DAGBB^D + P\begin{bmatrix} 0 & G_2 \\ 0 & 0 \end{bmatrix}Q^{-1}. \tag{3.5}$$

Therefore, we prove that each solution of the problem (3.3) is also a solution of the equation (3.5).

In what follows we show that from equation (3.5) follows the equation (3.1). If X satisfies (3.5), evidently, multiplying the equation (3.5) by A^{k_1} from the left and by B^{k_2} on the right, on the both hand sides, we obtain

$$A^{2k_1}XB^{2k_2} = A^{k_1}A^DAGBB^DB^{k_2} + P\begin{bmatrix} C_A^{k_1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1}P\begin{bmatrix} 0 & G_2 \\ 0 & 0 \end{bmatrix}Q^{-1}Q\begin{bmatrix} C_B^{k_2} & 0 \\ 0 & 0 \end{bmatrix}Q^{-1} = A^{k_1}GB^{k_2},$$

which proves that it satisfies (3.1).

Let us suppose that $\mathcal{R}(G|_{\mathcal{N}(B^{k_2})}) \subset \mathcal{N}(A^{k_1})$. Then, since $G_2 : \mathcal{N}(B^{k_2}) \to \mathcal{R}(A^{k_1})$, it follows that $G_2 = 0$, which implies that the equality (3.5) becomes

$$A^{k_1}XB^{k_2} = A^DAGBB^D. \tag{3.6}$$

Let X satisfies (3.1), i.e., $A^{2k_1}XB^{2k_2} = A^{k_1}GB^{k_2}$. By multiplying with $(A^{k_1})^D$ from the left and with $(B^{k_2})^D$ on the right and using the facts $(A^{k_1})^DA^{k_1} = A^DA$, $(B^{k_2})^DB^{k_2} = B^DB$, $A^DA^{k_1+1} = A^{k_1}$ and $B^DB^{k_2+1} = B^{k_2}$, we obtain (3.6). □

Corollary 3.1. *Let $A, B, G \in \mathbb{C}^{n \times n}$, where $k_1 = \mathrm{ind}(A)$, $k_2 = \mathrm{ind}(B)$. Let P be a matrix whose columns are a Jordan basis of the matrix A and Q be a matrix whose columns are a Jordan basis of matrix B. If $\hat{X}$ satisfies (3.3) then*

$$A^{k_1-1}\hat{X}B^{k_2-1} = A^D G B^D + A^{k_1-1}Y B^{k_2-1} - A^D A^{k_1} Y B^{k_2} B^D, \quad Y \in \mathbb{C}^{n \times n}.$$

Moreover, if

$$\mathcal{R}(\hat{X}) \subset \mathcal{R}(A^{k_1}), \quad \mathcal{N}(B^{k_2}) \subset \mathcal{N}(\hat{X})$$

then

$$A^{k_1-1}\hat{X}B^{k_2-1} = A^D G B^D.$$

Proof. If $\hat{X}$ satisfies (3.3), then it is a solution of the equation (3.1). Therefore, according to Lemma 3.1, it is of the form

$$\hat{X} = (A^{k_1})^D G (B^{k_2})^D + Y - A^D A Y B B^D, \quad Y \in \mathbb{C}^{n \times n}.$$

Then

$$
\begin{aligned}
A^{k_1-1}\hat{X}B^{k_2-1} &= A^{k_1-1}(A^{k_1})^D G (B^{k_2})^D B^{k_2-1} + A^{k_1-1}Y B^{k_2-1} - A^{k_1-1}A^D A Y B B^D B^{k_2-1} \\
&= A^D G B^D + A^{k_1-1}Y B^{k_2-1} - A^D A^{k_1} Y B^{k_2} B^D
\end{aligned}
\tag{3.7}
$$

Additionally, let $\mathcal{R}(\hat{X}) \subset \mathcal{R}(A^{k_1})$ and $\mathcal{N}(B^{k_2}) \subset \mathcal{N}(\hat{X})$. Then we have

$$
\begin{aligned}
&\mathcal{R}(A^{k_1-1}\hat{X}B^{k_2-1}) \subset \mathcal{R}(A^{k_1}), \\
&\mathcal{N}(B^{k_2}) \subset \mathcal{N}(A^{k_1-1}\hat{X}B^D B B^{k_2-1}) = \mathcal{N}(A^{k_1-1}\hat{X}B^{k_2-1}),
\end{aligned}
\tag{3.8}
$$

respectively.

Following (3.7) and (3.8) it can be easily deduced that

$$
\begin{aligned}
&\mathcal{R}(A^{k_1-1}Y B^{k_2-1}) \subset \mathcal{R}(A^{k_1}), \\
&\mathcal{N}(B^{k_2}) \subset \mathcal{N}(A^{k_1-1}Y B^{k_2-1})
\end{aligned}
$$

hold. Applying the well known result from [1] (page 62, Ex. 20), we get

$$P_{\mathcal{R}(A^{k_1}),\mathcal{N}(A^{k_1})} A^{k_1-1}Y B^{k_2-1} = A^{k_1-1}Y B^{k_2-1}$$

and

$$A^{k_1-1}Y B^{k_2-1} P_{\mathcal{R}(B^{k_2}),\mathcal{N}(B^{k_2})} = A^{k_1-1}Y B^{k_2-1}.$$

Finally we obtain,

$$
\begin{aligned}
A^D A^{k_1} Y B^{k_2} B^D &= A A^D A^{k_1-1}Y B^{k_2-1} B B^D = P_{\mathcal{R}(A^{k_1}),\mathcal{N}(A^{k_1})} A^{k_1-1}Y B^{k_2-1} P_{\mathcal{R}(B^{k_2}),\mathcal{N}(B^{k_2})} \\
&= A^{k_1-1}Y B^{k_2-1}
\end{aligned}
$$

and thus, $A^{k_1-1}\hat{X}B^{k_2-1} = A^D G B^D$, which completes the proof. $\square$

Considering these results, we are moving to the problem of computing the matrix $A^D G B^D$ in terms of expressing the matrix as a least-squares solution of a given problem. For that purpose let us focus on the minimization problem

$$\min_X f(X) = \min_X \|A^{2k_1} X B^{2k_2} - G\|_{P,Q}^2, \tag{3.9}$$

in order to find its solution. The characterization of the Drazin-inverse solution of the matrix equation (1.4), given in terms of the solution of the problem (3.9), is stated in the following theorem.

Theorem 3.2. *Let $A, B, G \in \mathbb{C}^{n \times n}$, where $k_1 = \text{ind}(A), k_2 = \text{ind}(B)$. Let P be a matrix whose columns are a Jordan basis of the matrix A and Q be a matrix whose columns are a Jordan basis of the matrix B. If $\hat{X}$ is a minimizer of the functional*

$$f(X) = \|A^{2k_1} X B^{2k_2} - G\|_{P,Q}^2, \tag{3.10}$$

then the following holds

$$A^{2k_1-1} \hat{X} B^{2k_2-1} = A^D G B^D. \tag{3.11}$$

Proof. Let the matrix $\hat{X}$ be a minimizer of the functional $f(X)$, then it is a PQ-norm least-squares solution of the system $A^{2k_1} X B^{2k_2} = G$. Thus $\hat{Y} = A^{k_1} \hat{X} B^{k_2}$ is the solution of the minimization problem

$$\min_Y \|A^{k_1} Y B^{k_2} - G\|_{P,Q}^2$$

which satisfies the conditions $\mathcal{R}(\hat{Y}) \subset \mathcal{R}(A^{k_1})$ and $\mathcal{N}(B^{k_2}) \subset \mathcal{N}(\hat{Y})$. According to Corollary 3.1, it follows that

$$A^D G B^D = A^{k_1-1} \hat{Y} B^{k_2-1} = A^{k_1-1} A^{k_1} \hat{X} B^{k_2} B^{k_2-1} = A^{2k_1-1} \hat{X} B^{2k_2-1},$$

which completes the proof. $\square$

Therefore, in order to find the Drazin inverse solution of the system $AXB = G$, stated by (1.4), we are moving to the problem of computing the PQ-norm least-square solution of the matrix equation

$$A^{2k_1} X B^{2k_2} = G. \tag{3.12}$$

In the following corollary we present some particular cases of the initial problem given by (1.4) and thus the same particular cases of the matrix equation (3.12) as a direct consequence of Theorem 3.2. By I we denote the identity matrix of an appropriate order.

Corollary 3.2. *Let us consider $A, B \in \mathbb{C}^{n \times n}$ whose indices are $k_1 = \text{ind}(A)$, $k_2 = \text{ind}(B)$. Let P be a matrix whose columns are a Jordan basis of the matrix A and Q be a matrix whose columns are a Jordan basis of B.*

a) *If $\hat{X} \in \mathbb{C}^{n \times n}$ is a minimizer of the functional*

$$f(X) = \|A^{2k_1} X - I\|_P^2 = \|P^{-1} A^{2k_1} X - P^{-1}\|_F^2,$$

then the Drazin inverse of A possesses the representation

$$A^D = A^{2k_1-1} \hat{X}. \tag{3.13}$$

b) *If $\hat{X} \in \mathbb{C}^{n \times n}$ is a minimizer of the functional*

$$f(X) = \|X B^{2k_2} - I\|_{I,Q}^2,$$

then the Drazin inverse of B can be represented as

$$B^D = \hat{X} B^{2k_2-1}. \tag{3.14}$$

c) *If $\hat{X}$ be the minimizer of the functional*

$$f(X) = \|A^{2k_1} X B^{2k_2} - I\|_{P,Q}^2.$$

Then

$$A^D B^D = A^{2k_1-1} \hat{X} B^{2k_2-1}. \tag{3.15}$$

Proposition 3.1. [12] *Let $S \in \mathbb{C}^{m\times n}$, $T \in \mathbb{C}^{k\times s}$ and $Q \in \mathbb{C}^{m\times s}$. Then the matrix $S^\dagger Q T^\dagger$ is the best–approximate solution of the matrix equation $SXT = Q$, i.e. it is a minimizer of the function*

$$g(X) = \|SXT - Q\|_F^2 \tag{3.16}$$

of the minimal norm.

The following result provides two new representations of the Drazin inverse of an arbitrary square matrix as well as the representation of the Drazin inverse solution of the initial problem (1.4).

Theorem 3.3. *Let $A, B, G \in \mathbb{C}^{n\times n}$, where $k_1 = \text{ind}(A)$, $k_2 = \text{ind}(B)$. If P is a matrix whose columns are a Jordan basis of A and Q is a matrix whose columns are a Jordan basis of B then the following representations are valid:*

a) $A^D = A^{2k_1-1}(P^{-1}A^{2k_1})^\dagger P^{-1}$,

b) $B^D = Q(B^{2k_2}Q)^\dagger B^{2k_2-1}$,

c) $A^D G B^D = A^{2k_1-1}(P^{-1}A^{2k_1})^\dagger P^{-1} G Q(B^{2k_2}Q)^\dagger B^{2k_2-1}$.

Proof. **a)** The minimum-norm solution of the functional

$$f(X) = \|A^{2k_1}X - I\|_P^2 = \|P^{-1}A^{2k_1}X - P^{-1}\|_F^2,$$

is the matrix

$$\hat{X} = (P^{-1}A^{2k_1})^\dagger P^{-1}.$$

The representation of the A^D follows from Corollary 3.2, part a).

b) The minimum-norm least-squares solution of the function

$$f(X) = \|XB^{2k_2} - I\|_{I,Q}^2 = \|XB^{2k_2}Q - Q\|_F^2$$

is the matrix

$$\hat{X} = Q(B^{2k_2}Q)^\dagger.$$

The rest is obvious from Corollary 3.2, part b).

c) This part of the proof is implied by Theorem 3.2 and Proposition 3.1. □

From the results exposed previously, it is obvious that the intentions and methodology for finding the general solution of a system of matrix equation lead us to a new representation of the Drazin inverse of a given matrix. A natural question arises: Whether the representations given with Theorem 3.3 are valid in more general case, or whether the Drazin inverse can be represented, for example, in the form $A^D = A^{l-1}(P^{-1}A^l)^\dagger P^{-1}$, where l belongs to some set of numbers. Although the presented theory cannot result with the answer to this question, in the sequel, we provide an alternative proof of Theorem 3.3 which is stated in more general form.

Theorem 3.4. *Let $A, G \in \mathbb{C}^{n\times n}$ and $l \geq k + 1$, where $k = \text{ind}(A)$. If P is a matrix whose columns are a Jordan basis of A then the following representations are valid:*

a) $A^D = A^{l-1}(P^{-1}A^l)^\dagger P^{-1}$,

b) $A^D = P(A^l P)^\dagger A^{l-1}$,

Proof. **a)** Let

$$P^{-1} = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^k) \\ N(A^k) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^k) \\ N(A^k) \end{bmatrix}$$

then we have

$$\begin{bmatrix} C_A^l & 0 \\ 0 & 0 \end{bmatrix} P^{-1} = \begin{bmatrix} C_A^l P_1 & C_A^l P_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^k) \\ \mathcal{N}(A^k) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^k) \\ \mathcal{N}(A^k) \end{bmatrix}. \tag{3.17}$$

Since $k = \mathrm{ind}(A)$ and $l \geq k + 1$ we know that A^l can be represented on the following way

$$A^l = P \begin{bmatrix} C_A^l & 0 \\ 0 & 0 \end{bmatrix} P^{-1}.$$

From here it follows that

$$\left(\begin{bmatrix} C_A^l & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \right) |_{\mathcal{N}(A^k)} = 0, \tag{3.18}$$

since if we suppose that $x \in \mathcal{N}(A^l) = \mathcal{N}(A^k)$, $x \neq 0$ we obtain the following equivalences

$$A^l x = 0 \iff P \begin{bmatrix} C_A^l & 0 \\ 0 & 0 \end{bmatrix} P^{-1} x = 0 \iff \begin{bmatrix} C_A^l & 0 \\ 0 & 0 \end{bmatrix} P^{-1} x = 0.$$

From (3.17) and (3.18) it follows that $C_A^l P_2 = 0$. Finally, the last equality with simple calculations reveals that

$$\left(\begin{bmatrix} C_A^l & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \right)^\dagger = \begin{bmatrix} P_1^{-1} C_A^{-l} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^k) \\ \mathcal{N}(A^k) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^k) \\ \mathcal{N}(A^k) \end{bmatrix}.$$

If we multiply the last equality with the matrix $P \begin{bmatrix} C_A^{l-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$ from the left-hand side and with the matrix P^{-1} from the right-hand side, we obtain

$$P \begin{bmatrix} C_A^{l-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \left(\begin{bmatrix} C_A^l & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \right)^\dagger P^{-1} = P \begin{bmatrix} C_A^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$$

$$\iff A^{l-1} (P^{-1} A^l)^\dagger P^{-1} = A^D,$$

completing the proof.

b) Similarly. $\square$

4. Examples

In this section we report some results as an illustration for the theory obtained in the previous section. The test matrices which are presented in the following examples are taken from the papers [5, 17]. MATLAB programming package is used for the matrix computation. In order ro provide some auxiliary result which we need for the computation of the Drazin inverse we use MATLAB built-in function *pinv*.

Example 4.1. *In this example we consider the given real matrix $A \in \mathbb{R}^{4 \times 4}$*

$$A = \begin{bmatrix} 2 & 4 & 6 & 5 \\ 1 & 4 & 5 & 4 \\ 0 & -1 & -1 & 0 \\ -1 & -2 & -3 & -3 \end{bmatrix},$$

with index $k = 2$, in order to compute the Drazin inverse A^D. According to the formula given by Theorem 3.3 a) we obtain

$$A^D = A^{2k-1}(P^{-1}A^{2k})^\dagger P^{-1}$$

$$= \begin{bmatrix} 3 & 11 & 14 & 14 \\ 2 & 9 & 11 & 11 \\ -1 & -4 & -5 & -5 \\ -1 & -4 & -5 & -5 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & -3/5 & 17/5 \\ 0 & 0 & 2/5 & -13/5 \\ 0 & 0 & -1/5 & 4/5 \\ 0 & 0 & -1/5 & 4/5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 3 & 2 \\ 1 & 1 & 2 & 3 \\ -1 & -1 & -2 & -2 \\ 0 & -1 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & 2 & 2 \\ 2 & 1 & 3 & 3 \\ -1 & 0 & -1 & -1 \\ -1 & 0 & -1 & -1 \end{bmatrix},$$

which is the exact value of A^D.

Theorem 3.4 a) *produces the same result for all values $l \geq k + 1$. Particularly, in the case $l = k + 1$ we obtain*

$$A^D = A^k(P^{-1}A^{k+1})^\dagger P^{-1}$$

$$= \begin{bmatrix} 3 & 8 & 11 & 11 \\ 2 & 7 & 9 & 9 \\ -1 & -3 & -4 & -4 \\ -1 & -3 & -4 & -4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & -3/5 & 14/5 \\ 0 & 0 & 2/5 & -11/5 \\ 0 & 0 & -1/5 & 3/5 \\ 0 & 0 & -1/5 & 3/5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 3 & 2 \\ 1 & 1 & 2 & 3 \\ -1 & -1 & -2 & -2 \\ 0 & -1 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & 2 & 2 \\ 2 & 1 & 3 & 3 \\ -1 & 0 & -1 & -1 \\ -1 & 0 & -1 & -1 \end{bmatrix}.$$

Example 4.2. *Let us compute the Drazin inverse of the given matrix*

$$B = \begin{bmatrix} 1 & -1 & 2 & 2 & 2 \\ 0 & 0 & -2 & -2 & -2 \\ 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & -1 & 1 \end{bmatrix}$$

with index $k = 2$ in order to confirm the formula given by Theorem 3.3 b). Thus we have

$$B^D = Q(B^{2k}Q)^\dagger B^{2k-1}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & -4 & -1 \\ 4 & -8 & 0 & 0 & 2 \\ -8 & -2 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 8 & 0 & 4 & 0 & 0 \end{bmatrix} \cdot \frac{1}{10}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 40 & 176 & -5 & -88 & 5 \\ -10 & -68 & 0 & 34 & 0 \\ 0 & 16 & 0 & -8 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 4 & 10 & 4 \\ 0 & 0 & 0 & -2 & 0 \\ 3 & -1 & 4 & 7 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ -3 & 1 & -4 & -7 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 4 & -6 & 4 \\ 0 & 0 & 0 & -2 & 0 \\ -1 & 3 & -12 & 11 & -13 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & -3 & 12 & -11 & 13 \end{bmatrix}$$

which is the exact Drazin inverse B^D.

Example 4.3. *In this test example we compute the Drazin inverse solution for the matrix equation*

$$AXB = G,$$

where the matrices are given as follows

$$A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 1/3 & -1 \\ -1 & -1 & -1 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 & -24 & 16 \\ -8 & 8 & -8 \\ 0 & -16 & 16 \end{bmatrix},$$

$k_1 = \mathrm{ind}(A) = 3$, $k_2 = \mathrm{ind}(B) = 1$. *Based on the formulae presented by Theorem* 3.3 c)

$$A^D G B^D = A^{2k_1-1}(P^{-1}A^{2k_1})^\dagger P^{-1} G Q (B^{2k_2}Q)^\dagger B^{2k_2-1}$$

$$= \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} \cdot \begin{bmatrix} 0 & -24 & 16 \\ -8 & 8 & -8 \\ 0 & -16 & 16 \end{bmatrix} \cdot \frac{1}{16} \begin{bmatrix} -1 & -6 & -1 \\ -6 & 12 & -6 \\ -1 & -6 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 6 & -3 \\ 9 & -18 & 9 \\ -6 & 12 & -6 \end{bmatrix},$$

which presents the exact Drazin–inverse solution of the matrix equation AXB = G.

5. Conclusion

In the present article, we show that the Drazin inverse of a given matrix, and more generally the Drazin inverse solution $A^D G B^D$ of the matrix equation $AXB = G$ possesses similar properties. Namely, we establish the minimal properties of the Drazin-inverse solution with respect to PQ-norm, which are similar to those of the Moore-Penrose inverse with respect to the Frobenius norm.

On that way, the technique presented in the article also leads us to new representation of the Drazin inverse expressible via the Moore-Penrose inverse and a Jordan basis of the matrix A. Later, we generalize the obtained result to get a whole set of representations of the Drazin inverse of a matrix.

The advantages of the presented ideas can be summarized as follows:

- obtained results may help us to further understand the matrix equation $AXB = D$ for arbitrary square matrices $A, B, G \in \mathbb{C}^{n \times n}$;
- the least-square properties of the Drazin inverse, and more generally the Drazin inverse solution of the matrix equation $AXB = D$, are established;
- a new relationship between the Drazin inverse and the Moore-Penrose inverse is introduced, which enables more closely to analyze the similar minimization properties of both inverses.

References

[1] A. Ben-Israel and T.N.E. Greville, Generalized Inverses: Theory and Applications, second ed., Springer, 2003.
[2] S.L. Campbell and C.D. Meyer Jr., *Generalized Inverse of Linear Transformation*, Pitman, London (1979), SIAM, 2009.
[3] Y. Chen, *A Cramer rule for solution of the general restricted linear equation*, Linear Multilinear Algebra **34** (1993), 177–186.
[4] Y. Chen, *Representations and Cramer rules for the solution of a restricted matrix equation*, Linear Multilinear Algebra **35** (1993), 339–354.
[5] L. Guo, X. Du, *Representations for the Drazin inverses of 2 × 2 block matrices*, Appl. Math. Comput. **217** (2010), 2833–2842.
[6] A.J. Laub, *Matrix Analysis for Scientists and Engineers*, SIAM, Philadelphia, (2005).
[7] J. Levine, R.E. Hartwig, *Applications of Drazin inverse to the Hill cryptographic systems*, Cryptologia **4** (1980), 71–85.
[8] S. Miljković, M. Miladinović, P.S. Stanimirović, Y. Wei, *Gradient methods for computing the Drazin-inverse solution*, J. Comput. Appl. Math. **253** (2013), 255–263.
[9] H. Neudecker, *A Note on Kronecker Matrix Products and Matrix Equation Systems*, SIAM J. Appl. Math. **17** (1969), 603–606.
[10] M. Nikuiea, M.K. Mirnia, Y. Mahmoudi, *Some results about the index of matrix and Drazin inverse*, Mathematical Sciences **4** (2010), 283–294.
[11] R. Penrose, *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc. **51** (1955), 406–413
[12] G. Wang, Y. Wei, S. Qiao, *Generalized Inverses: Theory and Computations*, Graduate Series in Mathematics 5, Science Press, Beijing/New York, April 2004. ISBN 7-03-012437-5.
[13] G. Wang, *A Cramer rule for finding the solution of a class of singular equations*, Linear Algebra Appl. **116** (1989), 27–34.
[14] G. Wang, Z. Xu, *Solving a kind of restricted matrix equations and Cramer rule*, Appl. Math. Comput. **162** (2005), 329–338.

[15] Y. Wei, *Index splitting for the Drazin inverse and the singular linear system*, Appl. Math. Comput. **95** (1998), 115–124.
[16] Y. Wei, H. Wu, *Convergence properties of Krylov subspace methods for singular linear systems with arbitrary index*, J. Comput. Appl. Math. **114** (2000), 305–318.
[17] Y. Wei, H. Wu, *The representation and approximation for Drazin inverse*, J. Comput. Appl. Math. **126** (2000), 417–432.
[18] Y. Wei, H. Wu, *Additional results on index splitting for Drazin inverse of singular linear system*, Electron. J. Linear Algebra **8** (2001), 83–93.
[19] N. Zhang, Y. Wei, *On the convergence of general stationary iterative methods for range-Hermitian singular linear systems*, Numer. Linear Algebra Appl. **17** (2010), 139–154.
[20] J. Zhou, Y. Wei, *A two-step algorithm for solving singular linear systems with index one*, Appl. Math. Comput. **8** (2001), 83–93.