Integer index in trees of diameter 4

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Abstract. We characterize when a tree of diameter 4 has integer index and we provide examples of infinite families of non-integral trees with integer index. We also determine a tight upper bound for the index of any tree of diameter 4 based on its maximum degree. Moreover, we present a new infinite family of integral trees of diameter 4.

1. Introduction

Let $G = (V, E)$ be a simple graph on n vertices and $A(G)$, its adjacency matrix. A graph G is called integral when all eigenvalues of $A(G)$ are integer numbers. Since 1974, when Harary and Schwenk [11] posed the question *Which graphs have integral spectra?*, the search for integral graphs has been done (see [1]). In [8], the authors pointed out the relevance of the study of integral graphs in applications to computer science. A tree T is a connected acyclic graph. The search for integral trees, specially of small diameter, has an important emphasis in the state of the art of this research, as we can see in [2–4, 12–14, 17–21]. More recently, it has been proved that exists an integral tree of arbitrary diameter [5, 9].

We see however that in many applications, just the index (the largest eigenvalue of $A(G)$) needs to be an integer. In [6], the authors introduced a new parameter based on the index of the graph, the *tightness of second type*. They claim that graphs with small tightness of second type are suitable for representing multiprocessors. The index is a parameter also related to several other graph invariants, such as chromatic number, clique number, maximum degree and others [7]. Most of these invariants are rational numbers, and in many cases an integer. It is therefore important to investigate conditions when these indices are integers, even if the other eigenvalues are irrational. We note that the eigenvalues are all integers or irrational numbers, as they are roots of monic polynomials with integer coefficients.

In this work we deal with the class of trees of diameter 4. In the second section, we determine an upper bound for the index of a diameter 4 tree, and we characterize the tree that attains this bound. The third section is devoted to the study of spectral properties of balanced trees of diameter 4, investigating conditions for integrality of these trees. Also in this section, we study how to increase the index in a special subfamily of 4 diameter trees, perturbing some parameters. Finally, in the last section, we obtain a new infinite family of integral trees of diameter 4.

2. Diameter 4 trees: upper bound for the index.

A tree of diameter 4 can be obtained by joining a vertex v to the central vertices of r stars ($r \geq 2$), $K_{1,a_1}, K_{1,a_2}, \ldots, K_{1,a_r}$, and to other b isolated vertices ($b \geq 0$). Let $a_1 \geq a_2 \geq \ldots \geq a_r \geq 1$, this tree will be denoted by $S(b, r, a_i)$. The vertex v will be its central vertex. If $a_1 = a_2 = \ldots = a_r = a$, the tree will be denoted by $R(b, r, a)$, as we can see in Figure 1.
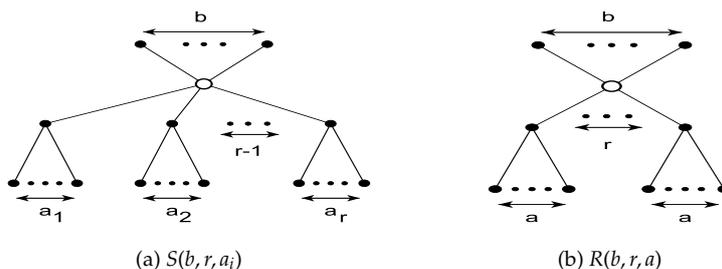

(a) $S(b, r, a_i)$ (b) $R(b, r, a)$

Figure 1: *Trees of diameter* 4

In this section we present a sharp upper bound for the index of a tree of diameter 4 in terms of its maximum degree Δ, determining the structure of the tree that reaches this quota. First, however, we present upper bounds for the index of $S(b, r, a_i)$, given as a function of Δ, found in the literature. The first one is the well known upper bound obtained by Godsil [10] in 1984: the index of a tree T, with $\Delta \geq 2$, satisfies

$$\lambda(T) < 2\sqrt{\Delta - 1}. \tag{1}$$

The proof of this result consists in estimating an upper bound for the index of a *Bethe tree*, whose definition, extracted from [16], is given below:

A (*rooted*) Bethe tree $\mathcal{B}_{\Delta,k}$, of maximum degree Δ and k levels, is obtained recursively. The tree $\mathcal{B}_{\Delta,1}$ consists of a single vertex, which is its root. For $k \geq 2$, the tree $\mathcal{B}_{\Delta,k}$ consists of a vertex u (that will be the root) adjacent to the roots of $\Delta - 1$ copies of trees $\mathcal{B}_{\Delta,k-1}$. Note that $\mathcal{B}_{\Delta,k}$ has diameter $2(k-1)$, its root has degree $\Delta - 1$, while the remaining non pendent vertices have degree Δ. Also, $\mathcal{B}_{\Delta,1} = \mathcal{B}_{1,k} = K_1$ and $\mathcal{B}_{\Delta,2} = K_{1,\Delta-1}$.

As each tree T of maximum degree Δ is a subgraph of a Bethe tree $\mathcal{B}_{\Delta,k}$, for some k, its index $\lambda(T)$ is bounded above by $\lambda(\mathcal{B}_{\Delta,k})$. In order to obtain a tight bound, we consider the smallest k possible. If one of the central vertices of T does not have maximum degree, choose the smallest k such that $2(k-1)$ is greater than or equal to the diameter of T. Otherwise, k is increased by one. Then, a tree of diameter 4 and maximum degree Δ is a *proper* subtree of $\mathcal{B}_{\Delta,4}$, as shown in Figure 2.
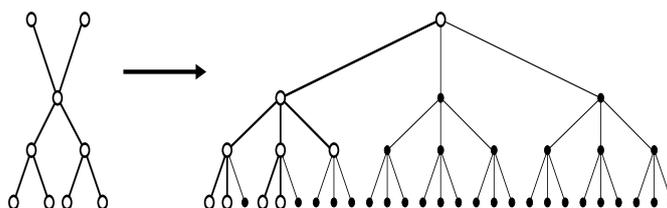

Figure 2: *Tree of diameter 4 is a proper subtree of* $\mathcal{B}_{\Delta,4}$

Robbiano and Rojo [15] calculated the index of a Bethe tree, namely:

$$\lambda(\mathcal{B}_{\Delta,k}) = 2\sqrt{\Delta - 1}\,\cos\left(\frac{\pi}{k+1}\right). \tag{2}$$

Thus, it follows that

$$\lambda(S(b, r, a_i)) < 2\sqrt{\Delta - 1}\cos\frac{\pi}{5}, \tag{3}$$

providing an upper bound for the index better than the previous one, given by Godsil. In Theorem 2.1 we improve this limit, obtaining an upper bound for the index of $S(b, r, a_i)$ as a function of Δ, and characterizing the extremal tree that attains this new quota, namely $R(0, \Delta, \Delta - 1)$.

Theorem 2.1. *Let $b \geq 0$, $r \geq 2$ and $a_1 \geq a_2 \geq \ldots \geq a_r \geq 1$ be integers. The tree $S(b, r, a_i)$, with maximum degree Δ, has index λ satisfying:*

$$\lambda \leq \sqrt{2\Delta - 1}\,; \tag{4}$$

equality occurs if and only if $b = 0$, $a_i = a$, for $1 \leq i \leq r$, and $\Delta = a + 1 = r$.

Proof. Considering $a = \max\limits_{1 \leq i \leq r} a_i$, we have that $S(b, r, a_i)$ is a connected subgraph of $R(b, r, a)$ and then its index λ is less than or equal to the index of $R(b, r, a)$; the equality occurring between the indices if and only if $S(b, r, a_i) = R(b, r, a)$. As $S(b, r, a_i)$ and $R(b, r, a)$ have the same maximum degree $\Delta = \max\{a + 1, b + r\}$, it is sufficient to prove the theorem for the tree $R(b, r, a)$.

If $v \in \mathbb{R}^n$ is *a Perron eigenvector* of $R(b, r, a)$, each of its coordinates v_i is a positive real number satisfying the equation

$$\lambda v_i = \sum_{j \sim i} v_j\,, \tag{5}$$

known as the *eigenvalue equation for the i-th vertex*. It follows from this that, if two vertices i e j have the same neighbors, the coordinates v_i and v_j are equal. Therefrom, we deduce that the coordinates of the pendent vertices that are adjacent to the center of each star $K_{1,a}$ are equal to each other. So, w.l.o.g., we assume that they are 1, obtaining the distribution of the coordinates of v shown in Figure 3.

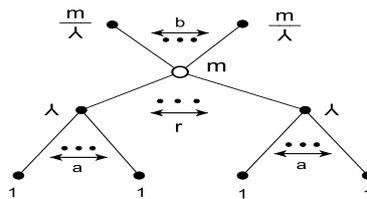Figure 3: *A Perron eigenvector of $R(b, r, a)$*

It follows that there exists a positive real number m satisfying the equations

$$\begin{cases} m = \lambda^2 - a \\ bm = \lambda^2(m - r) \end{cases}. \tag{6}$$

So, $\lambda^2(m - r) = b(\lambda^2 - a)$; hence, $\lambda^2(b + r - m) = ab$. Therefore, $m \leq b + r$ and we conclude:

$$\lambda^2 = a + m \leq (a + 1) + (b + r) - 1 \leq 2\Delta - 1\,. \tag{7}$$

Since $m = b + r$ if and only if $b = 0$, the equality $\lambda^2 = 2\Delta - 1$ occurs if and only if $b = 0$ and $a + 1 = r = \Delta$. ∎

From (7), it follows that the upper bound (4), given in the theorem above, can be improved if, besides Δ, we fix the parameter $\Delta' = \min\{a + 1, r + b\}$.

Corollary 2.1. *Let $2 \leq \Delta' \leq \Delta$ be integers. If $b \geq 0$, $r \geq 2$ and $a = a_1 \geq a_2 \geq \ldots \geq a_r \geq 1$ are integers such that $\max\{a + 1, r + b\} = \Delta$ and $\min\{a + 1, r + b\} = \Delta'$, then the index λ of $S(b, r, a_i)$ satisfies*

$$\lambda \leq \sqrt{\Delta + \Delta' - 1} \, ; \tag{8}$$

where equality occurs if and only if $b = 0$, $a_i = a$, for $1 \leq i \leq r$.

Note that the trees attaining both the quotas (4) and (8) are *balanced* trees $R(0, r, a)$. We devote the next section to the study of spectral properties of them.

3. Spectral properties of $R(0, r, a)$.

Watanabe and Schwenk, in [22], have determined the eigenvalues of a tree $R(0, r, a)$, with $r \geq 2$ and $a \geq 1$, proving that it is integral if and only if a and $a + r$ are perfect squares. In fact, from [1], we have that the positive eigenvalues of $R(b, r, a)$ are $\sqrt{a}$ and the positive roots of the equation

$$x^4 - (a + b + r)x^2 + ab = 0 \, . \tag{9}$$

So, if $b = 0$, its positive eigenvalues are $\sqrt{a}$ and $\sqrt{a + r}$, the latter being the index of the tree. Hence, the family of balanced trees of diameter 4 with integer index k is composed by the trees $R(0, k^2 - a, a)$, for integer $a \geq 1$ such that $k^2 \geq a + 2$.

Two distinct trees in this family, $R(0, k^2 - a, a)$ and $R(0, k^2 - c, c)$, have the same maximum degree Δ if and only if $c = k^2 - a - 1$. These trees also have the same number of vertices and may be integral or not. For example, if $k \geq 2$ and $a = 1$, we have that $R(0, k^2 - 1, 1)$ is integral, while $R(0, 2, k^2 - 2)$ is not. In fact, $R(0, 2, a)$ is never integral, since its eigenvalues, $\sqrt{a}$ and $\sqrt{a + 2}$, may not be both integers. Examples in which the two trees are integral are more difficult to find. In Proposition 3.1, we give conditions on a and k to ensure that this happens.

Proposition 3.1. *Let a and k be positive integers such that $k^2 - a \geq 2$ and $k^2 - 1 \neq 2a$. The non coespectral trees $R(0, k^2 - a, a)$ and $R(0, a + 1, k^2 - a - 1)$ are both integral if and only if k is odd and there are even positive integers x and y satisfying the equations $x^2 + y^2 = k^2 - 1$ and $a = x^2$.*

Proof. It is easy to verify that, if there are positive integers x and y such that $x^2 + y^2 = k^2 - 1$ and $a = x^2$, the trees $R(0, k^2 - a, a)$ and $R(0, a + 1, k^2 - a - 1)$ are integral, since $x = \sqrt{a}$ and $y = \sqrt{k^2 - a - 1}$ are integers. On the other hand, assuming that the trees $R(0, k^2 - a, a)$ and $R(0, a + 1, k^2 - a - 1)$ are integral, exist positive integers x and y such that $x^2 = a$ and $y^2 = k^2 - a - 1$; therefore, $x^2 + y^2 = k^2 - 1$. It remains to verify that, in this case, k is odd and x and y are even. Indeed, if k is even, $k^2 - 1 = x^2 + y^2$ is odd and, then, x and y have different parities. Supposing y odd, we have that $k^2 - x^2$ is a multiple of 4, but $y^2 + 1$ is not, resulting a contradiction. Then k is odd and we conclude, similarly, that x and y are even. ∎

From this proposition, we verify that we can easily obtain pairs of such trees where only one of them is integral. It is sufficient, for this, to consider k even or a odd. For example, if p is a positive integer, we see that, such for $k = 2p$ and $a = p^2$, as for $k = 2p + 3$ and $a = (2p + 1)^2$, the tree $R(0, k^2 - a, a)$ is integral, while $R(0, a + 1, k^2 - a - 1)$ is not. Although the case in which both trees are integral is more difficult to construct, we exhibit examples in the corollary below.

Corollary 3.1. *Given a positive integer p, then, for each pair of a and k below, the trees $R(0, k^2 - a, a)$ and $R(0, a + 1, k^2 - a - 1)$ are integrals:*

  (i) $a = 4p^2$ and $k = 2p^2 + 1$;

  (ii) $a = 4p^4$ and $k = 2p^2 + 1$.

From Theorem 2.1, we obtain that the trees $R(0, r, a)$, whose indices reach the upper bound $\sqrt{2\Delta - 1}$, have the two highest degrees equal, therefore they are *biregular*. Moreover, their index λ is integer if and only if it is odd. In this case, the tree is written in the form $R(0, 2p(p + 1) + 1, 2p(p + 1))$, for some positive integer p, and its index is $\lambda = 2p + 1$. The smallest examples of them are the trees $R(0, 5, 4)$ and $R(0, 13, 12)$, while the first is integral, the second one is not.

In the next corollary, we characterize when the trees $R(0, r, r - 1)$ are integral.

Corollary 3.2. *The tree $R(0, r, r - 1)$ is integral if and only if $x = \sqrt{r - 1} \geq 2$ and y are positive integer solutions of the equation $2x^2 - y^2 + 1 = 0$. More precisely, $x = x_k$ and $y = y_k$ are obtained recursively, for each positive integer k, through the expressions*

$$\begin{cases} x_{k+1} = 3x_k + 2y_k \\ y_{k+1} = 4x_k + 3y_k \end{cases},$$

from $x_1 = 2$, $y_1 = 3$.

As a consequence of Corollary 2.1, we conclude that, in the subfamily $R(0, r, a)$ with a maximum degree Δ, the index is a strictly increasing function of Δ'. However, in the family $R(b, r, a)$ with $b > 0$ this does not occur. Indeed, the index of $R(2, 2, 2)$ is smaller than the index of $R(1, 3, 2)$, and both trees have same $\Delta = 4$ and $\Delta' = 3$. We finalize this section by studying the variation of the index of the trees $R(b, r, a)$, due to certain perturbations in the parameters b, r, a that define them.

Proposition 3.2. *Let $a \geq 1$, $b \geq 0$, $r \geq 2$ and $t > 0$ be integers. Then:*
 (i) $\lambda(R(b, r, a)) < \lambda(R(b - t, r + t, a))$, if $t \leq b$.
 (ii) $\lambda(R(b, r, a)) \leq \lambda(R(b, r + t, a - t))$, if $t \leq a - 1$. The equality holds if and only if $b = 0$.
 (iii) $\lambda(R(b, r, a)) < \lambda(R(b + t, r, a - t))$, if $a - b < t \leq a - 1$; $\lambda(R(b, r, a)) > \lambda(R(b + t, r, a - t))$, if $t \leq a - 1$ and $t < a - b$.

Proof. Let λ, μ, ρ and α the respective indices of $R(b, r, a)$, $R(b - t, r + t, a)$, $R(b, r + t, a - t)$ and $R(b + t, r, a - t)$, where $t > 0$ satisfies the conditions determined in each item. From (9), we obtain that λ, μ, ρ and α are the respective largest roots of $p(x) = x^4 - (a + b + r)x^2 + ab$, $q(x) = p(x) - at$, $r(x) = p(x) - bt$ and $s(x) = p(x) - t(b - a + t)$. So, $q(\lambda) < 0$ and, then, $\lambda < \mu$. Similarly, $r(\lambda) \leq 0$ and, therefore, $\lambda < \rho$, if $b > 0$, and $\lambda = \rho$, if $b = 0$. Finally, $s(\lambda) < 0$ if $b - a + t > 0$; in this case, $\lambda < \alpha$. And, $p(\alpha) < 0$, if $b - a + t < 0$, which implies that $\alpha < \lambda$. ∎

4. New family of integral trees of diameter 4.

In [21], Wang and Liu provide a way to build an infinite family of integral trees from a given integral tree, all of them of diameter 4. Given a natural k and a tree $S(b, r, a_i)$, where each a_i is repeated c_i times, the authors consider the tree $S(bk^2, rk^2, a_ik^2)$, where each a_ik^2 is repeated c_ik^2 times, proving that $S(b, r, a_i)$ is integral if and only if $S(bk^2, rk^2, a_ik^2)$ is also integral. Furthemore, the authors give the distribution of the eigenvalues of $S(b, r, a_i)$.

Since the study of graphs with integer index includes the search for integral graphs, it is pertinent to note that the above result remains valid on condition of integrality given only by the index, and this follows from its proof (see Lemma 6 of [21]).

Remark 4.1. Let $b \geq 0$, $r \geq 2$ and $a_1 \geq \ldots \geq a_r > 1$ be integers. Given a positive integer k, we have that $S(b, r, a_i)$ has integer index if and only if $S(bk^2, rk^2, a_ik^2)$ has integer index.

In the special case where the sequence $\{a_i\}_{1 \leq i \leq r}$ is constant (equal to a), Wang proves, in Theorem 1.3.7 of [17], that, for $b \neq 0$, $R(b, r, a)$ is integral if and only if $R(a, r, b)$ is also integral. Again, we note that this result still holds if we just assume the integrality of the index. Actually, from (9), both trees have the same index.

In the next proposition we characterize when the tree $R(b, r, a)$ has integer index, as a consequence of (6). We then obtain a parametrization of a, b and r as a function of the coordinates of a Perron vector of $R(b, r, a)$.

Proposition 4.1. *Given integers $a \geq 1$, $b \geq 0$ and $r \geq 2$, a positive real λ is the index of the tree $R(b, r, a)$ if and only if there are m and v positive reals such that*

$$\begin{cases} a = \lambda^2 - m \\ b = \lambda^2 - v \\ r = \frac{mv}{\lambda^2} \end{cases}, \qquad (10)$$

and $\lambda^2 - m \geq 1$, $\lambda^2 - v \geq 0$ and $\frac{mv}{\lambda^2} \geq 2$. Moreover, λ is integer if and only if v is integer and $m + a$ is a perfect square.

Proof. From (6), it follows that a positive real λ is the index of $R(b, r, a)$ if and only if there exist a positive real m satisfying $m = \lambda^2 - a$ and $bm = \lambda^2(m - r)$. Considering $v = \lambda^2 - b$, we have $v > 0$ and $r = \frac{mv}{\lambda^2}$, obtaining (10). It also follows that λ is integer if and only if $m + a$ is a perfect square. It remains to verify that, if λ is integer, v is also an integer. And this actually happens, since $v = \lambda^2 - b$. ∎

Next, we show how to use Proposition 4.1 in order to obtain examples of non integral trees $R(b, r, a)$ with integer index. Additionally to the index, the other positive eigenvalues of $R(b, r, a)$ are

$$\sqrt{a} = \sqrt{\lambda^2 - m} \text{ and, if } b \geq 1, \ u_2 = \sqrt{\lambda^2 - (m + v - r)}.$$

Thus, it would be sufficient to ensure that $\sqrt{a}$ and u_2 are not both simultaneously integers. In Corollary 4.1, we exemplify this situation exhibiting infinite families. Two trees of these families are $R(0, 3, 6)$ and $R(3, 2, 6)$. Both of them have indices 3.

Corollary 4.1. *Let k, p and t be integers such that $a = k(k - p) \geq 1$, $b = k(k - t) \geq 0$ and $r = pt \geq 2$, and such that k and p satisfy one of the following conditions:*

(i) $p \equiv 1 \pmod 2$ and $k - 1 \geq (\frac{p+1}{2})^2$;

(ii) $p \equiv 0 \pmod 4$ and $k - 1 \geq \frac{p(p+4)}{8}$;

(iii) $p \equiv 2 \pmod 4$ and $k - 1 \geq \frac{(p+2)^2}{8}$.

Then the tree $R(b, r, a)$ is not integral and has integer index equal to k.

In the case where $a = b \geq 1$ and $r \geq 2$, Wang ([19], Theorem 15) determined an infinite family of integral trees $R(a, r, a)$, given by $a = zw$ and $r = (z - w)^2$, where z and w are positive integers such that $z > w$ and zw is a perfect square. For this family, the index is given by z. From Corollary 4.2, we can conclude that these are the *unique* integral trees $R(a, r, a)$. Indeed, the parametrization given by Wang follows from this, considering $p = z - w$.

Corollary 4.2. *Given integers $a \geq 1$ and $r \geq 2$, a positive real λ is the index of the tree $R(a, r, a)$ if and only if there is a real $2 \leq p < \lambda$ such that $a = \lambda(\lambda - p)$ and $r = p^2$. In this case, λ is integer if and only if p is integer and $p^2 + 4a$ is a perfect square. Further, if a is a perfect square, $R(a, r, a)$ is integral.*

In the above corollary, we characterize the trees $R(a, r, a)$ that have integer index. Note that the index of this tree must divide the coordinate m, relating to the central vertex, of its Perron eigenvector. In this family, the smallest example of non-integral tree with integer index is $R(3, 4, 3)$ and the smallest integral tree is $R(4, 9, 4)$.

Next, we study the trees $R(b, r, a)$, with integer index, where the index λ divides m. Applying Proposition 4.1, we obtain a new parametrization of a, b and r for this case:

As $m = \lambda p$, we have that $a = \lambda(\lambda - p)$, $b = \lambda^2 - v$ and $r = \frac{pv}{\lambda}$, and that λ also divides pv. Considering $d = \text{mdc}\{p, \lambda\}$, $p = dq$ and $\lambda = ds$, for some q and s relatively prime. As s divides qv, we conclude that $v = sk$, for some k. In this way, we obtain the parametrization of a, b and r enunciated by Proposition 4.2. This parametrization will allow us to construct a *new infinite family* of integral trees at the end of this section.

Proposition 4.2. *Let d, s, q and k be positive integers such that $s > q$, $d^2s \geq k$ and $kq \geq 2$. Then, for*

$$\begin{cases} a = d^2s(s - q) \\ b = s(d^2s - k) \\ r = kq \end{cases},$$

the tree $R(b, r, a)$ has integer index $\lambda = ds$. If s and q are relatively prime, this tree is integral if and only if s, $s - q$ and $d^2s - k$ are perfect squares.

Proof. From (9), the index λ of $R(b, r, a)$ satisfies $2\lambda^2 = (a + b + r) + \sqrt{(a + b + r)^2 - 4ab}$. As $(a + b + r)^2 - 4ab = (d^2sq + sk - kq)^2$, we conclude that λ is an integer. Calculating $a + b + r = 2d^2s^2 - d^2sq - sk + kq$, we deduce that $\lambda = ds$. The remaining positive eigenvalues of $R(b, r, a)$ are $\sqrt{a} = d\sqrt{s(s - q)}$ and, if $b \neq 0$, $u_2 = \sqrt{(s - q)(d^2s - k)}$. Then, if s, $s - q$ and $d^2s - k$ are perfect squares, we conclude that $\sqrt{a}$ and u_2 are integers. Conversely, if s and q are relatively prime and $\sqrt{a}$ and u_2 are integers, we have that s and $s - q$ are perfect squares. Therefore, $d^2s - k$ is also a perfect square. ∎

Note that, in the proof of Proposition 4.2, it is not necessary to assume that s and q are relatively prime to ensure that $R(b, r, a)$ is integral. Applying the parametrization of a, b and r enunciated in this proposition, we obtain an infinite family of integral trees $R(b, r, a)$:

As $a = d^2s(s - q)$, $b = s(d^2s - k)$ and $r = kq$, for some s, $s - q$ and $d^2s - k$ perfect squares, we will consider $s = (j + p)^2$, $q = j^2$, obtaining: $a = d^2(j + p)^2j^2$, $b = (j + p)^2(ds^2 - k)$ and $r = k(2jp + p^2)$. Assuming $d^2s - k$ in the form d^2t^2, we have that $k = d^2[(j + p)^2 - t^2]$. We then construct a family of integral trees considered in the following corollary.

Corollary 4.3. *Let d, j, p and t be positive integers such that $t < j + p$. Then, for*

$$\begin{cases} a = d^2(j + p)^2j^2 \\ b = d^2(j + p)^2t^2 \\ r = d^2[(j + p)^2 - t^2](2jp + p^2) \end{cases},$$

the tree $R(b, r, a)$ is integral.

We point out that there is an infinite subfamily of trees satisfying Corollary 4.3 that are distinct of all families obtained in [21]. In fact, it is sufficient to consider $d = 1$, $j = 2$, $t = 3$ and an odd $p \geq 3$ multiple of 3. Then, a tree $R(b, r, a)$ given by

$$\begin{cases} a = 4(p + 2)^2 \\ b = 9(p + 2)^2 \\ r = [(p + 2)^2 - 4] \cdot [(p + 2)^2 - 9] \end{cases} \tag{11}$$

does not satisfy Corollary 13 of [21]. And the same occurs with $R(a, r, b)$. Moreover, note that a, r and b are relatively prime, so both trees $R(b, r, a)$ and $R(a, r, b)$ are not obtained from another integral tree $R(b', r', a')$, multiplying a', b' and r' by k^2, for some integer k.

We claim that the family described in Corollary 4.3 is distinct from the families of integral trees of diameter 4, that we can find in [1, 13, 17, 19, 20]. All these families are given by the following parametrization, where the integers $z > y > w$ are their positive eigenvalues and satisfy:

$$a = y^2, \quad b = \frac{z^2w^2}{y^2} \quad \text{and} \quad r = z^2 + w^2 - y^2 - \frac{z^2w^2}{y^2}, \tag{12}$$

$$r + b = z^2 + w^2 - y^2 \quad \text{is a perfect square}. \tag{13}$$

Determining the value of $z^2 + w^2 - y^2$, for the trees of Corollary 4.3, we verify that it may not be a perfect square, when $j < t \leq p$. Indeed, in this case, $z = d(j + p)^2$, $y = d(j + p)j$ and $w = dtj$. Thus, $z^2 + w^2 - y^2 = d^2\alpha$, for $\alpha = (j + p)^2(2jp + p^2) + j^2t^2$.

If $j < t \leq p$, α is not a perfect square since, in this case, it is strictly between two consecutive perfect squares:
$$[j(j + p) + 2p^2 - 1]^2 < \alpha < [j(j + p) + 2p^2]^2 .$$

Note that a tree given by (11) satisfies $j = 2 < t = 3 \leq p$. Then, in this case, $r + b$ is not a perfect square. We verify that, for this tree, neither $r + a$ is a perfect square. So, both $R(b, r, a)$ and $R(a, r, b)$, for each tree in this subfamily, is distinct of all known in the literature. The smaller example of them, $R(225, 336, 100)$, has 34162 vertices. Consequently, the trees in this family have a large number of vertices, which makes difficult an exhaustive search of these trees.

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