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# Convenient adjacencies on $\mathbb{Z}^2$

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**Abstract.** We discuss graphs with the vertex set $\mathbb{Z}^2$ which are subgraphs of the 8-adjacency graph and have the property that certain natural cycles in these graphs are Jordan curves, i.e., separate $\mathbb{Z}^2$ into exactly two connected components. After considering graphs with the usual connectedness, we concentrate on a graph with a special one.

## 1. Introduction

It is one of the crucial problems of digital image processing to provide the digital plane $\mathbb{Z}^2$ with a convenient structure for the study of geometric and topological properties of (two-dimensional) digital images. Here, convenience means that such a structure behaves analogously to the Euclidean topology on the real plane $\mathbb{R}^2$. First of all, it is usually required that an analogue of the Jordan curve theorem be valid. (Recall that the classical Jordan curve theorem states that any simple closed curve in the Euclidean plane divides this plane into exactly two components). In the classical, graph theoretic approach to this problem (see e.g. [7] and [8]), the well-known binary relations of 4-adjacency and 8-adjacency are used for structuring $\mathbb{Z}^2$. Unfortunately, neither 4-adjacency nor 8-adjacency itself allows for an analogue of the Jordan curve theorem - cf. [5]. To eliminate this deficiency, a combination of the two adjacencies has to be used. Despite this inconvenience, the graph-theoretic approach proved to be useful for solving many problems of digital image processing and for creating efficient graphic software. In [3], a new, purely topological approach to the problem has been proposed which utilizes a convenient topology for structuring the digital plane, namely the Khalimsky topology. The topological approach was then developed by many authors - see, e.g., [2], [4]-[6] and [9]-[13].

Since the Khalimsky topological space is an Alexandroff space (i.e., has a completely additive closure), its connectedness coincides with the connectedness in a certain graph with the vertex set $\mathbb{Z}^2$, the so-called connectedness graph of the topology. Thus, when studying the connectedness of digital images, this graph, rather than the Khalimsky topology itself, may be used for structuring the digital plane. A well-known analogue of the Jordan curve theorem is then valid in the graph - cf. [3]. A disadvantage of this approach is that Jordan curves in the (connectedness graph of the) Khalimsky topology may never turn to form the acute angle $\frac{\pi}{4}$. It would therefore be useful to find some new, more convenient graphs with the vertex set $\mathbb{Z}^2$, i.e., graphs that would allow Jordan curves to turn, at some points, to form the acute angle $\frac{\pi}{4}$. In the

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present note, we will introduce such graphs. More precisely, we will define a certain natural graph with the vertex set $\mathbb{Z}^2$ whose cycles are eligible for Jordan curves in $\mathbb{Z}^2$ and we solve the problem of finding graphs with the vertex set $\mathbb{Z}^2$ with respect to which these cycles are Jordan curves. The results obtained propose new structures on $\mathbb{Z}^2$ with natural Jordan curves which may, for example, be used in digital image processing for solving problems related to boundaries such as boundary detection, contour filling, data compression, etc.

## 2. Preliminaries

For the graph-theoretic concepts used see, for instance, [1]. By a *graph* on a set $V$ we always mean an undirected simple graph without loops whose vertex set is $V$, i.e., a graph $(V, E)$ where $E \subseteq \{\{a, b\};\ a, b \in V,\ a \neq b\}$ is the set of edges of the graph. For an arbitrary vertex $a \in V$, we put $E(a) = \{b \in V;\ \{a, b\} \in E\}$.

A nonempty, finite and connected subset $C$ of $V$ is said to be a *simple closed curve* in $(V, E)$ if $E(a) \cap C$ has precisely two elements for every $a \in C$. Clearly, every simple closed curve is a cycle. A simple closed curve in $(V, E)$ is called a *Jordan curve* if it separates the set $V$ into precisely two components, i.e., if the induced subgraph $V - C$ has exactly two components.

In the sequel, we will consider graphs on $\mathbb{Z}^2$ only. For every point $(x, y) \in \mathbb{Z}^2$, we denote by $A_4(x, y)$ or $A_8(x, y)$ the sets of all points that are 4-adjacent or 8-adjacent to $(x, y)$, respectively. Thus, $A_4(x, y) = \{(x + i, y + j);\ i, j \in \{-1, 0, 1\},\ ij = 0,\ i + j \neq 0\}$ and $A_8(x, y) = A_4(x, y) \cup \{(x + i, y + j);\ i, j \in \{-1, 1\}\}$. The graphs $(\mathbb{Z}^2, A_4)$ and $(\mathbb{Z}^2, A_8)$ are called the *4-adjacency graph* and *8-adjacency graph*, respectively. An arbitrary subset $A \subseteq A_8$ is said to be an *adjacency* on $\mathbb{Z}^2$ and the graph $(\mathbb{Z}^2, A)$ is said to be an *adjacency graph*. Thus, adjacency graphs are exactly the graphs on $\mathbb{Z}^2$ that are subgraphs of the 8-adjacency graph, i.e., the graphs $(\mathbb{Z}^2, A)$ with the property $A(z) \subseteq A_8(z)$ for every $z \in \mathbb{Z}^2$. In an adjacency graph $(\mathbb{Z}^2, A)$, vertices $a, b \in \mathbb{Z}^2$ are said to be *adjacent* if $\{a, b\} \in A$, i.e., if they are joined by an edge. Note that the inclusion of adjacencies gives a partial order on the set of all adjacency graphs.

**Definition 1.** The *square-diagonal graph* is the adjacency graph in which two points $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in \mathbb{Z}^2$ are adjacent if and only if one of the following four conditions is fulfilled:

1. $|y_1 - y_2| = 1$ and $x_1 = x_2 = 4k$ for some $k \in \mathbb{Z}$,
2. $|x_1 - x_2| = 1$ and $y_1 = y_2 = 4l$ for some $l \in \mathbb{Z}$,
3. $x_1 - x_2 = y_1 - y_2 = \pm 1$ and $x_1 - 4k = y_1$ for some $k \in \mathbb{Z}$,
4. $x_1 - x_2 = y_2 - y_1 = \pm 1$ and $x_1 = 4l - y_1$ for some $l \in \mathbb{Z}$.

A portion of the square-diagonal graph is shown in Figure 1.

When studying digital images, it may be advantageous to equip $\mathbb{Z}^2$ with a structure with respect to which all or most of the cycles in the square-diagonal graph are Jordan curves. Such a convenient structure given by a topology was introduced in [10]. In this note, we focus on convenient structures given by adjacencies.

## 3. Adjacency graphs with the usual connectedness

In digital image processing, the 4-adjacency and 8-adjacency are the most frequently used adjacencies. But, since the late 1980's, another adjacency has been used too, namely the adjacency given by the connectedness graph of the Khalimsky topology on $\mathbb{Z}^2$ [3] (i.e., the graph on $\mathbb{Z}^2$ in which arbitrary distinct vertices $z_1, z_2 \in \mathbb{Z}^2$ are joined by an edge if and only if $\{z_1, z_2\}$ is a connected subset of the Khalimsky topological space). This adjacency will be called the *Khalimsky adjacency* and the corresponding adjacency graph will be called the *Khalimsky graph*.

The Khalimsky graph coincides with the adjacency graph $(\mathbb{Z}^2, K)$ given as follows:

For any $z = (x, y) \in \mathbb{Z}^2$,

$$K(z) = \begin{cases} A_8(z) \text{ if } x + y \text{ is even,} \\ A_4(z) \text{ if } x + y \text{ is odd.} \end{cases}$$

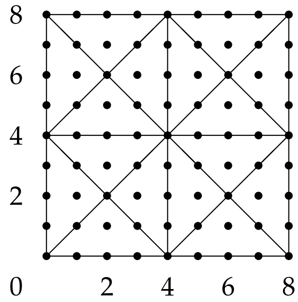



Figure 1: A portion of the square-diagonal graph.

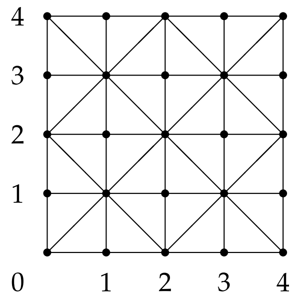



Figure 2: A portion of the Khalimsky graph.

A portion of the Khalimsky graph is shown in Figure 2.

It is readily verified that a cycle in the square-diagonal graph is a Jordan curve in the Khalimsky graph if and only if it does not turn, at any of its points, to form the acute angle $\frac{\pi}{4}$ - cf. [3]. It could therefore be useful to replace the Khalimsky graph with some more convenient adjacency graphs that allow Jordan curves to turn, at some points, to form the acute angle $\frac{\pi}{4}$.

We define an adjacency graph $(\mathbb{Z}^2, A)$ as follows:

For any point $z = (x, y) \in \mathbb{Z}^2$,

$$A(z) = \begin{cases} A_8(z) \text{ if } x = 4k,\ y = 4l,\ k, l \in \mathbb{Z}, \\ A_8(z) - A_4(z) \text{ if } x = 2 + 4k,\ y = 2 + 4l, \\ \quad k, l \in \mathbb{Z}, \\ \{(x-1, y), (x+1, y)\} \text{ if } x = 2 + 4k, \\ \quad y = 1 + 2l,\ k, l \in \mathbb{Z}, \\ \{(x, y-1), (x, y+1)\} \text{ if } x = 1 + 2k, \\ \quad y = 2 + 4l,\ k, l \in \mathbb{Z}, \\ A_4(z) \text{ if either } x = 4k \text{ and } y = 2 + 4l \text{ or} \\ \quad x = 2 + 4k \text{ and } y = 4l,\ k, l \in \mathbb{Z}. \end{cases}$$

A portion of the graph $(\mathbb{Z}^2, A)$ is shown in Figure 3.

**Theorem 1.** *Every cycle in the square-diagonal graph is a Jordan curve in the adjacency graph* $(\mathbb{Z}^2, A)$ *and* $(\mathbb{Z}^2, A)$ *is a minimal adjacency graph with this property.*

*Proof.* Clearly, any cycle in the square-diagonal graph is a simple closed curve in $(\mathbb{Z}^2, A)$. Let $z = (x, y) \in \mathbb{Z}^2$ be a point such that $x = 4k + p$ and $y = 4l + q$ for some $k, l, p, q \in \mathbb{Z}$ with $pq = \pm 2$. Then we define the *fundamental triangle T(z)* to be the nine-point subset of $\mathbb{Z}^2$ given below:

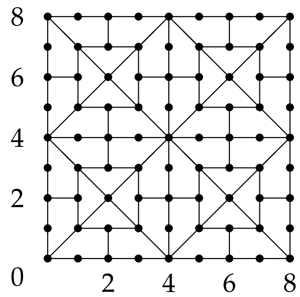
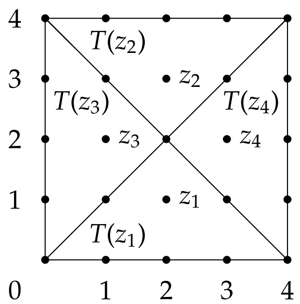


Figure 3: A portion of the graph $(\mathbb{Z}^2, A)$.

$$T(z) = \begin{cases} \{(r,s) \in \mathbb{Z}^2;\ y-1 \le s \le y+1-|r-x|\} \text{ if} \\ x = 4k+2 \text{ and } y = 4l+1 \text{ for some } k,l \in \mathbb{Z}, \\ \{(r,s) \in \mathbb{Z}^2;\ y-1+|r-x| \le s \le y+1\} \text{ if} \\ x = 4k+2 \text{ and } y = 4l-1 \text{ for some } k,l \in \mathbb{Z}, \\ \{(r,s) \in \mathbb{Z}^2;\ x-1 \le r \le x+1-|s-y|\} \text{ if} \\ x = 4k+1 \text{ and } y = 4l+2 \text{ for some } k,l \in \mathbb{Z}, \\ \{(r,s) \in \mathbb{Z}^2;\ x-1+|s-y| \le r \le x+1\} \text{ if} \\ x = 4k-1 \text{ and } y = 4l+2 \text{ for some } k,l \in \mathbb{Z}. \end{cases}$$

Graphically, the fundamental triangle $T(z)$ consists of the point $z$ and the eight points lying on the triangle surrounding $z$ - the four types of fundamental triangles are represented in the following figure:




Given a fundamental triangle, we speak about its sides - it is clear from the above picture which sets are understood to be the sides (note that each side consists of five or three points and that two different fundamental triangles may have at most one side in common).

Now, one can easily see that:

1. Every fundamental triangle is connected (so that the union of two fundamental triangles having a common side is connected) in $(\mathbb{Z}^2, A)$.
2. If we subtract from a fundamental triangle some of its sides, then the resulting set is still connected in $(\mathbb{Z}^2, A)$.
3. If $S_1, S_2$ are fundamental triangles having a common side $D$, then the set $(S_1 \cup S_2) - M$ is connected in $(\mathbb{Z}^2, A)$ whenever $M$ is the union of some sides of $S_1$ or $S_2$ different from $D$.
4. Every connected subset of $\mathbb{Z}^2$ with at most two points is a subset of a fundamental triangle.

We will now show that the following is also true:

5. For every cycle $C$ in the square-diagonal graph, there are sequences $\mathcal{S}_F, \mathcal{S}_I$ of fundamental triangles, $\mathcal{S}_F$ finite and $\mathcal{S}_I$ infinite, such that, whenever $\mathcal{S} \in \{\mathcal{S}_F, \mathcal{S}_I\}$, the following two conditions are satisfied:
(a) Each member of $\mathcal{S}$, excluding the first one, has a common side with at least one of its predecessors.

(b) $C$ is the union of those sides of fundamental triangles in $\mathcal{S}$ that are not shared by two different fundamental triangles from $\mathcal{S}$.

Put $C_1 = C$ and let $S_1^1$ be an arbitrary fundamental triangle with $S_1^1 \cap C_1 \neq \emptyset$. For every $k \in \mathbb{Z}$, $1 \leq k$, if $S_1^1, S_2^1, ..., S_k^1$ are defined, let $S_{k+1}^1$ be a fundamental triangle with the following properties: $S_{k+1}^1 \cap C_1 \neq \emptyset$, $S_{k+1}^1$ has a side in common with $S_k^1$ which is not a subset of $C_1$ and $S_{k+1}^1 \neq S_i^1$ for all $i$, $1 \leq i \leq k$. Clearly, there will always be a (smallest) number $k \geq 1$ for which no such fundamental triangle $S_{k+1}^1$ exists. Denoting by $k_1$ this number, we have defined a sequence $(S_1^1, S_2^1, ..., S_{k_1}^1)$ of fundamental triangles. Let $C_2$ be the union of those sides of fundamental triangles in $(S_1^1, S_2^1, ..., S_{k_1}^1)$ that are disjoint from $C_1$ and not shared by two different fundamental triangles in $(S_1^1, S_2^1, ..., S_{k_1}^1)$. If $C_2 \neq \emptyset$, we construct a sequence $(S_1^2, S_2^2, ..., S_{k_2}^2)$ of fundamental triangles in an analogous way to $(S_1^1, S_2^1, ..., S_{k_1}^1)$ by taking $C_2$ instead of $C_1$ (and obtaining $k_2$ analogously to $k_1$). Repeating this construction, we get sequences $(S_1^3, S_2^3, ..., S_{k_3}^3)$, $(S_1^4, S_2^4, ..., S_{k_4}^1)$, etc. We put $\mathcal{S} = (S_1^1, S_2^1, ..., S_{k_1}^1, S_1^2, S_2^2, ..., S_{k_2}^2, S_1^3, S_2^3, ..., S_{k_3}^3, ...)$ if $C_i \neq \emptyset$ for all $i \geq 1$ and $\mathcal{S} = (S_1^1, S_2^1, ..., S_{k_1}^1, S_1^2, S_2^2, ..., S_{k_2}^2, ..., S_1^l, S_2^l, ..., S_{k_l}^l)$ if $C_i \neq \emptyset$ for all $i$ with $1 \leq i \leq l$ and $C_i = \emptyset$ for $i = l + 1$.

Further, let $S_1' = T(z)$ be a fundamental triangle such that $z \notin S$ whenever $S$ is a member of $\mathcal{S}$. Having defined $S_1'$, let $\mathcal{S}' = (S_1', S_2', ...)$ be a sequence of fundamental triangles defined analogously to $\mathcal{S}$ (by taking $S_1'$ instead of $S_1^1$). Then one of the sequences $\mathcal{S}$, $\mathcal{S}'$ is finite and the other is infinite. Indeed, $\mathcal{S}$ is finite (infinite) if and only if its first member equals such a fundamental triangle $T(z)$ for which $z = (k, l) \in \mathbb{Z}^2$ has the property that (1) $k$ is even, $l$ is odd and the cardinality of the set $\{(x, l) \in \mathcal{Z}^2;\ x > k\} \cap C$ is odd (even) or (2) $k$ is odd, $l$ is even and the cardinality of the set $\{(k, y) \in \mathcal{Z}^2;\ y > l\} \cap C$ is odd (even). The same is true for $\mathcal{S}'$. If we put $\{\mathcal{S}_F, \mathcal{S}_I\} = \{\mathcal{S}, \mathcal{S}'\}$ where $\mathcal{S}_F$ is finite and $\mathcal{S}_I$ is infinite, then the conditions (a) and (b) are clearly satisfied.

Given a cycle $C$ in the square-diagonal graph, let $S_F$ and $S_I$ denote the union of all members of $\mathcal{S}_F$ and $\mathcal{S}_I$, respectively. Then $S_F \cup S_I = \mathbb{Z}^2$ and $S_F \cap S_I = C$. Let $\mathcal{S}_F^*$ and $\mathcal{S}_I^*$ be the sequences obtained from $\mathcal{S}_F$ and $\mathcal{S}_I$ by subtracting $C$ from each member of $\mathcal{S}_F$ and $\mathcal{S}_I$, respectively. Let $S_F^*$ and $S_I^*$ denote the union of all members of $\mathcal{S}_F^*$ and $\mathcal{S}_I^*$, respectively. Then $S_F^*$ and $S_I^*$ are connected by (1), (2) and (3) and it is clear that $S_F^* = S_F - C$ and $S_I^* = S_I - C$. So, $S_F^*$ and $S_I^*$ are the two components of $\mathbb{Z}^2 - C$ by (4) ($S_F - C$ is called the *inside* component and $S_I - C$ is called the *outside* component). We have proved that every cycle in the square-diagonal graph is a Jordan curve in $(\mathbb{Z}^2, A)$.

To show that $(\mathbb{Z}^2, A)$ is a minimal adjacency graph with this property, let $(\mathbb{Z}^2, B)$ be a subgraph of $(\mathbb{Z}^2, A)$ such that every cycle in the square-diagonal graph is a Jordan curve in $(\mathbb{Z}^2, B)$. Suppose that there is an edge $\{z_1, z_2\} \in A - B$. Since $(\mathbb{Z}^2, B)$ is a supergraph of the square-diagonal graph, there is a fundamental triangle $T(z)$ with $z \in \{z_1, z_2\}$. Thus, $\{z_1, z_2\}$ is one of the three edges incident with $z$ and the point $z' \in \{z_1, z_2\} - \{z\}$ lies on a side $D$ of $T(z)$. Let $S$ be the fundamental triangle different from $T(z)$ such that one of the sides of $S$ is $D$. Then the union $C$ of all sides of $T(z)$ and $S$ different from $D$ is a cycle in the square diagonal graph but it is not a Jordan curve in $(\mathbb{Z}^2, B)$ because the inside part of $C$, i.e., the set $(T(z) \cup S) - C$, is evidently not connected in the subgraph $\mathbb{Z}^2 - C$ of $(\mathbb{Z}^2, B)$. Thus, the subgraph $\mathbb{Z}^2 - C$ of $(\mathbb{Z}^2, B)$ has more than two components. This is a contradiction. Therefore, $A = B$ and the minimality of $(\mathbb{Z}^2, A)$ is proved. □

**Remark 1.** It follows from the proof of Theorem 1 that every cycle in the square-diagonal graph is a Jordan curve in $(\mathbb{Z}^2, B)$ whenever $(\mathbb{Z}^2, B)$ is an adjacency graph that is a supergraph of $(\mathbb{Z}^2, A)$ with the property that, for every edge $\{a, b\} \in B - A$, there exists a fundamental triangle $T$ with $\{a, b\} \subseteq T$ such that the union of all sides of $T$ is a simple closed curve in $(\mathbb{Z}^2, B)$. For example, every cycle in the square-diagonal graph is a Jordan curve in each of the two adjacency graphs portions of which are shown in Figure 4 (and also in the adjacency graph that is the union of the two graphs). Another adjacency graph having this property is given by the connectedness graph of the topology studied in [10] and [11]. This adjacency graph is demonstrated in Figure 5.

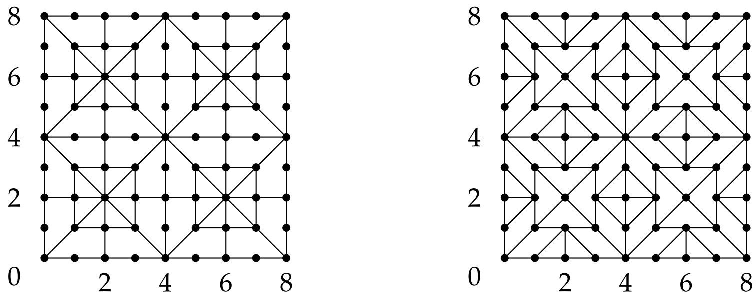



Figure 4: Portions of two adjacency supergraphs of $(\mathbb{Z}^2, A)$.

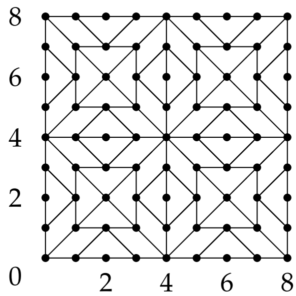



Figure 5: A portion of the adjacency graph that coincides with the connectedness graph of the topology introduced in [10].

## 4. An adjacency graph with a special connectedness

**Definition 2.** Let $(V, E)$ be a graph. A set $\mathcal{P}$ of paths of length 2 in $(V, E)$ is said to be a *path partition of type* 2, briefly a 2-*partition*, of $(V, E)$ if

(a) for every edge $\{a, b\} \in E$, there is exactly one path $(a_0, a_1, a_2) \in \mathcal{P}$ with the property that there exists $i \in \{1, 2\}$ such that $\{a, b\} = \{a_{i-1}, a_i\}$ and

(b) every pair of different paths belonging to $\mathcal{P}$ has at most one vertex in common.

Let $(V, E)$ be a graph with a 2-partition $\mathcal{P}$ and $U \subseteq V$ be an induced subgraph of $(V, E)$. Let $\mathcal{P}_U$ be the set of all paths belonging to $\mathcal{P}$ that are paths in $U$. If $\mathcal{P}_U$ is a 2-partition of $U$, then we say that $U$ is a $\mathcal{P}$-*subgraph* of $(V, E)$.

**Definition 3.** Let $(V, E)$ be a graph with a 2-partition $\mathcal{P}$. A sequence $C = (c_i |\, i \le n)$ of vertices of $(V, E)$ is called a $\mathcal{P}$-*path* in $(V, E)$ if every path $(a_0, a_1, a_2) \in \mathcal{P}$ satisfies the following two conditions:

(i) If there exists $i \in \{0, 1, ..., n-1\}$ such that $c_i = a_1$ and $c_{i+1} = a_2$, then $i > 0$ and $c_{i-1} = a_0$.

(ii) If there exists $i \in \{1, 2, ..., n\}$ such that $c_{i-1} = a_2$ and $c_i = a_1$, then $i < n$ and $c_{i+1} = a_0$.

Given a graph $(V, E)$ with a 2-partition $\mathcal{P}$, a subset $X \subseteq V$ is said to be $\mathcal{P}$-*connected* if, for every pair $a, b \in X$, there is a $\mathcal{P}$-path $(c_i |\, i \le n)$ in $(V, E)$ such that $c_0 = a$, $c_n = b$ and $c_i \in X$ for all $i \in \{0, 1, ..., n\}$. A maximal (with respect to set inclusion) $\mathcal{P}$-connected subset of $V$ is called a $\mathcal{P}$-*component* of the graph $(V, E)$.

**Definition 4.** Let $(V, E)$ be a graph with a 2-partition $\mathcal{P}$. A nonempty, finite and $\mathcal{P}$-connected subset $J$ of $V$ is said to be a $\mathcal{P}$-*simple closed curve* in $(V, E)$ if every path $(a_0, a_1, a_2) \in \mathcal{P}$ with $\{p_0, p_1\} \subseteq J$ satisfies $a_2 \in J$ and every $z \in J$ fulfills the following two conditions:

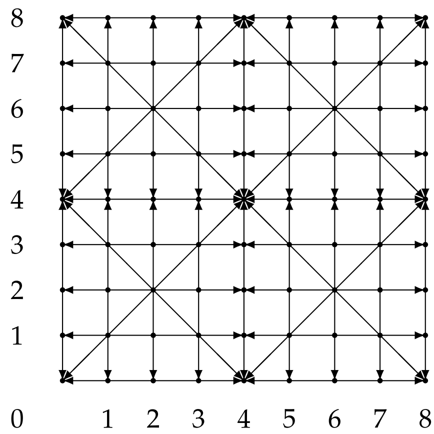



Figure 6: A portion of the graph $(\mathbb{Z}^2, C)$.

(1) There are exactly two paths $(a_0, a_1, a_2) \in \mathcal{P}$ satisfying both $\{a_0, a_1, a_2\} \subseteq J$ and $z \in \{a_0, a_2\}$ and there is no path $(b_0, b_1, b_2) \in \mathcal{P}$ satisfying both $\{b_0, b_1, b_2\} \subseteq J$ and $z = b_1$.
(2) There is exactly one path $(b_0, b_1, b_2) \in \mathcal{P}$ satisfying both $\{b_0, b_1, b_2\} \subseteq J$ and $z = b_1$ and there is no path $(a_0, a_1, a_2) \in \mathcal{P}$ satisfying both $\{a_0, a_1, a_2\} \subseteq J$ and $z \in \{a_0, a_2\}$.

Clearly, every $\mathcal{P}$-simple closed curve is a cycle.

**Definition 5.** In a graph $(V, E)$ with a 2-path partition $\mathcal{P}$, a $\mathcal{P}$-simple closed curve $J$ is called a *$\mathcal{P}$-Jordan curve* if the induced subgraph $V - J$ of $(V, E)$ is a *$\mathcal{P}$-subgraph* of $(V, E)$ consisting of precisely two $\mathcal{P}$-components.

Let $C$ be an adjacency on $(\mathbb{Z}^2, C)$ given as follows:
$C = A_4 \cup \{\{(4k+i, 4l+i), (4k+i+1, 4l+i+1)\};\ k, l, i \in \mathbb{Z}\} \cup \{\{(4k+i, 4l-i), (4k+i+1, 4l-i-1)\};\ k, l, i \in \mathbb{Z}\}$.

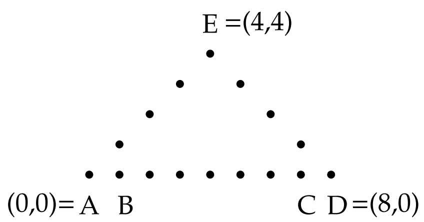
Put $Q = \{((x_i^1, x_i^2)|\ i \le 2) \in (\mathbb{Z}^2)^3;$ for every $j \in \{0, 1\}$, $x_0^j = x_1^j = x_2^j$ or there exists an odd number $k \in \mathbb{Z}$ fulfilling either $x_i^j = 2k + i + 1$ for all $i = 0, 1, 2$ or $x_i^j = 2k - i - 1$ for all $i = 0, 1, 2\} - \{((x_i^0, x_i^1)|\ i \le 2) \in (\mathbb{Z}^2)^3;\ x_0^j = x_1^j = x_2^j$ for every $j \in \{0, 1\}\}$.

It may easily be seen that $Q$ is a 2-partition on $(\mathbb{Z}^2, C)$. A portion of the graph $(\mathbb{Z}^2, C)$ is demonstrated in Figure 6. Each path $(a_0, a_1, a_2) \in Q$ in this portion is represented as an arrow whose initial, mid and terminal points are $a_0$, $a_1$ and $a_2$, respectively.

**Theorem 2.** *Every cycle in the square diagonal graph that does not turn at any point* $(4k+2, 4l+2)$, $k, l \in \mathbb{Z}$, *is a $Q$-Jordan curve in the adjacency graph* $(\mathbb{Z}^2, C)$.

*Proof.* In [9], a closure operation on $\mathbb{Z}^2$ is studied and, as the main result (Theorem 3.19), it is proved that every cycle in the square diagonal graph that does not turn at any point $(4k+2, 4l+2)$, $k, l \in \mathbb{Z}$, is a Jordan curve with respect to this closure operation. It may easily be seen that connectedness with respect to the closure operation from [9] is equivalent to $Q$-connectedness in the adjacency graph $(\mathbb{Z}^2, C)$. □

**Example 1.** Consider the following (digital picture of a) triangle:



While the triangle ADE is a $\mathcal{Q}$-Jordan curve in $(\mathbb{Z}^2, C)$, it is not a Jordan curve in the Khalimsky graph. For this triangle to be a Jordan curve in the Khalimsky graph, we have to delete the points A,B,C and D. But this will lead to a considerable deformation of the triangle.

## References


[1] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, 2008.

[2] D.N. Georgiou, S.-E. Han, Generalized topological function space and a classification of generalized computer topological space, Filomat 26 (2012) 539–552.

[3] E.D. Khalimsky, R. Kopperman, P.R. Meyer, Computer graphics and connected topologies on finite ordered sets, Topology Appl. 36 (1990) 1–17.

[4] E.D. Khalimsky, R. Kopperman, P.R. Meyer, Boundaries in digital planes, Jour. of Appl. Math. and Stoch. Anal. 3 (1990) 27–55.

[5] T.Y. Kong, R. Kopperman, P.R. Meyer, A topological approach to digital topology, Amer. Math. Monthly 98 (1991) 902–917.

[6] R. Kopperman, P.R. Meyer, R.G. Wilson, A Jordan surface theorem for three-dimensional digital spaces, Discr. and Comput. Geom. 6 (1991) 155–161.

[7] A. Rosenfeld, Digital topology, Amer. Math. Monthly 86 (1979) 621–630.

[8] A. Rosenfeld, Picture Languages, Academic Press, New York, 1979.

[9] J. Šlapal, A digital analogue of the Jordan curve theorem, Discr. Appl. Math. 139 (2004) 231–251.

[10] J. Šlapal, Digital Jordan curves, Topol. Appl. 153 (2006) 3255–3264.

[11] J. Šlapal, A quotient-universal digital topology, Theor. Comp. Sci. 405 (2008) 164–175.

[12] J. Šlapal, Jordan curve theorems with respect to certain pretopologies on $\mathbb{Z}^2$, Lect. Notes Comput. Sc. 5810 (2009) 252-262.

[13] J. Šlapal, Convenient closure operators on $\mathbb{Z}^2$, Lect. Notes Comput. Sc. 5852 (2009) 425-436.