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On the Edge Wiener Index

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Abstract. Let G be a simple connected graph. The Wiener index of G is the sum of all distances between vertices of G. Whereas, the edge Wiener index of G is defined as the sum of distances between all pairs of edges of G where the distance between the edges f and g in $E(G)$ is defined as the distance between the vertices f and g in the line graph of G. In this paper we will describe a new method for calculating the edge Wiener index. Then find this index for the triangular graphs. Also, we obtain an explicit formula for the Wiener index of the Cartesian product of two graphs using the group automorphisms of graphs.

1. Introduction

Graph invariants are properties of graphs that are invariant under graph isomorphisms. There are many examples of graph invariants, especially those based on distances, which are applicable in chemistry. Let G be a simple connected graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. The distance between the vertices u and v of G is denoted as $d(u, v\,|G)$ or $d(u, v)$. It is defined as the length of a minimum path connecting them. The first, and most well-known parameter, the Wiener index, was introduced in the late 1940s in an attempt to analyze the chemical properties of paraffins (alkanes) [27]. This is a distance-based index, whose mathematical properties and chemical applications have been widely researched. In our notation, it can be described as follows:

$$W(G) = \sum_{\{u,v\}\in V(G)} d(u, v\,|G) = \frac{1}{2} \sum_{u\in V(G)} d(u\,|G) \tag{1}$$

where $d(u\,|G) = \sum_{v\in V(G)} d(u, v\,|G)$. In mathematical research, the Wiener index has been first studied in [12], and for a long time mathematicians were not aware of the importance of the Wiener index in mathematical chemistry. However, because of the chemical facts about the Wiener index and also because it is an invariant of the graph, various researches found methods to calculate this index. Among the important works on finding the Wiener index of a general graph one is referred to the papers [7]–[11], [14]-[17],[24]-[26].

Another index that was recently defined in an analogous way, using the distance between the edges, is called the edge Wiener index [20]. Let $f = xy$ and $g = uv$ be two edges of G. The distance between f and g is denoted by $d_e(f, g\,|G)$ and defined as the distance between the vertices f and g in the line graph of G. This distance is equal to $min\,\{d(x, u), d(x, v), d(y, u), d(y, v)\} + 1$.

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Distance 1 means that the edges share a vertex, distance 2 means that at least two of the four end-vertices of two edges are adjacent. The edge Wiener index of the graph G is denoted by $W_e(G)$ and defined as the sum of distances between all pairs of edges of the graph G. That is,

$$W_e(G) \quad = \sum_{\{f,g\}\subseteq E(G)} d_e(f,g\,|G) = \frac{1}{2}\sum_{f\in E(G)} d_e(f\,|G) \tag{2}$$

where $d_e(f\,|G) = \sum_{g\in E(G)} d(f,g\,|G)$.

We encourage the reader to consult [1, 2], [6] and [21, 22] for computational techniques and mathematical properties of the edge Wiener index.

In this paper our aim is to use new method which applies group theory to graph theory. We improve only mathematically computation of the edge Wiener index in certain graphs by this method.

First we need some concepts from the theory of groups and graph theory that we will give on the following. Let $A = Aut(G)$ be the automorphism of a graph G which is a group. A act *transitive* on V (or E) if for any pair u, v of vertices (or f, g of edges) in G, there exists an automorphism σ such that $\sigma(u) = v$ (or $\sigma(f) = g$). In this case, G is called *vertex-transitive* (or *edge-transitive*) [3],[13].

The *triangular graph* $T(n)$ is the line graph of the complete graph K_n. The vertices of $T(n)$ may be identified with the 2-subsets of $\Omega = \{1, 2, \ldots, n\}$ i.e., $V = \{\{a, b\}\,|a, b \in \Omega,\ a \neq b\}$. Two distinct vertices $\{a, b\}$ and $\{c, d\}$ are adjacent iff the 2-subsets have a nonempty intersection. We have $|V| = \binom{n}{2}$, the degree of each vertex is $2n-4$ and hence $|E| = (n-2)\binom{n}{2}$. We can see easily that $T(n)$ is an edge-transitive graph [8]. We will express explicit formula for the edge Wiener index of this graph, based on the properties of the automorphism group of the graph.

The next family of graphs we are going to consider is the *hypercube of dimension r* define as the graph Q_r whose vertex set consists of all 0, 1 vectors $(v_1, v_2, \ldots, v_r)$, where two vertices are adjacent if and only if they differ in precisely one coordinate [18], [19]. Therefore Q_r is a graph with $|V| = 2^r$ vertices where the degree of each vertex is n. The number of edges is equal to $|E| = r2^{r-1}$. Acoording to concepts in graph theory, we will show that Q_r is vertex-transitive and using relation between Cartesian product and Wiener index we compute the Wiener index of Q_r by new way.

2. Main results

Let E' and E'' be two subsets of $E = E(G)$, we define $d_e(E', E'')$ as follows:

$$d_e(E', E'') = \sum_{f\in E'}\sum_{g\in E''} d_e(f, g). \tag{3}$$

According to the above notation, we can write:

$$W_e(G) = \frac{1}{2} d_e(E, E). \tag{4}$$

Define a *distance number* $\delta_e(\sigma)$ of $\sigma \in A = Aut(G)$ as follows:

$$\delta_e(\sigma) = \frac{1}{|E|}\sum_{f\in E} d_e(f, \sigma(f)). \tag{5}$$

If Γ is a subgroup of A, we denote $\delta_e(\Gamma)$ as follows:

$$\delta_e(\Gamma) = \frac{1}{|\Gamma|}\sum_{\sigma\in\Gamma} \delta_e(\sigma) = \frac{1}{|\Gamma||E|}\sum_{f\in E}\sum_{\sigma\in\Gamma} d_e(f, \sigma(f)). \tag{6}$$

When $\Gamma = A$, the distance number of a graph G is denoted by $\delta_e(A)$.

Since Γ act on the set E, we denote the orbits of this action by $E_i = O(f_i) = \{f_i^\sigma | \sigma \in A\}, 1 \leq i \leq r$. Therefore, we have $E = E_1 \cup \cdots \cup E_r$. We will need the following well-known lemma [13].

Lemma 2.1. *(Orbit-stabilizer) Let G be a permutation group acting on Ω and let ω be a point in Ω. Then $|G| = |G_\omega|\,\left|\omega^G\right|$ where $\omega^G = \left\{\omega^g \,\middle|\, g \in G\right\}$ is an orbit of G and $G_\omega = \{g \in G \,|\, \omega^g = \omega\}$ is the stabilizer of ω in G.*

So if $\Gamma_i = \left\{\sigma \in \Gamma \,\middle|\, {f_i}^\sigma = f_i\right\}$ denotes a stabilizer of an edge f_i from E_i then by the above Lemma, $|\Gamma| = |E_i|\,|\Gamma_i|$.
In the following Theorem, we shows that the edge Wiener index of an edge-transitive graph G can be expressed in terms of the distance number of G.

Theorem 2.2. *For a connected graph G with the edge set E we have:*

$$|E|\,\delta_e(A) = \sum_{i=1}^{r} \frac{2\,W_e(E_i)}{|E_i|}. \tag{7}$$

Proof. Take $\Gamma = A$ and for two edges f and g of G, let $\Sigma = \{\sigma \in \Gamma | f^\sigma = g\}$ and $n(f,g) = |\Sigma|$. We denote by Γ_f the stabilizer group for f. If f and g belong to the same orbit E_i, we can construct a bijection between Σ and Γ_f and so $n(f,g) = \left|\Gamma_f\right| = \left|\Gamma_g\right|\ (= |\Gamma_i|)$. Now we have

$$\begin{aligned}\delta_e(A) = \frac{1}{|\Gamma|}\sum_{\sigma\in\Gamma}\delta_e(\sigma) \quad &= \quad \frac{1}{|\Gamma|\,|E|}\sum_{f\in E}\sum_{\sigma\in\Gamma} d_e(f,\sigma(f)) = \frac{1}{|\Gamma|\,|E|}\sum_{f\in E}\sum_{g\in E} d_e(f,g)\,n(f,g)\\ &= \quad \frac{1}{|\Gamma|\,|E|}\sum_{i=1}^{r}\sum_{f\in E_i}\sum_{g\in E_i} d_e(f,g)\ |\Gamma_i| = \frac{1}{|E|}\sum_{i=1}^{r}\frac{|\Gamma_i|}{|\Gamma|}\,d(E_i,E_i) = \frac{1}{|E|}\sum_{i=1}^{r}\frac{1}{|E_i|}\,d(E_i,E_i).\end{aligned}$$

Therefore

$$|E|\ \delta_e(A) = \sum_{i=1}^{r}\frac{1}{|E_i|}d(E_i,E_i) = \sum_{i=1}^{r}\frac{2\,W_e(E_i)}{|E_i|},$$

and so we have the desired result. □

Corollary 2.3. *If G is an edge-transitive graph, then*

$$W_e(G) = \frac{|E|^2\,\delta_e(A)}{2}. \tag{8}$$

Now let us introduce another useful concept. For a subset U of E, define

$$\omega_e(U) = \frac{1}{|U|^2}\sum_{f\in U}\sum_{g\in U} d_e(f,g) = \frac{1}{|U|^2}\,d_e(U,U). \tag{9}$$

Clearly we have

$$\omega_e(U) = \frac{2\,W_e(U)}{|U|^2}. \tag{10}$$

By this definition and Theorem 2.2, $\delta_e(A)$ becomes a weighted average of $\omega_e(E_i)$, that is,

$$\delta_e(A) = \frac{1}{|E|}\sum_{i=1}^{r}|E_i|\ \omega_e(E_i). \tag{11}$$

Theorem 2.4. *If E_i, $1 \le i \le r$, is an orbit, then for each edge $f \in E_i$*

$$\omega_e(E_i) = \frac{1}{|E_i|}\,d_e(f\,|E_i). \tag{12}$$

Proof. We have

$$\omega_e(E_i) = \frac{1}{|E_i|^2} d_e(E_i, E_i).$$

However, if f and g are two elements from E_i, then for some $\sigma \in Aut(G)$, $f^\sigma = g$ and we have

$$d_e(f\,|E_i) = \sum_{h\in E_i} d(f,h) = \sum_{h\in E_i} d(f^\sigma, h^\sigma) = \sum_{h\in E_i} d(g, h^\sigma) = \sum_{h'\in E_i} d(g, h') = d_e(g\,|E_i)\,,$$

therefore $\omega_e(E_i) = \frac{1}{|E_i|^2}\,|E_i|\,d_e(f\,|E_i) = \frac{1}{|E_i|}\,d_e(f\,|E_i)$. □

Corollary 2.5. *If G is an edge-transitive graph, then for each $f \in E$, we have*

$$\delta_e(A) = \omega_e(E) = \frac{1}{|E|} d_e(f\,|G)\,. \tag{13}$$

Corollary 2.6. *If G is an edge-transitive graph, then for each $f \in E$, we have*

$$W_e(G) = \frac{1}{2}\,|E|\; d_e(f\,|G)\,. \tag{14}$$

Proof. According to Corollary 2.3 and Corollary 2.5, we have the desired result. □

In the next examples, we will consider a few well-known graphs [13], and find their edge Wiener index using the result mentioned in Corollary 2.6. For these graphs, this index was computed directly in [20]. Here we only change our computation.

Example 2.7. *The automorphism group of K_n is the symmetric group S_n. So the complete graph K_n is edge-transitive graph. Thus we have*

$$W_e(K_n) = \frac{1}{2}\,|E(K_n)|\; d_e(f\,|K_n) = \frac{1}{2}\binom{n}{2}(n^2-3n+2) = \binom{n}{2}\binom{n-1}{2}.$$

Example 2.8. *Suppose C_n be a cyclic of length n. We labeling its vertices by $1, 2, ..., n$. Therefore $\langle (12\ldots n)\rangle \leq Aut(C_n)$ is a cyclic group that acting transitive on edge set of C_n. So according to Corollary 2.6, $W_e(C_n) = \frac{1}{2} n\, d_e(f\,|G)$ where e is any edge of C_n. With a simple calculation we have:*

$$d_e(f\,|G) = \begin{cases} \dfrac{n^2}{4}, & n \text{ even} \\[2ex] \dfrac{n^2-1}{4}, & n \text{ odd} \end{cases}$$

Now, we can compute the Wiener index of C_n by substituting the above formula in $W_e(C_n) = \frac{1}{2} n\, d_e(f\,|G)$.

Example 2.9. *The Petersen graph is edge-transitive.*

We have $W_e(P) = \frac{1}{2}|E| d_e(f\,|P)$, where f is an arbitrary edge of P. One can see that $d_e(f\,|P) = 26$ and so we conclude that $W_e(P) = 195$.

All graphs expressed in Examples 2.7, 2.8 and 2.9 are both vertex- and edge-transitive. A graph that is both vertex- and edge-transitive is called a *symmetric graph*.

The graph O, in Figure 2, is vertex-transitive but not edge-transitive.

Let f_1 and f_2 be representatives of two orbits E_1 and E_1, respectively. We have:

$$W_e(P) = \frac{1}{2}[|E_1|\, d_e(f_1\,|O) + |E_2|\, d_e(f_2\,|O)] = \frac{1}{2}[6\cdot 12 + 3\cdot 12] = 54$$

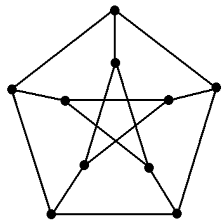

Figure 1: The Petersen graph (P).

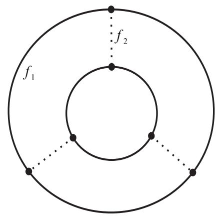

Figure 2: The graph O.

Another example of a vertex-transitive graph with two edge orbits is the Heawood graph [29]. There are also numerous examples of cubic vertex-transitive graphs with three edge orbits, called *zero-symmetric graphs* [5].

The next Theorem shows that an edge-transitive graph that is not vertex-transitive is necessarily bipartite.

Theorem 2.10. *([3] Theorem 15.1) If a connected graph G is edge-transitive but not vertex-transitive, then it is bipartite.*

The converse of the above theorem is false. For example, one can find many trees, such as any path of order $n \geq 4$, that are not vertex- and edge-transitive. Also, many molecular graph are not vertex- and edge-transitive. We refer to reader references [1, 2], [20, 22] for more details about the computation of the edge Wiener index of some molecular graphs.

Example 2.11. *Since the complete bipartite graph $K_{m,n}$ with $m \neq n$ is an edge-transitive, we have*

$$W_e(K_{m,n}) = \frac{1}{2} \left|E(K_{m,n})\right| \, d_e(f \left| K_{m,n}\right.) \tag{15}$$

where f is any edge of $K_{m,n}$. We have $d_e(f \left| K_{m,n}\right.) = \sum_{g \in E} d_e(f, g) = 2mn-n-m$ and so $W_e(K_{m,n}) = \dfrac{mn\,(2mn-n-m)}{2}$.

The complete bipartite graph $K_{m,n}$ with $m \neq n$ is a simple example of a graph which is edge-transitive but not vertex-transitive, although examples of this type are by no means easy to find [4]. However, the graph $K_{n,n}$ is a symmetric graph and according to the above formula when $m = n$, we have $W_e(K_{n,n}) = n^3\,(n-1)$.

Now in the following, we give the formula of the edge Wiener index of the triangular graph $T(n)$ according to concepts in transitive graph.

Theorem 2.12. *The edge Wiener index of the triangular graph $G = T(n)$ is*

$$W_e(G) = \frac{1}{4}(n-2)\binom{n}{2}(3n^3 - 18n^2 + 43n - 44). \tag{16}$$

Proof. The distance between any two distinct vertices of V is either 1 or 2. The vertices z whose distance from $u = \{a, b\} \in V$ is 1 should meet u in one element, hence the number of them is $2n-4$. If v is another vertex of V with $u \cap v = \emptyset$, then $v = \{c, d\}$, where c and d are distinct elements of Ω disjoint from a and b. Now if we take $w = \{a, c\}$, then $u \to w \to v$ is a path of length 2 from u to v.

Now according to definition of the distance between $f, g \in E$, we have $d_e(f, g|G) = 1$, 2 or 3. Fixing $f = uv \in E$ where $u = \{a, b\}$ and $v = \{a, c\}$. The edges whose distance from f is 1 should share with f in one vertex. So the number of these edges is $4n-10$. Suppose that $g = xy$ is a different edge from f. We let $x = \{r, s\}$ and $y = \{r, t\}$ where r, s, t are distinct elements of Ω disjoint from a, b, c. Therefore by the above arguments, $d(x, u) = d(x, v) = d(y, u) = d(y, v) = 2$ and so we have $d_e(f, g|G) = min\{d(x, u), d(x, v), d(y, u), d(y, v)\} + 1 = 3$. The number of these edges is equal to the number of selections of $g = xy$, which is equal to $(n-3)(n-4)(n-5)/2$. In this manner the number of edge at distance 2 of f is

$$(|E| - 1) - ((4n - 10) + (\frac{1}{2}(n - 3)(n - 4)(n - 5)) = \frac{1}{2}(9n^2 - 53n + 78),$$

therefore we have:

$$d_e(f|G) = (4n - 10) + 2\left[\frac{1}{2}(9n^2 - 53n + 78)\right] + 3\left[\frac{1}{2}(n - 3)(n - 4)(n - 5)\right] = \frac{1}{2}(3n^3 - 18n^2 + 43n - 44),$$

thus by Corollary 2.6,

$$W_e(G) = \frac{1}{2}|E|\ d_e(f|G) = \frac{1}{4}(n - 2)\binom{n}{2}(3n^3 - 18n^2 + 43n - 44),$$

and so we conclude the desired result. □

2.1. Cartesian product and Wiener index

Many interesting classes of graphs arise from simpler graphs via binary operations known as graph products. The Cartesian product of G and H is a graph, denoted by $G \times H$, with vertex set is $V(G \times H) = \{(u, v)|u \in V(G), v \in V(H)\}$. Two vertices (u, x) and (v, y) are adjacent precisely if $u = v$ and $xy \in E(H)$, or $uv \in E(G)$ and $x = y$. The graphs G and H are called factors of the product $G \times H$. The n^{th} power of G with respect to the Cartesian product is denoted as $G^{\times, n}$, that is, $G^{\times, n} = G \times G \times \cdots \times G$ (n factors) [18], [19].

The Wiener index of Cartesian product graphs, studied in [14], [28], is given by the formula $W(G \times H) = |V(G)|^2 W(H) + |V(H)|^2 W(G)$. In this part, we study the Cartesian product of vertex-transitive graphs and then give another formula for this product. This formula is better in vertex-transitive graphs. In [18], we have the following results in relation to this product.

Lemma 2.13. *([18]) Let G and H be two graphs. Then we have:*

1. $|V(G \times H)| = |V(G)||V(H)|$.
2. $|E(G \times H)| = |V(G)||E(H)| + |E(G)||V(H)|$.
3. *If (u, x) and (v, y) are vertices of $G \times H$, then*

$$d((u, x), (v, y)|G \times H) = d(u, v|G) + d(x, y|H).$$

Theorem 2.14. *([18]) A Cartesian product has transitive automorphism group if and only if every factor has transitive automorphism group.*

Corollary 2.15. *([18]) Let G and H be two vertex-transitive graphs. Then $G \times H$ is a vertex-transitive graph.*

Corollary 2.16. *([18]) For any transitive graph G, $G^{\times, n}$ is a vertex-transitive graph.*

Similar to computation of the edge Wiener index for edge-transitive graphs, we have the following Corollary for the Wiener index for vertex-transitive graphs.

Corollary 2.17. *([7]) Let G be a simple transitive graph. Then $W(G) = \frac{1}{2}|V|d(u|G)$ for any $u \in V$.*

Using the above Theorem, we compute the Wiener index of product of vertex-transitive graphs.

Theorem 2.18. *Let G and H be two vertex-transitive graphs. Then for each two vertices $w \in G$ and $x \in H$, we have:*

$$W(G \times H) = \frac{1}{2} \,|\, V(G \times H)|^2 \cdot \left(\frac{d(w\,|G)}{|\, V(G)|} + \frac{d(x\,|H)}{|\, V(H)|} \right). \tag{17}$$

Proof. For each vertex (w, z) in $G \times H$ first we have:

$$\begin{aligned} d((w,x)\,|G \times H) &= \sum_{(z,y) \in V(G \times H)} d((w,x),(z,y)) = \sum_{(z,y) \in V(G \times H)} (d(w,z) + d(x,y)) = \sum_{y \in V(H)} d(w\,|G) + \sum_{z \in V(G)} d(x\,|H) \\ &= |\, V(G \times H)| \left(\frac{d(w\,|G)}{|\, V(G)|} + \frac{d(x\,|H)}{|\, V(H)|} \right). \end{aligned}$$

Now, according to Corollary 2.15, $G \times H$ is a vertex-transitive graph, and so by Corollary 2.17 and the above formula, we have

$$\begin{aligned} W(G \times H) &= \tfrac{1}{2} \,|V(G \times H)| \cdot d((w,x)\,|G \times H) = \tfrac{1}{2} \,|V(G \times H)| \cdot \left(|\, V(G \times H)| \cdot \left(\frac{d(w\,|G)}{|\, V(G)|} + \frac{d(x\,|H)}{|V(H)|} \right) \right) \\ &= \tfrac{1}{2} \,|V(G \times H)|^2 \cdot \left(\frac{d(w\,|G)}{|V(G)|} + \frac{d(x\,|H)}{|V(H)|} \right). \end{aligned}$$

and the Theorem is proved. □

Corollary 2.19. *Let $G_1, G_2, \ldots, G_n$ be vertex-transitive graphs with $V_i = V(G_i)$, $1 \le i \le n$, and $V = V(G)$ such that $G = G_1 \times \cdots \times G_n$. Then*

$$W(G) = \frac{1}{2} \,|V|^2 \sum_{i=1}^{n} \frac{d(u_i\,|G_i)}{|V_i|}, \tag{18}$$

In particular, $W(G^{\times, n}) = \frac{n}{2} \,|V(G)|^{2n-1} \, d(u\,|G)$, for all u in $V(G)$.

Proof. Applying an induction argument and Theorem 2.18, we have:

$$\begin{aligned} W(G) &= W\left((G_1 \times \cdots \times G_{n-1}) \times G_n\right) \\ &= \frac{1}{2} \left[|V(G_1 \times \cdots \times G_{n-1})| \,|V(G_n)|\right] \left[|V(G_n)| \, d(w\,|G_1 \times \cdots \times G_{n-1})\right] \\ &+ \frac{1}{2} \left[|V(G_1 \times \cdots \times G_{n-1})| \,|V(G_n)|\right] \left[|V(G_1 \times \cdots \times G_{n-1})| \, d(u_n\,|G_n)\right] \\ &= \frac{1}{2} \,|V| \left(|V(G_n)| \,|V(G_1 \times \cdots \times G_{n-1})| \cdot \sum_{i=1}^{n-1} \frac{d(u_i\,|G_i)}{|V_i|} \right) \\ &+ \frac{1}{2} \,|V| \left(|V(G_1 \times \cdots \times G_{n-1})| \cdot d(u_n\,|G_n) \right) \\ &= \frac{1}{2} \,|V| \left(|V| \sum_{i=1}^{n-1} \frac{d(u_i\,|G_i)}{|V_i|} + |V| \,\frac{d(u_n\,|G_n)}{|V_n|} \right) \\ &= \frac{1}{2} \,|V|^2 \left(\sum_{i=1}^{n} \frac{d(u_i\,|G_i)}{|V_i|} \right). \end{aligned}$$

□

One of the graphs that is important in graph theory, is hypercube Q_r. In [7], by materials in group theory was showed that this graph is vertex-transitive and its Wiener index was computed. However now, acoording to concepts in graph theory, we can simply show that Q_r is vertex-transitive. So we have the following Theorem:

Theorem 2.20. *The Wiener index of Q_r is equal to $W(Q_r) = r2^{2r-2}$.*

Proof. We have $Q_r = K_2^{\times,r}$ and since K_2 is vertex-transitive, by Corollary 2.16, Q_r is also vertex-transitive. Now according to Corollary 2.19, for all u in $V(Q_r)$ we have $W(K_2^{\times,r}) = \frac{r}{2}|V(K_2)|^{2r-1}d(u|K_2)$ that is equal to $r2^{2r-2}$. □

Note that G and H may be both vertex- and edge-transitive and $G \times H$ need not be edge-transitive. For instance, let $G = K_2$, $H = K_3$. The Cartesian product $K_2 \times K_3$ is vertex-transitive but has six edges in one orbit and three edges in the other. Janez Žerovnik in [29] stated that for any vertex and edge-transitive graph G, the nth Cartesian power $G^{\times,n}$ is symmetric. So both vertex- and edge-transitivity are needed. Note that the path on three vertices P_3 is edge-transitive but $P_3 \times P_3$ is not. Also, $K_2 \times K_3$ is vertex-transitive but $K_2 \times K_3 \times K_2 \times K_3$ is not edge-transitive. However, according to results in the first part of Section 2.1, if G is a simple connected graph and $Aut(G)$ on $E(G)$ has orbits $E_1, E_2, \cdots, E_r$, with representatives $f_1, f_2, \cdots, f_r$, respectively, then we can say that $W_e(G) = \frac{1}{2}\sum_{i=1}^{r}|E_i|d_e(f_i|G)$. According to the above arguments, we express the following example:

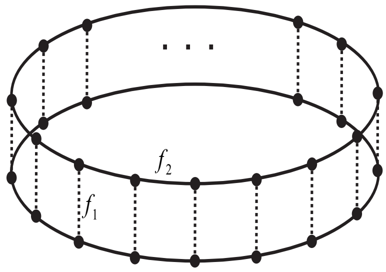

Figure 3: The Cartesian product $K_2 \times C_n$.

Example 2.21. *We compute the edge Wiener index of Cartesian product $K_2 \times C_n$. This graph is vertex-transitive but has n edges in one orbit and $2n$ edges in the other. Let f_1 and f_2 be representatives of two orbits E_1 and E_1, respectively. Thus when n is even we have:*

$$\begin{aligned} W_e(K_2 \times C_n) &= \tfrac{1}{2}\left(|E_1|\ d_e(f_1|K_2 \times C_n) + |E_2|\ d_e(f_2|K_2 \times C_n)\right) \\ &= \tfrac{1}{2}\left(n\left(\tfrac{3n^2}{4} + 2n - 1\right) + 2n\left(\tfrac{3n^2}{4} + \tfrac{3n}{2} + 1\right)\right) \\ &= \tfrac{1}{2}n\left(\tfrac{9n^2}{4} + 5n + 1\right). \end{aligned}$$

Finally, we close the paper with a discussion of several open problems. In this paper, we discussed about the edge Wiener index of edge-transitive graphs and computed explicit formul for the Wiener index of the Cartesian product $G \times H$ in vertex-transitive graphs. There are still many composite graphs not covered by our approach. It would be interesting to find explicit formula for the edge Wiener index of the Cartesian product of edge-transitive graphs. Besides the Cartesian product, there are also other types of operations resulting in composite graphs, such as: the strong product, the lexicographic product and also corona. Since all of them have interesting properties for transitive graphs, it could be useful to investigate the behavior of graph invariants for these composite graphs in transitive graphs.

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References

[1] M. Azari, A. Iranmanesh, A. Tehranian, A method for calculating an edge version of the Wiener number of a graph operation, Utilitas Mathematica 87 (2012) 151–164.

[2] M. Azari, A. Iranmanesh, Computation of the edge Wiener indices of the sum of graphs, Ars. Combin. 100 (2011) 113–128.
[3] N. Biggs, Algebraic Graph Theory, Cambridge University Press, 1974.
[4] I. Z. Bouwer, On edge but not vertex transitive regular graphs, J. Comb. Theory (B) 12 (1972) 32–40.
[5] H. S. M. Coxeter, R. Frucht, D. L. Powers, Zero-Symmetric Graphs, Academic Press, New York, 1981.
[6] P. Dankelmann, I. Gutman, S. Mukwembi, H. C. Swart, The edge- Wiener index of a graph, em Discrete Math. 309(10) (2009) 3452-3457.
[7] M. R. Darafsheh, Computation of Topological Indices of Some Graphs. Acta. Appl. Math. 110 (2010) 1225–1235.
[8] M.R. Darafsheh, The Wiener, Szeged and PI-index of the triangle graph, In Lee See Keong and Hailiza Kamarul Haili (eds) 5th Asian mathematica conference proceedings (vol.1) June 2009, pp. 8189, ISBN: 978-967-5417-53-5.
[9] A. A. Dobrynin, R. Entringer, I. Gutman, Wiener index of trees: theory and applications, Acta Appl. Math. 66 (2001) 211–249.
[10] A. A. Dobrynin, I. Gutman, S. Klavžar, P. Žigert, Wiener index of hexagonal systems, Acta Appl. Math. 72 (2002) 247–294.
[11] T. Došlić, Vertex-Weighted Wiener Polynomials for Composite Graphs, Ars Mathematica Contemporanea 1 (2008) 66–80.
[12] R. C. Entringer, D. E. Jackson, D. A. Snyder, Distance in graphs, Czech. Math. J. 26 (1976) 283–296.
[13] D. Godsil, G. Royle, Algebraic graph theory, Springer, New York, 2001.
[14] A. Graovac, T. Pisanski, On the Wiener index of a graph, em J. Math. Chem. 8 (1991) 53–62.
[15] I. Gutman, A new method for the calculation of the Wiener number of acyclic molecules, J. Mol. Struct. (Theochem) 285 (1993) 137–142.
[16] I. Gutman, Calculation the Wiener number: the Doyle-Graver method, J. Serb. Chem. Soc. 58 (1993) 745–750.
[17] I. Gutman, Y. N. Yeh, S. L. Lee, Y. L. Luo, Some recent results in the theory of the Wiener number, Indian J. Chem. 32A (1993) 651–661.
[18] R. Hammack, W. Imrich, S. Klavžar, Handbook of Product Graphs, (Second Edition), CRC Press, 2011.
[19] W. Imrich, S. Klavžar, Product graphs: structure and recognition, John Wiley & Sons, New York, USA, 2000.
[20] A. Iranmanesh, I. Gutman, O. Khormali, A. Mahmiani, The edge versions of Wiener index, MATCH Commun. Math. Computt. Chem. 61 (2009) 663–672.
[21] A. Iranmanesh, A. S. Kafrani, Computation of the first edge-Wiener index of $TUC_4C_8(S)$ nanotube, MATCH Commun. Math. Comput. Chem. 62(2) (2011) 311–352.
[22] A. Iranmanesh, A. S. Kafrani, O. Khormali, A New Version of Hyper-Wiener Index, MATCH Commun. Math. Comput. Chem. 65(1) (2011) 113–122.
[23] M. H. Khalifeh, H. Yousefi-Azari, A.R. Ashrafi, S. G. Wagner, Some new results on distance-based graph invariants, European J. Combin. 30 (2009) 1149–1163.
[24] H. Liu, X. F. Pan, On the Wiener index of trees with fixed diameter, MATCH Commun. Math. Comput. Chem. 60 (2008) 85–94.
[25] D. Stevanovic, Maximizing Wiener index of graphs with fixed maximum degree, MATCH Commun. Math. Comput. Chem. 60 (2008) 71–83.
[26] G. Wagner, A class of trees and its Wiener index, Acta Appl. Math. 91 (2006) 119–132.
[27] H. Wiener, Structural determination of paraffin boiling points, J. Am. Chem. Soc. 69 (1947) 17–20.
[28] Y. N. Yeh, I. Gutman, On the sum of all distances in composite graphs, Discrete Math. 135 (1994) 359–365.
[29] J. Žerovnik, Szeged Index of Symmetric Graphs, J. Chem. Inf. Comput. Sci. 39 (1999) 77–80.