

## Computational Method Based on Bernstein Operational Matrices for Multi-Order Fractional Differential Equations

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**Abstract.** In this paper, the Bernstein operational matrices are used to obtain solutions of multi-order fractional differential equations. In this regard we present a theorem which can reduce the nonlinear fractional differential equations to a system of algebraic equations. The fractional derivative considered here is in the Caputo sense. Finally, we give several examples by using the proposed method. These results are then compared with the results obtained by using Adomian decomposition method, differential transform method and the generalized block pulse operational matrix method. We conclude that our results compare well with the results of other methods and the efficiency and accuracy of the proposed method is very good.

### 1. Introduction

Fractional differential operators have a long history, having been mentioned by Leibniz in a letter to L'Hopital in 1695. A history of the development of fractional differential operators can be found in [15] and [18]. One of the most recent works on the subject of fractional calculus, i.e., the theory of derivatives and integrals of fractional (non-integer) order, is the book of Podlubny [19], which deals principally with fractional differential equations. Today many works have been done on fractional calculus as it is evident in the literature. See for example [1–8, 12–25]. Fractional differential equations (FDEs) have been of considerable interest in the recent years [2, 15, 18, 19, 21]. It is known that Laplace, Fourier or Mellin transforms can be extended to solve linear FDEs [19]. For nonlinear FDEs, however, one mainly resorts to numerical methods [4–6].

In this paper, we focus on providing numerical solutions to multi-order fractional differential equations where the highest order derivative may be greater than one. These problems arise, for instance, in the Basset equation [16] and the Bagley -Torvik equation [24].

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We consider the nonlinear multi-order fractional differential equation

$$D^\alpha y(t) - \sum_{j=1}^n a_j(t) D^{\beta_j} y(t) + \sum_{i=1}^k q_i y^i(t) = g(t), \quad y^{(i)}(0) = 0, \quad i = 0, \dots, [\alpha] - 1, \quad (1)$$

where  $k, n \in \mathbb{N}$  and  $q_i$  ( $i = 1, \dots, k$ ) are constants,  $\alpha > \beta_1 > \beta_2 > \dots > \beta_n > 0$  and  $\{a_j(t)\}_{j=1}^n, g(t)$  (as input signal) are known functions. Also, here  $D^\alpha$  denotes the Caputo fractional derivative of order  $\alpha$ . This problem has been considered earlier in the literature by various researchers [1-3,6-8,10,14,16,17,19-22]. For example, Arikoglu et al. [1] employed differential transform method, Jafari et al. used homotopy analysis method [7], Fractional sub-equation [9, 10], Momani [17] employed Adomian decomposition method, Sweilam et al. used variational iteration method [11, 23] and Li et al. [13] applied the generalized block pulse operational matrix to obtain approximate solutions of (1).

The purpose of this paper is to use Bernstein operational matrices to obtain solution of the multi-order fractional differential equation (1). To the best of our knowledge this is the first time that the Bernstein operational matrices are used to obtain solutions of the multi-order fractional differential equations. First we present a new theorem which can reduce the nonlinear multi-order fractional differential equations to a system of algebraic equations. The fractional derivative considered here is in the Caputo sense. The paper is organized as follows: In Section 2 we present some notations, definitions and results which will be used later in the paper. Section 3 deals with the statement and proof of the main theorem, which we utilize to reduce the nonlinear multi-order fractional differential equations to a system of algebraic equations. In Section 4 we give some examples, which will illustrate the efficiency and accuracy of the proposed method. Finally, concluding remarks will be presented in Section 5.

## 2. Preliminaries

In this section, we present some notations, definitions, Corollaries and Lemmas which are used latter in this paper. For details, see for example [2, 12, 14, 15, 18, 19, 21].

**Definition 2.1.** A real function  $f(t)$ ,  $t > 0$  is said to be in the space  $C_\alpha$ ,  $\alpha \in \mathbb{R}$  if there exists a real number  $p(> 0)$ , such that  $f(t) = t^p f_1(t)$  where  $f_1(t) \in C[0, \infty)$ . Clearly  $C_\alpha \subset C_\beta$  if  $\beta \leq \alpha$ .

**Definition 2.2.** A function  $f(t)$ ,  $t > 0$  is said to be in the space  $C_\alpha^m$ ,  $m \in \mathbb{N} \cup \{0\}$ , if  $f^{(m)} \in C_\alpha$ .

**Definition 2.3.** The (left sided) Riemann-Liouville fractional integral of order  $\mu > 0$ , of a function  $f \in C_\alpha$ ,  $\alpha \geq -1$  is defined as:

$$I^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t \frac{f(x)}{(t-x)^{1-\mu}} dx, \quad \mu > 0, \quad t > 0,$$

$$I^0 f(t) = f(t).$$

**Definition 2.4.** The (left sided) Caputo fractional derivative of  $f \in C_{-1}^m$ ,  $m \in \mathbb{N} \cup \{0\}$ , is defined as:

$$D^\mu f(t) = \begin{cases} I^{m-\mu} f^{(m)}(t) & m-1 < \mu < m, \quad m \in \mathbb{N}, \\ \frac{d^m}{dt^m} f(t) & \mu = m. \end{cases}$$

Note that

$$\begin{aligned} (i) \quad I^\mu t^\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\mu+1)} t^{\gamma+\mu}, & \mu > 0, \quad \gamma > -1, \quad t > 0, \\ (ii) \quad I^\mu D^\mu f(t) &= f(t) - \sum_{k=0}^{(m-1)} f^{(k)}(0^+) \frac{t^k}{k!}, & m-1 < \mu \leq m, \quad m \in \mathbb{N}, \\ (iii) \quad D^\beta f(t) &= I^{\alpha-\beta} D^\alpha f(t). \end{aligned} \quad (2)$$

**Definition 2.5.** The Bernstein polynomials (BPs) of degree  $m$  on the interval are defined by  $B_{i,m}(t) = \binom{m}{i} t^i (1-t)^{(m-i)}$ , ( $i = 0, 1, \dots, m$ ).

**Corollary 2.6.** The set  $\{B_{0,m}(x), B_{1,m}(x), \dots, B_{m,m}(x)\}$  is a complete basis in Hilbert space  $L^2[0, 1]$  and polynomials of degree  $m$  can be expanded in terms of linear combination of  $B_{i,m}(x)$  ( $i = 0, \dots, m$ ) as

$$P(x) = \sum_{i=0}^m c_i B_{i,m}(x). \tag{3}$$

If we define  $\Phi_m(x) = [B_{0,m}(x), B_{1,m}(x), \dots, B_{m,m}(x)]^T$ , then we can write

$$\Phi_m(x) = A T_m(x), \tag{4}$$

where  $T_m(x) = [1, x, x^2, \dots, x^m]^T$  and  $A = (a_{i,j})_{i,j=1}^{m+1}$  is an upper triangular matrix with

$$a_{i+1,j+1} = \begin{cases} (-1)^{j-i} \binom{m}{i} \binom{m-i}{j-i}, & i \leq j \\ 0, & i > j, \end{cases}$$

$$i, j = 0, 1, \dots, m.$$

**Lemma 2.7.** ([12]) Let  $L^2[0, 1]$  be a Hilbert space with the inner product  $\langle f, g \rangle = \int f(x)g(x) dx$  and  $S_m = \text{Span} \{B_{0,m}(x), B_{1,m}(x), \dots, B_{m,m}(x)\}$ ,  $y(x) \in L^2[0, 1]$ . Then, we can find the unique vector  $c = [c_0, c_1, \dots, c_m]^T$  such that

$$y(x) \approx \sum_{i=0}^m c_i B_{i,m}(x) = c^T \Phi_m(x). \tag{5}$$

**Corollary 2.8.** In Lemma 2.7, we have  $c^T = \langle f, \Phi_m \rangle Q^{-1}$ , such that

$$\langle f, \Phi_m \rangle = \int_0^1 f(x) \Phi_m(x)^T dx = [\langle f, B_{0,m} \rangle, \dots, \langle f, B_{m,m} \rangle]^T$$

and  $Q = (q_{i,j})_{i,j=1}^{m+1}$  is given by

$$q_{i,j} = \int_0^1 B_{i,m}(x) B_{j,m}(x) dx = \frac{\binom{m}{i} \binom{m}{j}}{(2m+1) \binom{2m}{i+j}}, \quad i, j = 0, \dots, m. \tag{6}$$

### 3. Bernstein operational matrices for multi-order fractional differential equations

In this section, we first state the main theorem. The proof of this theorem involves four steps. The whole idea of the theorem is to reduce the nonlinear multi-order fractional differential equations to a system of nonlinear algebraic equations.

We first note that in view of (3) we can rewrite (1) as

$$D^\alpha y(t) = \sum_{j=1}^n a_j(t) I^{\alpha-\beta_j} D^\alpha y(t) + \sum_{i=1}^k q_i (I^\alpha D^\alpha y(t))^i + g(t),$$

by taking  $u(t) = D^\alpha y(t)$  we have the following fractional integral equation:

$$u(t) = \sum_{j=1}^n a_j(t) I^{\alpha-\beta_j} u(t) + \sum_{i=1}^k q_i (I^\alpha u(t))^i + g(t). \quad (7)$$

We can now state our main theorem.

**Theorem 3.1.** Suppose  $\Phi_m(t) = [B_{0,m}(t), \dots, B_{m,m}(t)]^T$ , then we can obtain the new operational matrix  $F_\alpha$  by using Bernstein Polynomials, for fractional integration where  $I^\alpha \Phi_m(t) \approx F_\alpha \Phi_m(t)$ . Also, we can reduce the problem (7) to the nonlinear system of algebraic equation in term of the vector  $C$  as follows:

$$C = \sum_{j=1}^n (\hat{A}_j F_{\alpha-\beta_j}^T C) + \sum_{i=1}^k (q_i \hat{C}_\alpha^{i-1} C) + G. \quad (8)$$

Then by solving this nonlinear system we obtain  $u_m(t) = C^T \Phi_m(t)$  and  $y_m(t) = C^T F_\alpha \Phi_m(t)$  that are the approximate solutions for the problem (6) and (1), respectively.

*Proof.* We need the following steps for proving the theorem:

Step 1: We obtain the Bernstein operational matrix of product.

Step 2: We get the approximate function by Bernstein Polynomials.

Step 3: The Bernstein operational matrix prevails for fractional integration.

Step 4: Finally, we get the reduced nonlinear system of algebraic equation for the problem.

Below we derive the above steps for proving the main theorem.

#### Step 1:

Suppose that  $c = c_{(m+1) \times 1}$  is an arbitrary vector. Now, we obtain the matrix  $\hat{C}_{(m+1) \times (m+1)}$  where

$$c^T \Phi_m(t) \Phi_m(t)^T \approx \Phi_m(t)^T \hat{C}. \quad (9)$$

From (8) we have

$$\begin{aligned} c^T \Phi_m(t) \Phi_m(t)^T &= c^T \Phi_m(t) T_m(t)^T A^T = [c^T \Phi_m(t), t c^T \Phi_m(t), \dots, t^m c^T \Phi_m(t)] A^T \\ &= \left[ \sum_{i=0}^m c_i B_{i,m}(t), \sum_{i=0}^m t c_i B_{i,m}(t), \dots, \sum_{i=0}^m t^m c_i B_{i,m}(t) \right] A^T. \end{aligned}$$

Now, we approximate all functions  $t^k B_{i,m}(t)$  in terms of Bernstein basis. Thus we define  $e_{k,i} = [e_{k,i}^0, e_{k,i}^1, \dots, e_{k,i}^m]^T$ , then by Lemma 2.7 we can write

$$t^k B_{i,m}(t) \approx e_{k,i} \Phi_m(t), \quad i, k = 0, 1, \dots, m.$$

So we get

$$\begin{aligned} e_{k,i} &= Q^{-1} \left( \int_0^1 t^k B_{i,m}(t) \Phi_m(t) dt \right) \\ &= Q^{-1} \left[ \int_0^1 t^k B_{i,m} B_{0,m}(t) dt, \int_0^1 t^k B_{i,m} B_{1,m}(t) dt, \dots, \int_0^1 t^k B_{i,m} B_{m,m}(t) dt \right]^T \end{aligned}$$

$$= \frac{Q^{-1} \binom{m}{i}}{2m+k+1} \left[ \frac{\binom{m}{0}}{\binom{2m+k}{i+k}}, \frac{\binom{m}{1}}{\binom{2m+k}{i+k+1}}, \dots, \frac{\binom{m}{m}}{\binom{2m+k}{i+k+m}} \right]^T, \quad i, k = 0, 1, \dots, m.$$

Then, we have

$$\begin{aligned} \sum_{i=0}^m c_i t^k B_{i,m}(t) &\simeq \sum_{i=0}^m c_i \left( \sum_{j=0}^m e_{k,i}^j B_{j,m}(t) \right) = \sum_{j=0}^m B_{j,m}(t) \sum_{i=0}^m c_i e_{k,i}^j = \Phi_m(t)^T \left[ \sum_{i=0}^m c_i e_{k,i}^0, \sum_{i=0}^m c_i e_{k,i}^1, \dots, \sum_{i=0}^m c_i e_{k,i}^m \right]^T \\ &= \Phi_m(t)^T [e_{k,i}, e_{k,i}, \dots, e_{k,i}] c = \Phi_m(t)^T V_{k+1} c, \end{aligned}$$

where  $V_{k+1}$  ( $k = 0, 1, \dots, m$ ) is an  $(m+1) \times (m+1)$  matrix that has vectors  $e_{k,i}$  ( $i = 0, 1, \dots, m$ ) for each columns. If we define  $\bar{C} = [V_1 c, V_2 c, \dots, V_m c]$ , then we get

$$c^T \Phi_m(t) \Phi_m(t)^T \simeq \Phi_m(t)^T \bar{C} A^T, \tag{10}$$

and therefore we obtain the operational matrix of product  $\hat{C} = \bar{C} A^T$ .

**Step 2:**

Now, by using (9) and mathematical induction, we can obtain approximation for  $y(t)^k$  ( $k \in \mathbb{N}$ ) as follows:

For  $k = 1$ , by (9) we have  $y(t) \simeq c^T \Phi_m(t)$ . Also for  $k = 2$ , by (9) we obtain  $y(t)^2 \simeq c^T \Phi_m(t) \Phi_m(t)^T c \simeq \Phi_m(t)^T \hat{C} c$ , where  $\hat{C}$  is operational matrix of product.

Then for  $k = 3$ , we get  $y(t)^3 \simeq c^T \Phi_m(t) \Phi_m(t)^T \hat{C}^2 c$ .

So, by mathematical induction we can write

$$y(t)^k \simeq c^T \Phi_m(t) \Phi_m(t)^T \hat{C}^{k-2} c \simeq \Phi_m(t)^T \tilde{C}_k, \tag{11}$$

where  $\tilde{C}_k = \hat{C}^{k-1} c$ .

**Step 3:**

In this step, we want to obtain the operational matrix for the fractional integration. We can write

$$I^\alpha \Phi_m(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * \Phi_m(t), \quad 0 \leq t \leq 1,$$

where  $*$  denotes the convolution product and

$$t^{\alpha-1} * \Phi_m(t) = \left[ t^{\alpha-1} * B_{0,m}(t), t^{\alpha-1} * B_{1,m}, \dots, t^{\alpha-1} * B_{m,m}(t) \right]^T.$$

From (8), we have

$$t^{\alpha-1} * \Phi_m(t) = t^{\alpha-1} * (A T_m(t)) = A \left( t^{\alpha-1} * T_m(t) \right).$$

using (2) we get

$$\begin{aligned} t^{\alpha-1} * T_m(t) &= [t^{\alpha-1} * 1, t^{\alpha-1} * t, \dots, t^{\alpha-1} * t^m]^T = \Gamma(\alpha) [I^\alpha 1, I^\alpha t, \dots, I^\alpha t^m]^T \\ &= \Gamma(\alpha) \left[ \frac{0!}{\Gamma(\alpha+1)} t^\alpha, \frac{1!}{\Gamma(\alpha+2)} t^{\alpha+1}, \dots, \frac{m!}{\Gamma(\alpha+m+1)} t^{\alpha+m} \right]^T = \Gamma(\alpha) D \bar{T}, \end{aligned}$$

where  $D = (d_{i,j})_{i,j=1}^{m+1}$  and  $\bar{T}_{(m+1) \times (m+1)}$  are given by

$$d_{i,j} = \begin{cases} \frac{i!}{\Gamma(\alpha+i+1)} & i = j, \\ 0 & i \neq j, \end{cases} \quad i, j = 0, 1, \dots, m, \quad \bar{T} = [t^\alpha, t^{\alpha+1}, \dots, t^{\alpha+m}]^T.$$

Now, we need to approximate  $t^{\alpha+i}$  ( $i = 0, 1, \dots, m$ ). In this case we have

$$t^{\alpha+i} \simeq E_i^T \Phi_m(t),$$

where  $E_i = [E_{i,0}, E_{i,1}, \dots, E_{i,m}]^T$ ,  $i = 0, 1, \dots, m$ . We thus obtain

$$\begin{aligned} E_i &= Q^{-1} \left( \int_0^1 t^{\alpha+i} \Phi_m(t) dt \right) \\ &= Q^{-1} \left[ \int_0^1 t^{\alpha+i} B_{0,m}(t) dt, \int_0^1 t^{\alpha+i} B_{1,m}(t) dt, \dots, \int_0^1 t^{\alpha+i} B_{m,m}(t) dt \right]^T = Q^{-1} \bar{E}_i, \end{aligned}$$

where  $\bar{E}_i = [\bar{E}_{i,0}, \bar{E}_{i,1}, \dots, \bar{E}_{i,m}]^T$  and

$$\bar{E}_{i,j} = \int_0^1 t^{\alpha+i} B_{j,m}(t) dt = \frac{m! \Gamma(i+j+\alpha+1)}{j! \Gamma(i+m+\alpha+2)}, \quad i, j = 0, 1, \dots, m.$$

Now, we suppose that  $E$  is an  $(m+1) \times (m+1)$  matrix that has vector  $Q^{-1} \bar{E}_i$ ,  $i = 0, 1, \dots, m$ , for  $i$ -th column. Finally, we obtain

$$I^\alpha \Phi_m(t) \simeq F_\alpha \Phi_m(t), \tag{12}$$

where  $F_\alpha = ADE^T$ .

**Step 4:** By using (5), the input signal  $g(t)$ ,  $u(t)$ , and  $a_j(t)$  ( $j = 0, 1, \dots, k$ ) in (7) may be expanded as follows:

$$g(t) \simeq G^T \Phi_m(t), \tag{13}$$

$$u(t) \simeq C^T \Phi_m(t), \tag{14}$$

$$a_j(t) \simeq A_j^T \Phi_m(t), \tag{15}$$

where  $G$ ,  $A_j$  are known  $(m+1) \times 1$  column vectors and  $C$  is an unknown  $(m+1) \times 1$  column vector. From (12) and (13), we have

$$I^{\alpha-\beta_j} u(t) \simeq I^{\alpha-\beta_j} (C^T \Phi_m(t)) = C^T I^{\alpha-\beta_j} \simeq C^T F_{\alpha-\beta_j} \Phi_m(t). \tag{16}$$

Now, substituting (13)-(16) and  $y(t) \simeq C_\alpha^T \Phi_m(t)$  in (1) and using (11) we have

$$c^T \Phi_m(t) = \sum_{j=1}^n (A_j^T \Phi_m(t) \Phi_m(t)^T F_{\alpha-\beta_j}^T C) + \sum_{i=1}^k q_i (C_\alpha^T \Phi_m(t))^i + G^T \Phi_m(t),$$

where  $C_\alpha^T = C^T F_\alpha$ . If  $\hat{C}_\alpha$  is operational matrix of product that is obtained in the previous step, then we have

$$c^T \Phi_m(t) = \sum_{j=1}^n (A_j^T \Phi_m(t) \Phi_m(t)^T F_{\alpha-\beta_j}^T C) + \sum_{i=1}^k q_i \Phi_m(t)^T \hat{C}_\alpha^{i-1} C_\alpha + G^T \Phi_m(t).$$

Also, by (13) we get

$$c^T \Phi_m(t) = \sum_{j=1}^n (\Phi_m(t)^T \hat{A}_j F_{\alpha-\beta_j}^T C) + \sum_{i=1}^k q_i \Phi_m(t)^T \hat{C}_\alpha^{i-1} C_\alpha + G^T \Phi_m(t).$$

Finally, we obtain the following nonlinear system of algebraic equation

$$C = \sum_{j=1}^n \hat{A}_j F_{\alpha-\beta_j}^T C + \sum_{i=1}^k q_i C_\alpha^{i-1} C_\alpha + G.$$

Now, by solving this system we can obtain the vector  $C$ . So, we can get the approximate solution  $u_m(t) = C^T \Phi_m(t)$  for problem (7) and then we obtain the approximate solution  $y_m(t)$  for problem (1) as

$$y_m(t) \simeq I^\alpha u_m(t) = C^T I^\alpha \simeq C^T F_\alpha \Phi_m(t).$$

This completes the proof of the theorem.  $\square$

#### 4. Numerical examples

In this section we consider some examples and apply this method to find numerical solution of multi-order fractional differential equations. We define  $y_m(t)$  as the approximate solution of (1).

**Example 4.1.** Consider the following nonlinear fractional differential equation

$$D^2 y(t) - \sqrt[5]{t} D^{0.75} y(t) - y(t)^2 = 8\pi t^2 - \frac{2048}{585} \sqrt{2} \Gamma\left(\frac{3}{4}\right) t^{\frac{69}{20}} - \frac{4}{9} \pi^2 t^8, \quad 0 < t \leq 1,$$

with the initial conditions

$$y(0) = 0, \quad y'(0) = 0.$$

It can easily be verified that the exact solution of this problem is  $y(t) = \frac{2\pi}{3} t^4$ . The absolute error for this example is given in Table 1 and Fig 1 shows the behavior of  $y_m(t)$  for  $m = 2, 3$  and 4.

Table 1. The absolute error in Example 4.1.

| t   | Absolute error |
|-----|----------------|
| 0.1 | 1.4719710-6    |
| 0.3 | 3.0411110-6    |
| 0.5 | 3.3678210-6    |
| 0.7 | 7.3413510-7    |
| 0.9 | 5.6348110-6    |

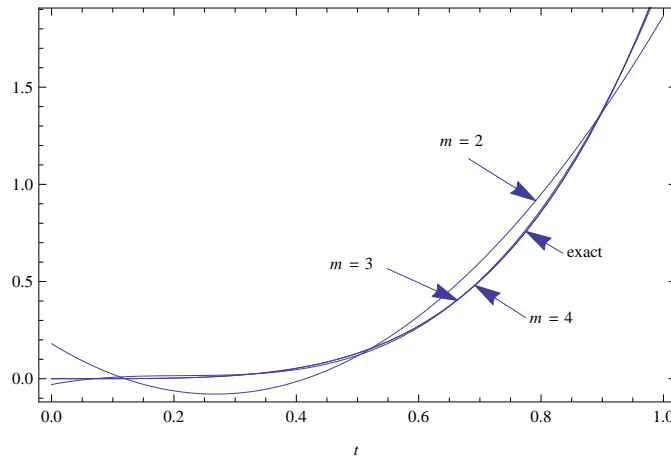


Figure 1. Approximate solution  $y_m(t)$  for  $m = 2, 3, 4$  and exact solution of Example 4.1.

**Example 4.2.** Consider the nonlinear fractional differential equation [1, 13, 22]

$$D^\alpha y(t) = y(t)^2 + 1, \quad 0 < t \leq 1, \quad n - 1 < \alpha \leq n,$$

with the initial conditions

$$y^{(k)}(0) = 0, \quad k = 0, 1, \dots, n - 1.$$

This example has been solved by using Adomian decomposition method (ADM) in [22], fractional differential transform method (FDTM) in [1] and block pulse operational matrix (BPOM) in [13]. Our results for  $\alpha = 1.5$  and  $\alpha = 2.5$  are compared with the results obtained by the three other methods and are given in Tables 2 and 3 respectively. It can be seen that the results obtained by our method are in good agreement with the results in Refs. [1, 13, 22]. Thus we see that our method is very effective and accurate. Also, Fig 2 below shows the plots of  $y_m(t)$  for different values of  $\alpha$ .

Table 2. Numerical results for  $\alpha = 1.5$  in Example 4.2

| t   | BPs                 | ADM [19]  | FDTM [1] | BPOM [10] |
|-----|---------------------|-----------|----------|-----------|
| 0.1 | 0.02377872315717216 | 0.0237904 | 0.23790  | 0.023800  |
| 0.2 | 0.06733597110753813 | 0.06733   | 0.067330 | 0.067335  |
| 0.3 | 0.12388621731117708 | 0.123896  | 0.123896 | 0.123900  |
| 0.4 | 0.19137284173893865 | 0.191362  | 0.191362 | 0.191368  |
| 0.5 | 0.2688507671446437  | 0.268856  | 0.268856 | 0.268862  |
| 0.6 | 0.3562348883374099  | 0.356238  | 0.356238 | 0.356244  |
| 0.7 | 0.4539578407504274  | 0.45395   | 0.453950 | 0.453956  |
| 0.8 | 0.562998749586631   | 0.563007  | 0.563007 | 0.563014  |
| 0.9 | 0.6850657717229061  | 0.685056  | 0.685056 | 0.685067  |
| 1   | 0.822540114146267   | 0.822511  | 0.822509 | 0.822525  |



Table 3. Numerical results for  $\alpha = 2.5$  in Example 4.2

| $t$ | BPs                   | ADM [22]   | FDTM [1] |
|-----|-----------------------|------------|----------|
| 0.1 | 0.0009518229596908563 | 0.00095153 | 0.000952 |
| 0.2 | 0.005382525957573449  | 0.00538269 | 0.005383 |
| 0.3 | 0.014833245151805435  | 0.014833   | 0.014833 |
| 0.4 | 0.03044962922797871   | 0.0304499  | 0.030450 |
| 0.5 | 0.053196681902190235  | 0.0531966  | 0.053197 |
| 0.6 | 0.0839245775747845    | 0.0839245  | 0.083925 |
| 0.7 | 0.12341162875028754   | 0.123412   | 0.123412 |
| 0.8 | 0.17239130998668684   | 0.172391   | 0.172391 |
| 0.9 | 0.23157382380896946   | 0.231574   | 0.231574 |
| 1   | 0.30167562488464017   | 0.301676   | 0.301676 |

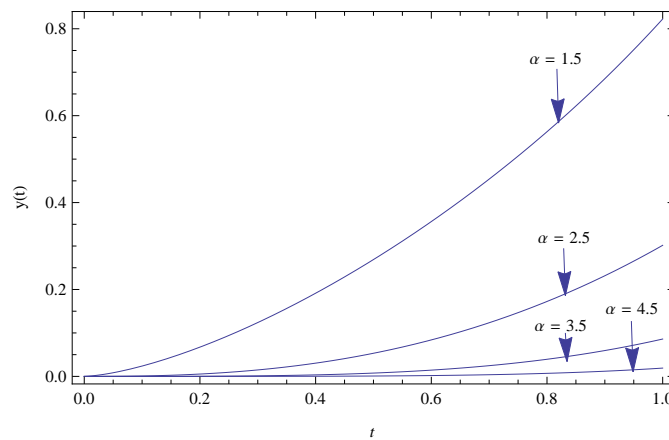


Figure 2. Plot of  $y_{10}$  for different  $\alpha$  in Example 4.2.

**Example 4.3.** We consider the nonlinear multi-order fractional differential equation

$$D^{2.5}y(t) + D^{1.25}y(t) + y(t) + y^2(t) - y^3(t) = 12\sqrt{\frac{t}{\phi}} + \frac{32t^{\frac{7}{4}}}{7\Gamma(\frac{3}{4})} + t^3 + t^6 - t^9, \quad 0 < t \leq 1,$$

with the initial conditions  $y(0) = 0, y'(0) = 0, y''(0) = 0$ .

The exact solution of the above problem is  $y(t) = t^3$ . The results obtained by using BPs are reported in Table 4 and Fig 3 shows the plots of  $y_m(t)$  for  $m = 2, 3$  and 4.

Table 4. Our results for Example 4.1.

| $t$ | Absolute error |
|-----|----------------|
| 0.1 | 1.5698510-7    |
| 0.3 | 8.4488210-7    |
| 0.5 | 5.1554710-7    |
| 0.7 | 1.9067410-7    |
| 0.9 | 9.6385710-7    |

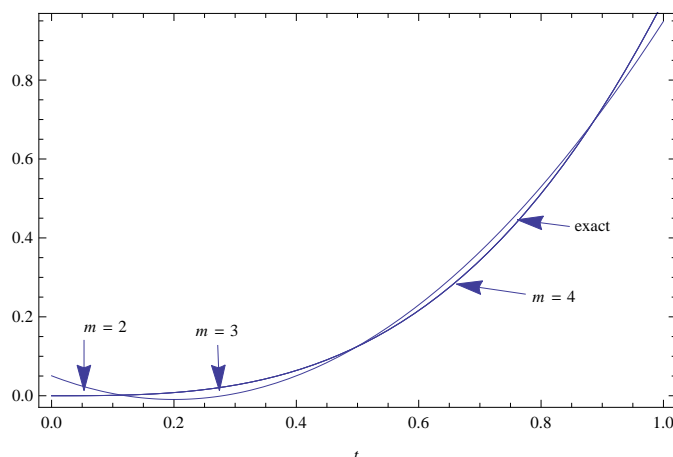


Figure 3. Approximate solution  $y_m(t)$  for  $m = 2, 3, 4$  and exact solution of Example 4.3.

## 5. Conclusion

In this paper Bernstein operational matrices were used to obtain solutions of multi-order fractional differential equations. We understand that the first time that Bernstein Polynomials operational matrices were used in determining solutions of multi-order fractional differential equations. A new theorem which can reduce the nonlinear fractional differential equations to a system of algebraic equations was presented first. Then, several examples were presented in which this new method was used. It can be seen that our results were in good agreement with the results obtained in the literature by other methods. This proved the efficiency and accuracy of the proposed method.

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