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Band and Semilattice Decompositions of $\mathcal{X}_k$-Simple Ordered Semigroups

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Abstract. In this paper, we define various types of k-regularity of ordered semigroups and using generalizations of some Green's relations and its properties, are described the structure of band and complete semilattice of $\mathcal{L}_k$-simple, $\mathcal{I}_k$-simple, $\mathcal{B}_k$-simple, $\mathcal{H}_k$-simple ordered semigroups.

1. Introduction

The band and semilattice congruences play an important role in studying the decomposition of semigroups without order. Semigroups having a decomposition into a band and semilattice of some special type semigroups have been studied in many papers (cf., for example, [1], [2], [3], [4], [5], [11]). In the case of ordered semigroups, the same role is played by the complete semilattice and band congruences. The complete semilattice and band congruences have been also proved to be useful in studying the structure, especially the decomposition of ordered semigroups (cf., for example, [8]). But the concept of band congruence without order is different from the case in an ordered semigroup. The characterization of bands of ordered semigroups of a given type $\mathcal{T}$ has been considered in [9], where it is shown the way that band congruences on an ordered semigroup S of type $\mathcal{T}$ decompose the ordered semigroup S into ordered subsemigroups of S of type $\mathcal{T}$. In this paper, we define various types of k-regularity of ordered semigroups and using generalizations of some Green's relations and its properties, are described the structure of band and complete semilattice decomposition of $\mathcal{L}_k$-simple ($\mathcal{I}_k$-simple, $\mathcal{B}_k$-simple, $\mathcal{H}_k$-simple) ordered semigroups.

2. Preliminaries

Throughout this paper, Z^+ will denote the set of all positive integers. Let $(S, \cdot, \leq)$ be an ordered semigroup, for $A, B \subseteq S$, let $AB := \{ab | a \in A, b \in B\}$. Let H be a nonempty subset of S, we denote: $(H] := \{t \in S | t \leq h$ for some $h \in H\}$. Then we have $H \subseteq (H]$ and $A \subseteq B$ implies $(A] \subseteq (B]$, for any nonempty subsets A, B of S. We denote by $I(a), L(a), R(a)$ and $B(a)$ the ideal, the left ideal, the right ideal and the bi-ideal of S generated by an element a of S, respectively. One can easily prove that: $L(a) = (a \bigcup Sa], R(a) = (a \bigcup aS], I(a) =$

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$(a \bigcup aS \bigcup Sa \bigcup SaS](\forall a \in S)$[6], $B(a) = (a \bigcup a^2 \bigcup aSa]$ [13]. By $\mathcal{L}$, $\mathcal{R}$, $\mathcal{I}$ and $\mathcal{H}$ we denote the well known Green's relations. We define the relation $\mathcal{B}$ as follows: $(a,b) \in \mathcal{B}$ if and only if $B(a) = B(b)$ [13].

Let $(S, \cdot, \leq)$ be an ordered semigroup. S is called a *band* if every element of S is idempotent, i.e. $a^2 = a$ for every $a \in S$. As in [9], a congruence σ on S having the properties $(a, a^2) \in \sigma$ and $(ab, ba) \in \sigma$ for all $a, b \in S$, is called a *semilattice congruence* on S. A semilattice congruence σ on S is called *complete* if $a \leq b$ implies $(a, ab) \in \sigma$. A congruence σ on S is called a *band congruence* if $(a, a^2) \in \sigma$ and $a \leq b$ implies $(a, ab) \in \sigma$ and $(a, ba) \in \sigma$. One can easily see that a complete semilattice congruence is a band congruence. An ordered semigroup S is called a *band* (*complete semilattice*) *of semigroups of a given type, say* $\mathcal{T}$, if there exists a band (complete semilattice) congruence σ on S such that the σ-class $(x)_\sigma$ of S is a subsemigroup of S of type $\mathcal{T}$ for each $x \in S$. In [9] N. Kehayopulu and M. Tsingelis proved that an ordered semigroup S is a band B of semigroups S_α of a given type $\mathcal{T}$ if and only if it is decomposable into pairwise disjoint subsemigroups S_α, $\alpha \in B$, of S of type $\mathcal{T}$. Also, then the following holds (i) $S_\alpha S_\beta \subseteq S_{\alpha\beta}$, for all $\alpha, \beta \in B$ and (ii) If $S_\alpha \bigcap (S_\beta] \neq \phi$, for $\alpha, \beta \in B$, then $\alpha = \alpha\beta = \beta\alpha$. If σ is a band congruence on S, then the set $S/\sigma = \{(x)_\sigma | x \in S\}$ with the multiplication " $\circ$ " on S/σ defined by $(a)_\sigma \circ (b)_\sigma := (ab)_\sigma$ is a band [9].

Remark 2.1 [9] *Let S be an ordered semigroup and σ a band congruence on S. Then we have the following*:
(i) $(\forall a \in S)(a, a^2) \in \sigma$ *and* $(a^2, a) \in \sigma$;
(ii) $(\forall a \in S)(\forall n \in Z^+)(a^n, a) \in \sigma$ *and* $(a, a^n) \in \sigma$;
(iii) *if* $a \leq b$, *then* $(ab, ba) \in \sigma$;
(iv) $(S/\sigma, \circ)$ *is a band;*
(v) $(a)_\sigma$ *is a subsemigroup of S for all* $a \in S$.

For an equivalence relation σ on an ordered semigroup S, by $\sigma^\flat$ we denote the greatest congruence relation on S contained in σ. The next lemma is a well-known result and its proof will be omitted.

Lemma 2.2 *On any ordered semigroup S,*
$$\sigma^\flat =: \{(a,b) | (\forall x, y \in S^1)(xay, xby) \in \sigma\}.$$

Lemma 2.3. *Let S be an ordered semigroup. Then $\sigma^\flat$ is a band congruence if and only if S satisfying the condition: if $a \leq b$, then $(xay, xaby) \in \sigma$ and $(xay, xbay) \in \sigma$ for $x, y \in S^1$.*

Proof Let $\sigma^\flat$ be a band on S, and let $a \leq b$ and $x, y \in S^1$. Since $\sigma^\flat$ is a band, we have $(a, ab), (a, ba) \in \sigma^\flat$. Since $\sigma^\flat$ is a congruence on S and $x, y \in S^1$, we have $(xay, xaby), (xay, xbay) \in \sigma^\flat$.

Conversely, let $a, b \in S, x, y \in S^1$. By Lemma 2.2, $\sigma^\flat$ is a congruence on S. We prove that the relation $\sigma^\flat$ is a band congruence on S. Assume $a \leq b$. By hypothesis, we have $(xay, x(ab)y) \in \sigma$ and $(xay, x(ba)y) \in \sigma$. Thus, we get $(a, ab) \in \sigma^\flat$ and $(a, ba) \in \sigma^\flat$, so $\sigma^\flat$ is a band congruence on S.

The concept of k-regular semigroups were introduced by K. S. Harinath in [10]. The other types of semigroups were introduce for the first time by S. Bogdanović, Ž. Popović and M. Ćirić in [4]. In this paper we extend the concepts of k-regular and k-Archimedean in case of ordered semigroups. Let $k \in Z^+$ be a fixed integer. An ordered semigroup S is: *k-regular* if $(\forall a \in S) a^k \in (a^k S a^k]$, *left k-regular* if $(\forall a \in S) a^k \in (Sa^{k+1}]$, *right k-regular* if $(\forall a \in S) a^k \in (a^{k+1} S]$, *completely k-regular* if $(\forall a \in S) a^k \in (a^{k+1} S a^{k+1}]$, *intra k-regular* if $(\forall a \in S) a^k \in (Sa^{2k} S]$, *t-k-regular* if $(\forall a \in S) a^k \in (Sa^{k+1}] \bigcap (a^{k+1} S]$. *k-Archimedean* if $(\forall a, b \in S) a^k \in (S^1 b S^1]$, *left k-Archimedean* if $(\forall a, b \in S) a^k \in (S^1 b]$, *right k-Archimedean* if $(\forall a, b \in S) a^k \in (bS^1]$ and *t-k-Archimedean* if $(\forall a, b \in S) a^k \in (bS^1] \bigcap (S^1 b]$.

Let $k \in Z^+$ and $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{I}, \mathcal{B}\}$, where $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{I}$ and $\mathcal{B}$ be Green's relations on an ordered semigroup S. On S we define the following relations by

$$\mathcal{X}_k =: \{(a,b) \in S \times S | (a^k, b^k) \in \mathcal{X}\},$$
$$\mathcal{X}_k^\flat =: \{(a,b) \in S \times S | (\forall x, y \in S^1)(xay, xby) \in \mathcal{X}_k\}.$$

Then it is easy to verify that $\mathcal{X}_k$ is an equivalence relation on an ordered semigroup S. Also, it is evident that $\mathcal{B}_k \subseteq \mathcal{L}_k \bigcap \mathcal{R}_k$ and $\mathcal{L}_k \bigcap \mathcal{R}_k = \mathcal{H}_k$. So, in this case these equivalences $\mathcal{X}_k$ are very similar to Green's equivalences and they can be consider as its generalizations. An ordered semigroup S is $\mathcal{X}_k$-*simple* if

$(a, b) \in \mathcal{X}_k$ for all $a, b \in S$. It is clear that an ordered semigroup S is $\mathcal{L}_k$- ($\mathcal{R}_k$-) simple if and only if S is left (right) k-Archimedean. S is $\mathcal{I}_k$-simple if and only if S is k-Archimedean. S is $\mathcal{H}_k$-simple if it is both $\mathcal{L}_k$-simple and $\mathcal{R}_k$-simple, and conversely.

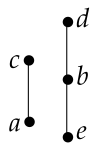
Example 2.4 We consider the ordered semigroup $S = \{a, b, c, d, e\}$, defined by multiplication " $\cdot$ " and the order " $\leq$ " below [7]:

.	a	b	c	d	e
a	a	b	c	b	b
b	b	b	b	b	b
c	a	b	c	b	b
d	d	b	d	b	b
e	e	e	e	e	e

$$\leq := \{(a, a), (a, c), (b, b), (b, d), (c, c), (d, d), (e, b), (e, d), (e, e)\}.$$

We give the covering relation "<" and the figure of S.

$$<=: \{(a, c), (b, d), (e, b)\}.$$



A) By $Reg(S) = \{a, b, c, e\}$ we denote the set of all regular elements of a semigroup S, since $d \notin Reg(S)$, then S is not regular. Since $d^2 = b \in Reg(S)$ and $\{a^2, b^2, c^2, d^2, e^2\} = \{a, b, c, e\} = Reg(S)$, then S is a 2-regular ordered semigroup. Also, S is not left regular, right regular and intra-regular ($\nexists x, y, u, v \in S : d \leq xd^2, d \leq d^2y, d \leq ud^2v$), but S is left 2-regular, right 2-regular and intra-2-regular.

B) It is easy to see that S is not Archimedean ($a^k \notin (S^1bS^1] = \{b, e\}$ for any $k \in Z^+$), but the subsemigroup $\{b, d, e\}$ of S is 2-Archimedean.

C) It is not hard to verify that the relation on S:

$$\mathcal{I} = \mathcal{I}_1 = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, c), (c, a), (b, e), (e, b)\},$$

is a congruence and it is not a band congruence and semilattice congruence, since $(d, d^2) = (d, b) \notin \mathcal{I}$. Further, the relation

$$\mathcal{I}_2 = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, c), (c, a), (b, d), (d, b), (b, e), (e, b), (d, e), (e, d)\},$$

is a band congruence on S, and S is a band of 2-Archimedean semigroups $\{a, c\}$ and $\{b, d, e\}$.

3. Bands and complete semilattice of $\mathcal{X}_k$-simple ordered semigroups

In this section we give structural characterizations of bands and complete semilattices for different types of simple ordered semigroups.

Theorem 3.1 *Let $k \in Z^+$ and $\mathcal{X}_k \in \{\mathcal{L}_k, \mathcal{R}_k, \mathcal{I}_k, \mathcal{H}_k, \mathcal{B}_k\}$. Then the following conditions on an ordered semigroup S are equivalent*:
(i) *S is a band of $\mathcal{X}_k$-simple ordered semigroups;*
(ii) *if $a \leq b$, then $(xay, xaby) \in \mathcal{X}_k$ and $(xay, xbay) \in \mathcal{X}_k$ for $x, y \in S^1$;*
(iii) *$\mathcal{X}_k^\flat$ is a band congruence on S.*

Proof (i)⇒(ii) Let S be a band Y of $\mathcal{X}_k$-simple ordered semigroups $S_\alpha, \alpha \in Y$. Assume $a, b, x, y \in S$ (in a similar way we can prove the cases with $x = 1$ or $y = 1$). Let $a \le b$ and $a \in S_\alpha, b \in S_\beta, x \in S_\gamma, y \in S_\delta$. Then since $a \le b$ and Y is a band, we have $\alpha = \alpha\beta = \beta\alpha$, so $xay, xaby, xbay \in S_{\gamma\alpha\delta}$ and since $S_{\gamma\alpha\delta}$ is $\mathcal{X}_k$-simple, then $(xay, xaby) \in \mathcal{X}_k$ and $(xay, xbay) \in \mathcal{X}_k$.

(ii)⇒(iii) This follows immediately by Lemma 2.3.

(iii)⇒(i) Let $\mathcal{X}_k^\flat$ be a band congruence on S and S a band of $\mathcal{X}_k^\flat$-classes. Suppose $S = \bigcup_{\alpha \in Y} S_\alpha$, Y is a band and $S_\alpha, \alpha \in Y$ is a $\mathcal{X}_k^\flat$-class of S, then $S_\alpha, \alpha \in Y$ is $\mathcal{X}_k$-simple. In fact: Let $a, b \in S_\alpha, \alpha \in Y$. Then $(a, b) \in \mathcal{X}_k^\flat$, and since $\mathcal{X}_k^\flat$ is the greatest congruence on S contained in $\mathcal{X}_k$, then we have that $(a, b) \in \mathcal{X}_k$, that is, $S_\alpha, \alpha \in Y$ is an $\mathcal{X}_k$-simple semigroup. Thus, S is a band of $\mathcal{X}_k$-simple semigroups.

Lemma 3.2 *Let $k \in Z^+$. Then the following conditions on an ordered semigroup S are equivalent*:
(i) *S is $\mathcal{L}_k$-simple;*
(ii) *S is left Archimedean and left k-regular;*
(iii) *S is left k-Archimedean.*

Proof (i)⇒(ii) Let S be $\mathcal{L}_k$-simple and $a, b \in S$. Then $(a, b) \in \mathcal{L}_k$, so $a^k \in (S^1 b^k] \subseteq (Sb]$, that is S is left Archimedean. Also, since $a, a^2 \in S$, we have $(a, a^2) \in \mathcal{L}_k$, so $a^k \in (S^1 a^{2k}] \subseteq (Sa^{k+1}]$, that is S is left k-regular.

(ii)⇒(iii) Let S be left Archimedean and left k-regular. Assume $a, b \in S$. By S is left k-regular, we have $a^k \in (Sa^{k+1}] \subseteq (S(Sa^{k+1}]a] \subseteq (Sa^k a^2]$, continuous use of this process, we have $a^k \in (Sa^k a^n] \subseteq (Sa^n]$ for all $n \in Z^+$. Since S is left Archimedean, we have $a^m \in (Sb]$ for some $m \in Z^+$, so $a^k \in (Sa^m] \subseteq (S(Sb]] \subseteq (Sb]$. Hence S is left k-Archimedean.

(iii)⇒(i) This implication immediately follows.

By the following theorem we describe the structure of an ordered semigroup which can be decompose into a band of $\mathcal{L}_k$-simple ordered semigroups.

Theorem 3.3 *Let $k \in Z^+$. Then the following conditions on an ordered semigroup S are equivalent*:
(i) *S is a band of $\mathcal{L}_k$-simple ordered semigroups;*
(ii) *$\mathcal{L}_k^\flat$ is a band congruence on S;*
(iii) *S is a band of left Archimedean and left k-regular ordered semigroups;*
(iv) *S is a band of left k-Archimedean ordered semigroups.*

Proof (i)⇔(ii) This equivalence immediately follows by Theorem 3.1.

(iii)⇔(iv) and (iv)⇔(i) These equivalences follow by Lemma 3.2.

Lemma 3.4 *Let $k \in Z^+$ and $\mathcal{X}_k \in \{\mathcal{L}_k, \mathcal{R}_k, \mathcal{I}_k, \mathcal{H}_k, \mathcal{B}_k\}$. Assume S is a complete semilattice Y of $\mathcal{X}_k$-simple $S_\alpha, \alpha \in Y$ and $a \in S_\alpha, b \in S_\beta$ with $\alpha, \beta \in Y$. If $(a, b) \in \mathcal{X}_k$, then $\alpha = \beta$.*

Proof We deal with only the $\mathcal{L}_k$-simple. The proof is similar for other cases. Let $a, b \in S$ such that $(a, b) \in \mathcal{L}_k$. Assume $a \in S_\alpha, b \in S_\beta$ for some $\alpha, \beta \in Y$, and since $a^k \le xb^k$ and $b^k \le ya^k$, for some $x \in S_\delta^1, y \in S_\theta^1$, where $\delta, \theta \in Y$, then we obtain that $\alpha = \alpha\delta\beta$ and $\beta = \beta\theta\alpha$. By this we have that $\alpha\beta = \alpha\delta\beta \cdot \beta = \alpha\delta\beta = \alpha$ and $\beta\alpha = \beta\theta\alpha \cdot \alpha = \beta\theta\alpha = \beta$. Since Y is a complete semilattice then it follows that $\alpha = \alpha\beta = \beta\alpha = \beta$.

By the following theorem we describe the structure of a semigroup which can be decompose into a complete semilattice of $\mathcal{L}_k$-simple semigroups. Also, here should be to emphasize that a semilattice of $\mathcal{L}_k$-simple ordered semigroups is coincident with a complete semilattice of $\mathcal{L}_k$-simple ordered semigroups.

Theorem 3.5 *Let $k \in Z^+$. Then the following conditions on an ordered semigroup S are equivalent*:
(i) *S is a complete semilattice of $\mathcal{L}_k$-simple ordered semigroups;*
(ii) *$\mathcal{L}_k^\flat = \mathcal{L}_k$ is a complete semilattice congruence on S;*
(iii) *$\mathcal{L}_k$ is a semilattice congruence on S;*
(iv) *S is a complete semilattice of left Archimedean and left k-regular ordered semigroups .*

Proof (i)⇒(ii) Let S be a complete semilattice Y of $\mathcal{L}_k$-simple ordered semigroups $S_\alpha, \alpha \in Y$. Assume $a, b, c \in S$ such that $(a, b) \in \mathcal{L}_k$. Let $a \in S_\alpha, b \in S_\beta, c \in S_\gamma$ for some $\alpha, \beta, \gamma \in Y$. By Lemma 3.4, we have $\alpha = \beta$, so $a, b \in S_\alpha, \alpha \in Y$. So, $ac, bc \in S_{\alpha\gamma}, \alpha, \gamma \in Y$, and since $S_{\alpha\gamma}$ is an $\mathcal{L}_k$-simple semigroup, then $(ac, bc) \in \mathcal{L}_k$. Similarly we prove that $(ca, cb) \in \mathcal{L}_k$. Thus $\mathcal{L}_k$ is a congruence relation on S. Further, $a, a^2 \in S_\alpha, \alpha \in Y$ and S_α is $\mathcal{L}_k$-simple, then $(a, a^2) \in \mathcal{L}_k$, for every $a \in S$. Also, since Y is a complete semilattice, let $ab, ba \in S_\alpha, \alpha \in Y$ and S_α is $\mathcal{L}_k$-simple, then $(ab, ba) \in \mathcal{L}_k$ for all $a, b \in S$, whence $\mathcal{L}_k$ is a semilattice congruence on S. Moreover, if $a \leq b$, then $a, ab \in S_\alpha, \alpha \in Y$ and S_α is $\mathcal{L}_k$-simple, then $(a, ab) \in \mathcal{L}_k$, whence $\mathcal{L}_k$ is a complete semilattice congruence on S. Since $\mathcal{L}_k^\flat$ is the greatest congruence relation on S contained in $\mathcal{L}_k$, Thus $\mathcal{L}_k^\flat = \mathcal{L}_k$.

(ii)⇒(iii) This implication immediately follows.

(iii)⇒(ii) Let $\mathcal{L}_k$ be a semilattice congruence on S. Since $\mathcal{L}_k^\flat$ is the greatest congruence relation on S contained in $\mathcal{L}_k$, thus $\mathcal{L}_k^\flat = \mathcal{L}_k$. Let $a \leq b$. Since $\mathcal{L}_k$ is a semilattice congruence, then $(a, a^2) \in \mathcal{L}_k$, so $a^k \in L(a^{2k})$, that is $a^k \leq xa^{2k}$ for some $x \in S^1$. Since $a \leq b$, then we have that $a^k \leq xa^{2k} \leq x(ab)^k \in L((ab)^k)$, so $L((a)^k) \subseteq L((ab)^k)$. Conversely, since $\mathcal{L}_k$ is a semilattice congruence, then we have $(ab, (ba)^3) \in \mathcal{L}_k$, so $(ab)^k \in (S^1(ba)^{3k}] = (S^1b(ab)^{2k-1}(ab)^ka] \subseteq (S^1b(ab)^{2k-1}(S^1(ba)^{3k}]a] \subseteq (S(ab)^ka^2]$, continuous use of this process, we have $(ab)^k \in (S(ab)^ka^k] \subseteq (Sa^k]$. Thus $L((ab)^k) \subseteq L(a^k)$. Therefore, $L(a^k) = L((ab)^k)$, i.e., $(a, ab) \in \mathcal{L}_k$.

(ii)⇒(iv) Let (ii) hold. Then S is a complete semilattice of $\mathcal{L}_k$-classes. Let $a, b \in A$, where A is an arbitrary $\mathcal{L}_k$-class of S. Since $(a^3, b) \in \mathcal{L}_k$, whence $(a^{3k}, b^k) \in \mathcal{L}$, so $b^k \in (S^1a^{3k}] \subseteq (Sa^2]$, that is $b^k \leq xa^2$, for some $x \in S$. Seeing that $\mathcal{L}_k$ is a complete semilattice congruence, then $(b, b^k) \in \mathcal{L}_k$, $(b^k, b^kxa^2) \in \mathcal{L}_k$ and $(b^kxa^2, b^kxa) \in \mathcal{L}_k$, so $(b, b^kxa) \in \mathcal{L}_k$. Since $(b^kxa, bxa) \in \mathcal{L}_k$, we get $(b, bxa) \in \mathcal{L}_k$, so $bxa \in A$ and $b^{k+1} \leq (bxa)a \in (Aa]$. Therefore, A is a left Archimedean ordered subsemigroup. Also, since $a, a^2 \in A$, we have $(a, a^2) \in \mathcal{L}_k$, so $a^k \in (S^1a^{2k}] \subseteq (Sa^{k+1}]$, that is A is left k-regular.

(iv)⇒(i) This follows by Lemma 3.2.

By the following result we describe the structure of an ordered semigroup which can be decompose into a complete semilattice of $\mathcal{I}_k$-simple ordered semigroups, and a band of $\mathcal{I}_k$-simple ordered semigroups is coincident with a complete semilattice of $\mathcal{I}_k$-simple ordered semigroups.

Theorem 3.6 *Let $k \in Z^+$. Then the following conditions on an ordered semigroup S are equivalent*:
(i) *S is a complete semilattice of $\mathcal{I}_k$-simple ordered semigroups;*
(ii) *$\mathcal{I}_k$ is a band congruence;*
(iii) *$\mathcal{I}_k$ is a complete semilattice congruence;*
(iv) *S is a complete semilattice of Archimedean and intra k-regular ordered semigroups;*
(v) *S is a complete semilattice of k-Archimedean ordered semigroups.*

Proof (i)⇒(ii) Let S be a complete semilattice Y of $\mathcal{I}_k$-simple ordered semigroups $S_\alpha, \alpha \in Y$. Assume $a, b, c \in S$, then $a \in S_\alpha, b \in S_\beta$ and $c \in S_\gamma$ for some $\alpha, \beta, \gamma \in Y$. Let $(a, b) \in \mathcal{I}_k$, then $(a^k, b^k) \in \mathcal{I}$, whence $\alpha = \beta$, i.e. $a, b \in S_\alpha$. Also, $ac, bc \in S_{\alpha\gamma}$ and $ca, cb \in S_{\gamma\alpha}$. Hence, $(ac, bc) \in \mathcal{I}_k$ and $(ca, cb) \in \mathcal{I}_k$, i.e. $\mathcal{I}_k$ is a congruence. Since $\mathcal{I}_k^\flat$ is the greatest congruence relation on S contained in $\mathcal{I}_k$, thus $\mathcal{I}_k = \mathcal{I}_k^\flat$, by Theorem 3.1, $\mathcal{I}_k = \mathcal{I}_k^\flat$ is a band congruence on S.

(ii)⇒(iii) Let $\mathcal{I}_k$ be a band congruence and $a, b \in S$. Then $(ab, (ab)^2) \in \mathcal{I}_k$, that is $((ab)^k, (ab)^{2k}) \in \mathcal{I}$, so $(ab)^k \in I((ab)^{2k}) \subseteq (S^1(ba)^kS^1] = I((ba)^k)$. In a similar way we can prove that $(ba)^k \in I((ab)^k)$. Thus $I((ba)^k) = I((ab)^k)$, that is $(ab, ba) \in \mathcal{I}_k$, i.e. $\mathcal{I}_k$ is a semilattice congruence on S. Moreover, let $a \leq b$. Since $\mathcal{I}_k$ is a band congruence on S, so $(a, ab) \in \mathcal{I}_k$. Hence, $\mathcal{I}_k$ is a complete semilattice congruence.

(iii)⇒(iv) Let S be a complete semilattice Y of $\mathcal{I}_k$-classes semigroups $S_\alpha, \alpha \in Y$. Assume $a, b \in S_\alpha$ for some $\alpha \in Y$. Then $(a, b) \in \mathcal{I}_k$, so $a^k \in I(a^k) = I(b^k) = (S_\alpha^1b^kS_\alpha^1] \subseteq (S_\alpha bS_\alpha]$, that is S_α is Archimedean. For $a, a^4 \in S_\alpha$, we have $(a, a^4) \in \mathcal{I}_k$, so $a^k \in (S_\alpha^1a^{4k}S_\alpha^1] \subseteq (S_\alpha a^{2k}S_\alpha]$, that is S_α is intra k-regular.

(iv)⇒(v) Let S be a complete semilattice Y of Archimedean and intra k-regular ordered semigroups $S_\alpha, \alpha \in Y$. Assume $a, b \in S_\alpha$. By hypothesis, we have $a^k \in (S_\alpha a^{2k}S_\alpha]$, that is $a^k \leq ua^{2k}v$ for some $u, v \in S_\alpha$, so $a^k \leq u^2a^k(a^kv)^2 \leq \cdots \leq u^na^k(a^kv)^n$ for any $n \in Z^+$. For $a^kv, b \in S_\alpha$, since S_α is Archimedean, we have

$(a^kv)^m \leq wbd$ for some $m \in Z^+$ and $w, d \in S_\alpha$, so $a^k \leq u^m a^k(a^kv)^m \leq u^m a^k(wbd) \in (S_\alpha b S_\alpha]$. Thus S_α is k-Archimedean.

(v)⇒(i) This follows immediately.

For an arbitrary relation ρ on an ordered semigroup S, the radical $R(\rho)$ of ρ is a relation on S defined by:

$$(a, b) \in R(\rho) \Leftrightarrow (\exists m, n \in Z^+)(a^m, b^n) \in \rho.$$

The radical $R(\rho)$ was introduced by L. N. Shevrin in [12] and it is clear that $\rho \subseteq R(\rho)$.

Theorem 3.7 *Let $k \in Z^+$. Then $\mathcal{I}_k$ is a semilattice congruence on S if and only if $R(\mathcal{I}_k) = \mathcal{I}_k$ and $\mathcal{I}_k$ is a congruence on S.*

Proof Let $\mathcal{I}_k$ be a semilattice congruence on S. Since the inclusion $\mathcal{I}_k \subseteq R(\mathcal{I}_k)$ always holds, then it remains to prove the opposite inclusion. Also, let $a \in S$, by $\mathcal{I}_k$ is a semilattice congruence on S, then we have that $(a, a^n) \in \mathcal{I}_k$ for all $n \in Z^+$. Now assume $a, b \in S$ such that $(a, b) \in R(\mathcal{I}_k)$. Then $(a^i, b^j) \in \mathcal{I}_k$, for some $i, j \in Z^+$, and by previous we have that $(a, a^i) \in \mathcal{I}_k$, $(b^j, b) \in \mathcal{I}_k$. Thus $(a, b) \in \mathcal{I}_k$. Therefore, $R(\mathcal{I}_k) \subseteq \mathcal{I}_k$, i.e. $R(\mathcal{I}_k) = \mathcal{I}_k$.

Conversely, let $\mathcal{I}_k = R(\mathcal{I}_k)$ and $\mathcal{I}_k$ be a congruence on S. Since $\mathcal{I}_k$ is reflexive, then for every $a \in S$ we have that $(a^2, a^2) \in \mathcal{I}_k$, so $((a^1)^2, (a^2)^1) \in \mathcal{I}_k$, that is $(a, a^2) \in R(\mathcal{I}_k)$. Hence $(a, a^2) \in \mathcal{I}_k$. Further, let $a, b \in S$. Then $(ab, (ab)^2) \in \mathcal{I}_k$, that is $((ab)^k, (ab)^{2k}) \in \mathcal{I}$, so $(ab)^k \in I((ab)^{2k}) \subseteq (S^1(ba)^kS^1] = I((ba)^k)$. In a similar way we can prove that $(ba)^k \in I((ab)^k)$. Thus $I((ba)^k) = I((ab)^k)$, that is $(ab, ba) \in \mathcal{I}_k$, i.e. $\mathcal{I}_k$ is a semilattice congruence on S.

By the following theorems we describe the structure of a semigroup which can be decompose into a band and complete semilattice of $\mathcal{B}_k$-simple semigroups.

Theorem 3.8 *Let $k \in Z^+$. Then the following conditions on an ordered semigroup S are equivalent*:
(i) *S is a band of $\mathcal{B}_k$-simple ordered semigroups;*
(ii) *$\mathcal{B}_k^\flat$ is a band congruence on S;*
(iii) *S is a band of t-Archimedean and completely k-regular ordered semigroups;*
(iv) *S is a band of t-k-Archimedean and k-regular ordered semigroups.*

Proof (i)⇔(ii) This equivalence immediately follows by Theorem 3.1.

(ii)⇒(iii) Let (ii) hold. Let S be a band of $\mathcal{B}_k^\flat$-clesses. Assume $a, b \in A$, where A is an $\mathcal{B}_k^\flat$-class of S. By Remark 2.1, the $\mathcal{B}_k^\flat$-class A is a subsemigroup of S. Since $(a^3, b) \in \mathcal{B}_k^\flat \subseteq \mathcal{B}_k$, whence $(a^3, b) \in \mathcal{B}_k$, so $b^k \in B(a^{3k})$, that is $b^k \leq a^{3k}$ or $b^k \in (a^{3k}S^1a^{3k}]$. If $b^k \leq a^{3k}$, then $b^k \in (Aa] \bigcap (aA]$, thus A is t-Archimedean. If $b^k \in (a^{3k}S^1a^{3k}] \subseteq (a^{3k}Sa^2]$, that is $b^k \leq a^{3k}xa^2$, for some $x \in S$. Seeing that $\mathcal{B}_k^\flat$ is a band congruence, then $(b, b^k) \in \mathcal{B}_k^\flat$, $(b^k, b^ka^{3k}xa^2) \in \mathcal{B}_k^\flat$ and $(b^ka^{3k}xa^2, b^ka^{3k}xa) \in \mathcal{B}_k^\flat$, so $(b, b^ka^{3k}xa) \in \mathcal{B}_k^\flat$. Since $(b^ka^{3k}xa, ba^{3k}xa) \in \mathcal{B}_k^\flat$, we get $(b, ba^{3k}xa) \in \mathcal{B}_k^\flat$, so $ba^{3k}xa \in A$ and $b^{k+1} \leq (ba^{3k}xa)a \in (Aa]$. Similarly we prove that $b^{k+1} \in (aA]$. Therefore, A is a t-Archimedean ordered subsemigroup. Also, Since $(a, a^2) \in \mathcal{B}_k^\flat \subseteq \mathcal{B}_k$, then $a^k \in B(a^{2k})$, so $a^k \leq a^{2k}$ or $a^k \leq a^{2k}ua^{2k}$ for some $u \in S^1$. If $a^k \leq a^{2k}$, clearly, A is completely k-regular. If $a^k \leq a^{2k}ua^{2k}$, then $a^k \leq a^{k+1} \cdot a^{2k-1}(ua^{2k})^2ua^{2k-1} \cdot a^{k+1}$. In a previous similar way we can prove that $a^{2k-1}(ua^{2k})^2ua^{2k-1} \in A$, so A is completely k-regular.

(iii)⇒(iv) Let S be a band Y of t-Archimedean and completely k-regular ordered semigroups $S_\alpha, \alpha \in Y$. Let $a, b \in S_\alpha$ for some $\alpha \in Y$. Since S_α is completely k-regular, then S_α is k-regular, and $a^k \in (a^{k+1}S_\alpha a^{k+1}] \subseteq (a(a^{k+1}S_\alpha a^{k+1}]S_\alpha(a^{k+1}S_\alpha a^{k+1}]a] \subseteq (a^2 \cdot a^kS_\alpha a^k \cdot a^2]$, continuous use of this process, we have $a^k \in (a^n \cdot a^kS_\alpha a^k \cdot a^n] \subseteq (a^nS_\alpha a^n]$ for all $n \in Z^+$. For $a, b \in S_\alpha$, since S_α is t-Archimedean, we have $a^m \in (bS_\alpha] \bigcap (S_\alpha b]$ for some $m \in Z^+$, so $a^k \in (a^mS_\alpha a^m] \subseteq ((bS_\alpha]S_\alpha(S_\alpha b]] \subseteq (bS_\alpha b]$. Thus $a^k \in (S_\alpha b] \bigcap (bS_\alpha]$, that is S_α is t-k-Archimedean.

(iv)⇒(i) Let S be a band Y of t-k-Archimedean and k-regular $S_\alpha, \alpha \in Y$. Assume $a, b \in S_\alpha$. For $a^k, b^k \in S_\alpha$, by hypothesis, we have $a^k \in (b^kS_\alpha] \bigcap (S_\alpha b^k]$. Since S_α is k-regular, then $a^k \in (a^kS_\alpha a^k]$, so $a^k \in ((b^kS_\alpha]S_\alpha(S_\alpha b^k]] \subseteq (b^kS_\alpha b^k] \subseteq B(b^k)$. Similarly, we can prove that $b^k \in B(a^k)$. Hence $(a^k, b^k) \in \mathcal{B}$, so $(a, b) \in \mathcal{B}_k$. Therefore S_α is $\mathcal{B}_k$-simple.

Theorem 3.9 *Let $k \in Z^+$. Then the following conditions on an ordered semigroup S are equivalent*:
(i) *S is a complete semilattice of $\mathcal{B}_k$-simple ordered semigroups;*
(ii) $\mathcal{B}_k^\flat = \mathcal{B}_k$ and $\mathcal{B}_k$ *is a complete semilattice congruence on S;*
(iii) *$\mathcal{B}_k$ is a semilattice congruence on S;*
(iv) *S is a complete semilattice of t-Archimedean and completely k-regular ordered semigroups.*

Proof (i)⇒(ii) Let S be a complete semilattice Y of $\mathcal{B}_k$-simple semigroups $S_\alpha, \alpha \in Y$. Assume $a, b, c \in S$ such that $(a, b) \in \mathcal{B}_k$. Let $a \in S_\alpha, b \in S_\beta, c \in S_\gamma$ for some $\alpha, \beta, \gamma \in Y$. By Lemma 3.4, we have $\alpha = \beta$, so $a, b \in S_\alpha, \alpha \in Y$. So, $ac, bc \in S_{\alpha\gamma}, \alpha, \gamma \in Y$, and since $S_{\alpha\gamma}$ is an $\mathcal{B}_k$-simple semigroup, then $(ac, bc) \in \mathcal{B}_k$. Similarly, we can prove that $(ca, cb) \in \mathcal{B}_k$. Thus $\mathcal{B}_k$ is a congruence relation on S. Further, $a, a^2 \in S_\alpha, \alpha \in Y$ and S_α is $\mathcal{B}_k$-simple, then $(a, a^2) \in \mathcal{B}_k$, for every $a \in S$. Also, by Y is a complete semilattice, let $ab, ba \in S_\alpha, \alpha \in Y$. Since S_α is $\mathcal{B}_k$-simple, then $(ab, ba) \in \mathcal{B}_k$ for all $a, b \in S$, whence $\mathcal{B}_k$ is a semilattice congruence on S. Moreover, if $a \le b$, then $a, ab \in S_\alpha, \alpha \in Y$ and S_α is $\mathcal{B}_k$-simple, then $(a, ab) \in \mathcal{B}_k$, whence $\mathcal{B}_k$ is a complete semilattice congruence on S. Since $\mathcal{B}_k^\flat$ is the greatest congruence relation on S contained in $\mathcal{B}_k$, Thus $\mathcal{B}_k^\flat = \mathcal{B}_k$.

(ii)⇒(iv) Let (ii) hold. Then S is a complete semilattice of $\mathcal{B}_k$-classes. Let $a, b \in A$, where A is an arbitrary $\mathcal{B}_k$-class of S. Since $(a^3, b) \in \mathcal{B}_k$, whence $(a^{3k}, b^k) \in \mathcal{B}$, that is $b^k \le a^{3k}$ or $b^k \in (a^{3k}S^1a^{3k}]$. As same as (ii)⇒(iii) of Theorem 3.8 we can prove that every complete semilattice component A of S is a t-Archimedean and completely k-regular ordered subsemigroup.

(iv)⇒(i) Let S be a complete semilattice Y of t-k-Archimedean and completely k-regular $S_\alpha, \alpha \in Y$. As same as (iv)⇒(i) of Theorem 3.8, we can prove that S_α is $\mathcal{B}_k$-simple.

(ii)⇒(iii) This implication immediately follows.

(iii)⇒(ii) Let $\mathcal{B}_k$ be a semilattice congruence on S. Since $\mathcal{B}_k^\flat$ is the greatest congruence relation on S contained in $\mathcal{B}_k$, Thus $\mathcal{B}_k^\flat = \mathcal{B}_k$. Let $a \le b$. Since $\mathcal{B}_k$ is a semilattice congruence, then $(a, a^2) \in \mathcal{B}_k$, so $a^k \in B(a^{2k})$, that is $a^k \le a^{2k}$ or $a^k \le a^{2k}xa^{2k}$ for some $x \in S^1$. By $a \le b$, we have $a^k \le (ab)^k \in B((ab)^k)$ or $a^k \le (ab)^kx(ab)^k \in B((ab)^k)$, so $B(a^k) \subseteq B((ab)^k)$. Conversely, since $\mathcal{B}_k$ is a semilattice congruence, then we have that $(ab, (ba)^{3k}) \in \mathcal{B}_k$ and $((ab)^{3k}, ba) \in \mathcal{B}_k$, so $(ab)^k \in B((ba)^{3k})$ and $(ba)^k \in B((ab)^{3k})$. For $(ab)^k \in B((ba)^{3k})$, if $(ab)^k \le (ba)^{3k}$, then $(ab)^k \le (ba)^{3k} \in (((ab)^{3k} \bigcup (ab)^{3k}S^1(ab)^{3k}](ba)^{2k} \subseteq ((ab)^{2k}S(ba)^{2k}]$. If $(ab)^k \in ((ba)^{3k}S^1(ba)^{3k}]$, also we have $(ab)^k \in ((ab)^{2k}S(ba)^{2k}]$. For $(ba)^k \in B((ab)^{3k})$, similarly, we can prove that $(ba)^k \in ((ab)^{2k}S(ba)^{2k}]$. Therefore,

$$\begin{aligned}&(ab)^k \in ((ab)^{2k}S(ba)^{2k}] = (a(ba)^{2k-1}bSb(ab)^{2k-1}a]\\&\subseteq (a((ab)^{2k}S(ba)^{2k}](ba)^{k-1}bSb(ab)^{k-1}((ab)^{2k}S(ba)^{2k}]a] \subseteq (a^2(ba)^kS(ab)^ka^2],\end{aligned}$$

continuous use of this process, we have $(ab)^k \in (a^k(ba)^kS(ab)^ka^k] \subseteq (a^kSa^k]$. Thus $B((ab)^k) \subseteq B(a^k)$. Therefore, $B(a^k) = B((ab)^k)$, i.e., $(a, ab) \in \mathcal{B}_k$.

By Theorem 3.3, Theorem 3.5 and their dual we describe the structure of a semigroup which can be decompose into a band and complete semilattice of $\mathcal{H}_k$-simple semigroups.

Theorem 3.10 *Let $k \in Z^+$. Then the following conditions on an ordered semigroup S are equivalent*:
(i) *S is a band of $\mathcal{H}_k$-simple ordered semigroups;*
(ii) *$\mathcal{H}_k^\flat = \mathcal{H}_k$ is a band congruence on S;*
(iii) *S is a band of t-Archimedean and t-k-regular ordered semigroups;*
(iv) *S is a band of t-k-Archimedean ordered semigroups.*

Theorem 3.11 *Let $k \in Z^+$. Then the following conditions on an ordered semigroup S are equivalent*:
(i) *S is a complete semilattice of $\mathcal{H}_k$-simple ordered semigroups;*
(ii) *$\mathcal{H}_k$ is a complete semilattice congruence on S;*
(iii) *$\mathcal{H}_k$ is a semilattice congruence on S;*
(iv) *S is a complete semilattice of t-Archimedean and t-k-regular ordered semigroups;*
(v) *S is a complete semilattice of t-k-Archimedean ordered semigroups.*

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