Complete Convergence for Weighted Sums
of a Class of Random Variables

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Abstract. Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of real numbers and $\{X_n, n \ge 1\}$ be a sequence of random variables satisfying the Rosenthal type inequality, which is stochastically dominated by a random variable X. Under mild conditions, we present some results on complete convergence for weighted sums $\sum_{i=1}^{n} a_{ni} X_i$ of random variables satisfying the Rosenthal type inequality. The results obtained in the paper generalize some known ones in the literatures.

1. Introduction

In many stochastic models, the assumption that random variables are independent is not plausible. So it is of interest to extend the results from independent framework to dependent variables. The emphasis of the paper is to consider a class of random variables satisfying the Rosenthal type inequality.

Let $\{Z_n, n \ge 1\}$ be a sequence of random variables. The Rosenthal type inequality is expressed as follows: for any $r \ge 2$ and every $n \ge 1$,

$$E\left(\max_{1 \le j \le n} \left|\sum_{i=1}^{j}(Z_i - EZ_i)\right|^r\right) \le C\left[\sum_{i=1}^{n} E|Z_i - EZ_i|^r + \left(\sum_{i=1}^{n} E(Z_i - EZ_i)^2\right)^{r/2}\right]. \tag{1.1}$$

The main purpose of the paper is to study the complete convergence for a class of random variables satisfying the Rosenthal type inequality. The concept of complete convergence was introduced by Hsu and Robbins [1] as follows.

Definition 1.1. *A sequence $\{X_n, n \ge 1\}$ of random variables converges completely to the constant θ (write $X_n \to \theta$ completely) if for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} P(|X_n - \theta| > \varepsilon) < \infty.$$

In view of the Borel-Cantelli lemma, this implies that $X_n \to \theta$ almost surely. Therefore, the complete convergence is a very important tool in establishing almost sure convergence. Hsu and Robbins [1] proved that the sequence of arithmetic means of independent and identically distributed random variables converges completely to the expected value if the variance of the summands is finite. Erdos [2] proved the converse. The result of Hsu-Robbins-Erdos is a fundamental theorem in probability theory and has been generalized and extended in several directions by many authors. One of the most important generalizations is Baum and Katz [3] for the strong law of large numbers. Due to Baum and Katz [3], one has:

Theorem 1.1. *Let $p > 1/\alpha$ and $1/2 < \alpha \le 1$. Let $\{X_n, n \ge 1\}$ be a sequence of independent and identically distributed random variables with $EX_1 = 0$. Then the following statements are equivalent:*

(i) $E|X_1|^p < \infty$;

(ii) $\sum\limits_{n=1}^{\infty} n^{p\alpha-2} P\left(\max\limits_{1 \le k \le n} \left| \sum\limits_{i=1}^{k} X_i \right| > \varepsilon n^\alpha \right) < \infty$ *for all $\varepsilon > 0$.*

Motivated by applications to sequential analysis of time series, Lai [4] extended this theorem from i.i.d. case to other dependent cases namely for some classes of φ-mixing and strong mixing sequences of random variables satisfying the following additional assumption: there exist some $\beta > 1$ and positive integer m such that as $x \to \infty$,

$$\sup_{i > m} P\left(|X_1| > x, |X_i| > x\right) = O\left(P^\beta(|X_1| > x)\right). \tag{1.2}$$

Peligrad [5] proved that the equivalence of (*i*) and (*ii*) in Theorem 1.1 holds for φ-mixing sequence without the additional assumption (1.2), but under the condition of strict stationarity. Wang [6] proved the equivalence of (*i*) and (*ii*) in Theorem 1.1 under the condition of stationarity, which is weaker than that of Peligrad [5]. Shao [7] generalized and improved the corresponding results of Wang [6] under the condition of identical distributions. Sung [8] obtained the weighted version of Baum and Katz type result for identically distributed ρ^*-mixing random variables. Wu [9] extended the well-known Baum and Katz [3] complete convergence theorem from the i.i.d. case to negatively dependent random variables and Wu [10] improved and extended this type theorem from the i.i.d. case to weighted sums of pairwise negative quadrant dependent random variables. For more details about the Baum and Katz type result, one can refer to Wang and Hu [11] and Shen [12]. The main purpose of the paper is to prove the Baum and Katz type result for weighted sums of a class of random variables satisfying the Rosenthal type inequality, which are stochastically dominated by a random variable X. The concept of stochastic domination is as follows.

Definition 1.2. *A sequence $\{X_n, n \ge 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that*

$$P(|X_n| > x) \le CP(|X| > x)$$

for all $x \ge 0$ and $n \ge 1$.

Throughout the paper, let $I(A)$ be the indicator function of the set A. The symbol C denotes a positive constant which is not necessarily the same one in each appearance, $\lfloor x \rfloor$ denotes the integer part of x and $a \wedge b \doteq \min\{a, b\}$. $a_n = O(b_n)$ stands for $|a_n/b_n| \le C$.

2. Main Results

Our main results are as follows.

Theorem 2.1. *Let $\{X_n, n \ge 1\}$ be a sequence of mean zero random variables, which is stochastically dominated by a random variable X with $E|X|^p < \infty$ for some $p > 1/\alpha$ and $1/2 < \alpha \le 1$. Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of real numbers satisfying $|a_{ni}| \le C$ for $1 \le i \le n$ and $n \ge 1$, where C is a positive constant. Suppose that (1.1) holds for any $r \ge 2$ and every $n \ge 1$, where $Z_i = a_{ni} X_i I(|X_i| \le n^\alpha)$ for $1 \le i \le n$. Then*

$$\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1 \le j \le n} \left| \sum_{i=1}^{j} a_{ni} X_i \right| > \varepsilon n^\alpha \right) < \infty, \text{ for all } \varepsilon > 0. \tag{2.1}$$

Theorem 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of mean zero random variables, which is stochastically dominated by a random variable X with $E|X|^p < \infty$ for some $p > 1/\alpha$ and $1/2 < \alpha \leq 1$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers satisfying $|a_{ni}| > 1$. Suppose that (1.1) holds for any $r \geq 2$ and every $n \geq 1$, where $Z_i = a_{ni}X_iI(|a_{ni}X_i| \leq n^\alpha)$ for $1 \leq i \leq n$. If*

$$\sum_{i=1}^{n} |a_{ni}|^q \leq n \text{ for some } q > p, \tag{2.2}$$

Then (2.1) holds true.

Together with Theorems 2.1 and 2.2, we can get the following result.

Theorem 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of mean zero random variables, which is stochastically dominated by a random variable X with $E|X|^p < \infty$ for some $p > 1/\alpha$ and $1/2 < \alpha \leq 1$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^q = O(n) \text{ for some } q > p. \tag{2.3}$$

Suppose that for any $r \geq 2$ and every $n \geq 1$, (1.1) holds for $Z_i = a_{ni}X_iI(|X_i| \leq n^\alpha)$ and $Z_i = a_{ni}X_iI(|a_{ni}X_i| \leq n^\alpha)$ for $1 \leq i \leq n$. Then (2.1) holds true.

If the condition stochastic domination is replaced by identical distribution, then we can get the following result on complete convergence for weighted sums of identically distributed random variables satisfying the Rosenthal type inequality.

Theorem 2.4. *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed random variables with $EX_1 = 0$. Let $p > 1/\alpha$, $1/2 < \alpha \leq 1$ and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers satisfying (2.3). Suppose that for any $r \geq 2$ and every $n \geq 1$, (1.1) holds for $Z_i = a_{ni}X_iI(|X_i| \leq n^\alpha)$ and $Z_i = a_{ni}X_iI(|a_{ni}X_i| \leq n^\alpha)$ for $1 \leq i \leq n$. If $E|X_1|^p < \infty$, then (2.1) holds true.*

Conversely, suppose that for every $n \geq 1$,

$$E\left|\sum_{i=1}^{n}(Z_i - EZ_i)\right|^2 \leq C\sum_{i=1}^{n} E(Z_i - EZ_i)^2, \tag{2.4}$$

where $Z_i = I(|X_i| > n^\alpha)$ for $1 \leq i \leq n$. If (2.1) holds for any array $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ satisfying (2.3), then $E|X_1|^p < \infty$.

Remark 2.1. There are many sequences of random variables satisfying the Rosenthal type inequality (1.1). See for example, independent random variables, φ-mixing identically distributed random variables with $\sum_{n=}^{\infty} \varphi^{1/2}(2^n) < \infty$ (see Shao [7]), ρ-mixing identically distributed random variables with $\sum_{n=}^{\infty} \rho^{2/p}(2^n) < \infty$ (see Shao [13]), negatively associated random variables (see Shao [14]), $\tilde{\rho}$-mixing random variables (see Utev and Peligrad [15]), φ-mixing random variables with $\sum_{n=}^{\infty} \varphi^{1/2}(n) < \infty$ (see Wang et al. [16]), asymptotically almost negatively associated random variables (see Yuan and An [17]), negatively superadditive-dependent random variables (see Wang et al. [18]). Therefore, the main results of the paper also hold for these sequences of random variables.

Remark 2.2. Bai and Cheng [19] presented a necessary and sufficient condition for the almost sure convergence of weighted sums of independent and identically distributed random variables as follows (see Theorem 2.1 in Bai and Cheng [19]): let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with mean zero and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers. Denote $T_n = \sum_{i=1}^{n} a_{ni}X_i$. Suppose that $E|X|^\beta < \infty$ and

$$\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n) \tag{2.5}$$

holds for some $1 < p < 2$ and $1/p = 1/\alpha + 1/\beta$. Then

$$T_n/n^{1/p} \to 0 \quad a.s. \tag{2.6}$$

Conversely, if (2.6) is true for any coefficient arrays satisfying (2.5), then $E|X|^\beta < \infty$.

Bai and Cheng's assumptions reduce to requiring the existence of a moment of order $\beta > 1/\alpha$ and the assumption on the weights becomes $q > 1/\alpha$. Compared with Theorem 2.1 in Bai and Cheng [19], Theorem 2.4 in our paper presents a necessary and sufficient condition for the complete convergence of weighted sums of a class of random variables satisfying the Rosenthal type inequality, which is more general than almost sure convergence. It is easily seen that if $p\alpha \geq 2$, then (2.1) implies (2.6). Hence, the result of Theorem 2.4 in the paper generalizes and improves the corresponding one of Theorem 2.1 in Bai and Cheng [19].

3. Proof of the Main results

To prove the main results of the paper, we need the following useful lemma. For the proof, one can refer to Wu ([20], [21]), Shen ([22], [23]) or Shen and Wu [24].

Lemma 3.1. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. Then for any $\alpha > 0$ and $b > 0$,*

$$E|X_n|^\alpha I\left(|X_n| \leq b\right) \leq C_1 \left[E|X|^\alpha I\left(|X| \leq b\right) + b^\alpha P\left(|X| > b\right)\right] \tag{3.1}$$

and

$$E|X_n|^\alpha I\left(|X_n| > b\right) \leq C_2 E|X|^\alpha I\left(|X| > b\right), \tag{3.2}$$

where C_1 and C_2 are positive constants. Consequently, $E|X_n|^\alpha \leq CE|X|^\alpha$.

Proof of Theorem 2.1.

For $1 \leq i \leq n$ and $n \geq 1$, define $X'_{ni} = X_i I(|X_i| \leq n^\alpha)$. Since $EX_i = 0$ and $|a_{ni}| \leq C$, we have that

$$
\begin{aligned}
n^{-\alpha} \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a_{ni} E X'_{ni} \right| &= n^{-\alpha} \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a_{ni} E X_i I(|X_i| > n^\alpha) \right| \\
&\leq n^{-\alpha} \max_{1 \leq j \leq n} \sum_{i=1}^{j} |a_{ni} E X_i I(|X_i| > n^\alpha)| \\
&\leq n^{-\alpha} \sum_{i=1}^{n} |a_{ni}| E|X_i| I(|X_i| > n^\alpha) \\
&\leq C n^{-\alpha} \sum_{i=1}^{n} |a_{ni}| E|X| I(|X| > n^\alpha) \\
&\leq C n^{1-\alpha} E|X| I(|X| > n^\alpha) \\
&\leq C n^{1-p\alpha} E|X|^p \to 0, \quad \text{as } n \to \infty.
\end{aligned}
$$

Hence for all n large enough, we have

$$n^{-\alpha} \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a_{ni} E X'_{ni} \right| < \frac{\varepsilon}{2}.$$

It follows that

$$
\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon n^{\alpha}\right)
$$

$$
\le \sum_{n=1}^{\infty} n^{p\alpha-2}\sum_{i=1}^{n} P(|X_i| > n^{\alpha}) + \sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X'_{ni}\right| > \varepsilon n^{\alpha}\right)
$$

$$
\le C\sum_{n=1}^{\infty} n^{p\alpha-1} P(|X| > n^{\alpha}) + C\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}(X'_{ni} - EX'_{ni})\right| > \frac{\varepsilon n^{\alpha}}{2}\right)
$$

$$
\doteq CI + CJ. \tag{3.3}
$$

It is easily seen that

$$
I = \sum_{n=1}^{\infty} n^{p\alpha-1}\sum_{i=n}^{\infty} P(i^{\alpha} < |X| \le (i+1)^{\alpha})
$$

$$
= \sum_{i=1}^{\infty} P(i^{\alpha} < |X| \le (i+1)^{\alpha})\sum_{n=1}^{i} n^{p\alpha-1}
$$

$$
\le C\sum_{i=1}^{\infty} P(i^{\alpha} < |X| \le (i+1)^{\alpha}) i^{p\alpha}
$$

$$
\le CE|X_1|^p < \infty. \tag{3.4}
$$

Next, we show $J < \infty$. We have by Markov inequality and (1.1) that for any $r \ge 2$,

$$
J \le \left(\frac{2}{\varepsilon}\right)^r \sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2} E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}(X'_{ni} - EX'_{ni})\right|^r\right)
$$

$$
\le C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\left\{\left(\sum_{i=1}^{n} a_{ni}^2 E(X'_{ni} - EX'_{ni})^2\right)^{r/2} + \sum_{i=1}^{n} |a_{ni}|^r E|X'_{ni} - EX'_{ni}|^r\right\}
$$

$$
\le C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\left\{\left(\sum_{i=1}^{n} a_{ni}^2 E(X'_{ni})^2\right)^{r/2} + \sum_{i=1}^{n} |a_{ni}|^r E|X'_{ni}|^r\right\}
$$

$$
\le C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\left(\sum_{i=1}^{n} E|X_i|^2 I(|X_i| \le n^{\alpha})\right)^{r/2} + C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\sum_{i=1}^{n} E|X_i|^r I(|X_i| \le n^{\alpha})
$$

$$
\doteq CJ_1 + CJ_2 \tag{3.5}
$$

In the last inequality, we used the fact that $|a_{ni}| \le C$ for $1 \le i \le n$ and $n \ge 1$.

If $p \ge 2$, then we take $r > \max\{(p\alpha - 1)/(\alpha - 1/2), p\}$. Taking into account Lemma 3.1, we have that

$$
J_1 \le C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\left[\sum_{i=1}^{n}(EX^2 I(|X| \le n^{\alpha}) + n^{2\alpha}P(|X| > n^{\alpha}))\right]^{r/2}
$$

$$
\le C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\left[\sum_{i=1}^{n}(EX^2 I(|X| \le n^{\alpha}) + EX^2 I(|X| > n^{\alpha}))\right]^{r/2}
$$

$$
\le C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2+r/2} < \infty. \tag{3.6}
$$

Since $r > p$, it follows by Lemma 3.1 again and (3.4) that

$$
\begin{aligned}
J_2 &\leq C \sum_{n=1}^{\infty} n^{p\alpha - r\alpha - 2} \sum_{i=1}^{n} \left(E|X|^r I(|X| \leq n^\alpha) + n^{r\alpha} P(|X| > n^\alpha) \right) \\
&= C \sum_{n=1}^{\infty} n^{p\alpha - r\alpha - 1} E|X|^r I(|X| \leq n^\alpha) + C \sum_{n=1}^{\infty} n^{p\alpha - 1} P(|X| > n^\alpha) \\
&\leq C \sum_{n=1}^{\infty} n^{p\alpha - r\alpha - 1} \sum_{i=1}^{n} E|X|^r I((i-1)^\alpha < |X| \leq i^\alpha) + C \\
&= C \sum_{i=1}^{\infty} E|X|^r I((i-1)^\alpha < |X| \leq i^\alpha) \sum_{n=i}^{\infty} n^{p\alpha - r\alpha - 1} + C \\
&\leq C \sum_{i=1}^{\infty} E|X|^r I((i-1)^\alpha < |X| \leq i^\alpha) i^{p\alpha - r\alpha} + C \\
&\leq CE|X|^p + C < \infty.
\end{aligned}
\tag{3.7}
$$

If $p < 2$, then we take $r = 2$. Since $r > p$, (3.7) still holds, and so $J_1 = J_2 < \infty$. This completes the proof of the theorem.

Proof of Theorem 2.2.

If $p < 2$, then we can take $\delta > 0$ such that $p < p + \delta < \min\{2, q\}$. Since $|a_{ni}| > 1$, we have that $\sum_{i=1}^{n} |a_{ni}|^{p+\delta} \leq \sum_{i=1}^{n} |a_{ni}|^q \leq n$. Thus we may assume that (2.2) holds for some $p < q < 2$ when $p < 2$.

Let $S'_{nj} = \sum_{i=1}^{j} a_{ni} X_i I(|a_{ni} X_i| \leq n^\alpha)$ for $1 \leq j \leq n$ and $n \geq 1$. In view of $EX_i = 0$, we have by Lemma 3.1 that

$$
\begin{aligned}
n^{-\alpha} \max_{1 \leq j \leq n} \left| ES'_{nj} \right| &\leq n^{-\alpha} \sum_{i=1}^{n} E|a_{ni} X_i| I(|a_{ni} X_i| > n^\alpha) \\
&\leq n^{-p\alpha} \sum_{i=1}^{n} E|a_{ni} X_i|^p \\
&\leq C n^{-p\alpha} \sum_{i=1}^{n} |a_{ni}|^p E|X|^p \\
&\leq C n^{-p\alpha} \left(\sum_{i=1}^{n} |a_{ni}|^q \right)^{p/q} n^{1 - p/q} E|X|^p \\
&\leq C n^{1 - p\alpha} E|X|^p \to 0, \quad \text{as } n \to \infty.
\end{aligned}
$$

Hence for all n large enough, we have that

$$
n^{-\alpha} \max_{1 \leq j \leq n} \left| ES'_{nj} \right| < \varepsilon/2.
$$

It follows that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon n^{\alpha}\right) \\
\le\ & \sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le i\le n}|a_{ni}X_i| > n^{\alpha}\right) + \sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}\left|S'_{nj}\right| > \varepsilon n^{\alpha}\right) \\
\le\ & C\sum_{n=1}^{\infty} n^{p\alpha-2}\sum_{i=1}^{n} P(|a_{ni}X| > n^{\alpha}) + C\sum_{n=1}^{\infty} P\left(\max_{1\le j\le n}\left|S'_{nj} - ES'_{nj}\right| > \frac{\varepsilon n^{\alpha}}{2}\right) \\
\doteq\ & CI + CJ
\end{aligned}
\tag{3.8}
$$

For $1 \le j \le n-1$ and $n \ge 2$, let

$$
I_{nj} = \left\{1 \le i \le n : n^{1/q}(j+1)^{-1/q} < |a_{ni}| \le n^{1/q} j^{-1/q}\right\}.
\tag{3.9}
$$

Thus, $\{I_{nj}, 1 \le j \le n-1\}$ are disjoint and $\bigcup_{j=1}^{n-1} I_{nj} = \{1 \le i \le n : a_{ni} \ne 0\}$. It follows by (2.2) that

$$
n \ge \sum_{\{1\le i\le n: a_{ni}\ne 0\}} |a_{ni}|^q = \sum_{j=1}^{n-1}\sum_{i\in I_{nj}} |a_{ni}|^q \ge n\sum_{j=1}^{k}\frac{1}{j+1}\#I_{nj} \ge \frac{n}{k+1}\sum_{j=1}^{k}\#I_{nj},
\tag{3.10}
$$

which implies that $\sum_{j=1}^{k} \#I_{nj} \le k+1$ for $1 \le k \le n-1$.

Denote $t = 1/(\alpha - 1/q)$. It is easily checked that,

$$
\begin{aligned}
I\ \le\ & C\sum_{n=2}^{\infty} n^{p\alpha-2}\sum_{j=1}^{n-1}\sum_{i\in I_{nj}} P(|a_{ni}X| > n^{\alpha}) \\
\le\ & C\sum_{n=2}^{\infty} n^{p\alpha-2}\sum_{j=1}^{n-1} P\left(|X|^t > nj^{t/q}\right)\#I_{nj} \\
\le\ & C\sum_{n=2}^{\infty} n^{p\alpha-2}\sum_{j=1}^{n-1}\#I_{nj}\sum_{k\ge\lfloor nj^{t/q}\rfloor} P\left(k < |X|^t \le k+1\right) \\
\le\ & C\sum_{n=2}^{\infty} n^{p\alpha-2}\sum_{k=n}^{\infty} P\left(k < |X|^t \le k+1\right)\sum_{j=1}^{(n-1)\wedge\lfloor((k+1)/n)^{q/t}\rfloor}\#I_{nj} \\
\le\ & C\sum_{n=2}^{\infty} n^{p\alpha-2}\sum_{k=n}^{\infty} P\left(k < |X|^t \le k+1\right)\left(n \wedge \left(\left\lfloor\left(\frac{k+1}{n}\right)^{q/t}\right\rfloor + 1\right)\right) \\
\le\ & C\sum_{n=1}^{\infty} n^{p\alpha-2}\sum_{k=n}^{\lceil n^{1+t/q}\rceil} P\left(k < |X|^t \le k+1\right)\left\lfloor\left(\frac{k+1}{n}\right)^{q/t} + 1\right\rfloor \\
& + C\sum_{n=1}^{\infty} n^{p\alpha-1}\sum_{k=\lfloor n^{1+t/q}\rfloor+1}^{\infty} P\left(k < |X|^t \le k+1\right) \\
=:\ & I_1 + I_2.
\end{aligned}
\tag{3.11}
$$

Since $p\alpha - 2 - q/t = -\alpha(q-p) - 1 < -1$, we obtain

$$
\begin{aligned}
I_1 &\leq C \sum_{n=1}^{\infty} n^{p\alpha-2-q/t} \sum_{k=n}^{\lfloor n^{1+t/q}\rfloor} P\left(k < |X|^t \leq k+1\right) k^{q/t} \\
&\leq C \sum_{k=1}^{\infty} P\left(k < |X|^t \leq k+1\right) k^{q/t} \sum_{n=\lfloor k^{q/(q+t)}\rfloor}^{k} n^{p\alpha-2-q/t} \\
&\leq C \sum_{k=1}^{\infty} P\left(k < |X|^t \leq k+1\right) k^{q/t-q\alpha(q-p)/(q+t)} \\
&\leq CE|X|^p < \infty.
\end{aligned}
\tag{3.12}
$$

We also obtain

$$
\begin{aligned}
I_2 &\leq C \sum_{k=1}^{\infty} P\left(k < |X|^t \leq k+1\right) \sum_{n=1}^{\lfloor k^{q/(t+q)}\rfloor} n^{p\alpha-1} \\
&\leq C \sum_{k=1}^{\infty} P\left(k < |X|^t \leq k+1\right) k^{p(\alpha-1/q)} \\
&\leq CE|X|^p < \infty.
\end{aligned}
\tag{3.13}
$$

From $I_1 < \infty$ and $I_2 < \infty$, we have $I < \infty$. Thus, it remains to show that $J < \infty$. We have by Markov inequality and (1.1) that for any $r \geq 2$,

$$
\begin{aligned}
J &\leq C \sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2} E\left(\max_{1\leq j\leq n}\left|S'_{nj} - ES'_{nj}\right|^r\right) \\
&\leq C \sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2} \left(\sum_{i=1}^{n} E|a_{ni}X_i|^2 I(|a_{ni}X_i| \leq n^{\alpha})\right)^{r/2} \\
&\quad + C \sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2} \sum_{i=1}^{n} E|a_{ni}X_i|^r I(|a_{ni}X_i| \leq n^{\alpha}) \\
&\doteq J_1 + J_2.
\end{aligned}
\tag{3.14}
$$

Observe that for $r \geq q$ and $n > m$,

$$
n \geq \sum_{j=1}^{n-1}\sum_{i\in I_{nj}} |a_{ni}|^q \geq n \sum_{j=1}^{n-1} \frac{1}{j+1} \#I_{nj} \geq n(m+1)^{r/q-1} \sum_{j=m}^{n-1} (j+1)^{-r/q} \#I_{nj},
$$

which implies that $\sum_{j=m}^{n-1} j^{-r/q} \#I_{nj} \leq Cm^{-(r/q-1)}$ for $r \geq q$ and $n > m$. For J_1 and J_2, we consider the following two cases.

(i) If $p \geq 2$, then we take $r > \max\{(p\alpha-1)/(\alpha-1/2), q\}$. Taking into account Lemma 3.1, we have that

$$
\begin{aligned}
J_1 &\leq C \sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2} \left(\sum_{i=1}^{n} |a_{ni}|^2 EX^2\right)^{r/2} \\
&\leq C \sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2} \left(\sum_{i=1}^{n} |a_{ni}|^q\right)^{r/2} \\
&\leq C \sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2+r/2} < \infty.
\end{aligned}
\tag{3.15}
$$

The second inequality follows by the fact that $|a_{ni}| > 1$.

It follows by Lemma 3.1 and $I < \infty$ that

$$
\begin{aligned}
J_2 &\leq C \sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2} \sum_{i=1}^{n} \left(E|a_{ni}X|^r I(|a_{ni}X| \leq n^\alpha) + n^{r\alpha} P(|a_{ni}X| > n^\alpha) \right) \\
&\leq C \sum_{n=2}^{\infty} n^{p\alpha-r\alpha-2} \sum_{j=1}^{n-1} \sum_{i \in I_{nj}} E|a_{ni}X|^r I(|a_{ni}X| \leq n^\alpha) \\
&\leq C \sum_{n=2}^{\infty} n^{p\alpha-r\alpha-2+r/q} \sum_{j=1}^{n-1} j^{-r/q} \#I_{nj} E|X|^r I\left(|X|^t \leq n(j+1)^{t/q} \right) \\
&\leq C \sum_{n=2}^{\infty} n^{p\alpha-r\alpha-2+r/q} \sum_{j=1}^{n-1} j^{-r/q} \#I_{nj} \sum_{0 \leq k \leq \lfloor n(j+1)^{t/q} \rfloor} E|X|^r I\left(k < |X|^t \leq k+1 \right) \\
&= C \sum_{n=2}^{\infty} n^{p\alpha-r\alpha-2+r/q} \sum_{j=1}^{n-1} j^{-r/q} \#I_{nj} \sum_{k=0}^{2n} E|X|^r I\left(k < |X|^t \leq k+1 \right) \\
&\quad + C \sum_{n=2}^{\infty} n^{p\alpha-r\alpha-2+r/q} \sum_{j=1}^{n-1} j^{-r/q} \#I_{nj} \sum_{k=2n+1}^{\lfloor n(j+1)^{t/q} \rfloor} E|X|^r I\left(k < |X|^t \leq k+1 \right) \\
&\doteq C J_3 + C J_4.
\end{aligned}
\tag{3.16}
$$

Since $p\alpha - r\alpha - 2 + r/q < q\alpha - r\alpha - 2 + r/q = -(r-q)(\alpha - 1/q) - 1 < -1$ and $q > p$, we have that

$$
\begin{aligned}
J_3 &= \sum_{n=2}^{\infty} n^{p\alpha-r\alpha-2+r/q} \sum_{k=0}^{2n} E|X|^r I\left(k < |X|^t \leq k+1 \right) \sum_{j=1}^{n-1} j^{-r/q} \#I_{nj} \\
&\leq C \sum_{n=2}^{\infty} n^{p\alpha-r\alpha-2+r/q} \sum_{k=0}^{2n} E|X|^r I\left(k < |X|^t \leq k+1 \right) \\
&\leq C \sum_{k=1}^{\infty} E|X|^r I\left(k < |X|^t \leq k+1 \right) \sum_{n=\lfloor k/2 \rfloor}^{\infty} n^{p\alpha-r\alpha-2+r/q} \\
&\leq C \sum_{k=1}^{\infty} E|X|^r I\left(k < |X|^t \leq k+1 \right) k^{p\alpha-r\alpha-1+r/q} \\
&\leq C \sum_{k=1}^{\infty} P\left(k < |X|^t \leq k+1 \right) k^{p\alpha-1} \\
&\leq C E|X|^p < \infty.
\end{aligned}
\tag{3.17}
$$

Since $1/t + 1/q - \alpha = 0$ and $p\alpha - 2 - q/t = -\alpha(q - p) - 1 < -1$, we also have that

$$
\begin{aligned}
J_4 &\leq C \sum_{n=2}^{\infty} n^{p\alpha - r\alpha - 2 + r/q} \sum_{k=2n+1}^{\lfloor n^{(q+t)/q} \rfloor} E|X|^r I\left(k < |X|^t \leq k+1\right) \sum_{j=\lfloor (k/n)^{q/t} \rfloor - 1}^{n-1} j^{-r/q} \# I_{nj} \\
&\leq C \sum_{n=2}^{\infty} n^{p\alpha - r\alpha - 2 + r/q} \sum_{k=2n+1}^{\lfloor n^{(q+t)/q} \rfloor} E|X|^r I\left(k < |X|^t \leq k+1\right) \left(\left\lfloor \left(\frac{k}{n}\right)^{q/t} \right\rfloor - 1\right)^{-(r/q-1)} \\
&\leq C \sum_{k=5}^{\infty} E|X|^r I\left(k < |X|^t \leq k+1\right) k^{-(r-q)/t} \sum_{n=\lfloor k^{q/(q+t)} \rfloor}^{\lfloor k/2 \rfloor} n^{p\alpha - 2 - q/t} \\
&\leq C \sum_{k=5}^{\infty} E|X|^r I\left(k < |X|^t \leq k+1\right) k^{-(r-q)/t - (\alpha - 1/q)(q-p)} \\
&\leq CE|X|^p < \infty.
\end{aligned}
\tag{3.18}
$$

From $J_3 < \infty$ and $J_4 < \infty$, we have $J_2 < \infty$.

(ii) If $p < 2$, then we take $r = 2$. As noted above, we may assume that $p < q < 2$. Since $r > q$, as in the case $p \geq 2$, we have $J_1 = J_2 \leq CE|X_1|^p < \infty$. This completes the proof of the theorem.

Proof of Theorem 2.3.

The assumption (2.3) implies that $\sum_{i=1}^{n} |a_{ni}|^q \leq Cn$ for some $q > p$, where C is a positive constant. For $n \geq 1$, let

$$
A_n = \{1 \leq i \leq n : |a_{ni}|/C^{1/q} \leq 1\}, B_n = \{1 \leq i \leq n : |a_{ni}|/C^{1/q} > 1\},
\tag{3.19}
$$

and let $a'_{ni} = a_{ni}/C^{1/q}$ if $i \in A_n, a'_{ni} = 0$ otherwise, and $a''_{ni} = a_{ni}/C^{1/q}$ if $i \in B_n, a''_{ni} = 0$ otherwise. Then

$$
\max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} \frac{a_{ni}}{C^{1/q}} X_i \right| \leq \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a'_{ni} X_i \right| + \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a''_{ni} X_i \right|.
\tag{3.20}
$$

It follows that

$$
\begin{aligned}
\sum_{n=1}^{\infty} n^{p\alpha - 2} P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a_{ni} X_i \right| > \varepsilon n^{\alpha}\right) &\leq \sum_{n=1}^{\infty} n^{p\alpha - 2} P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a'_{ni} X_i \right| > \frac{\varepsilon n^{\alpha}}{2C^{1/q}}\right) \\
&+ \sum_{n=1}^{\infty} n^{p\alpha - 2} P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a''_{ni} X_i \right| > \frac{\varepsilon n^{\alpha}}{2C^{1/q}}\right) \\
&\doteq I + J.
\end{aligned}
\tag{3.21}
$$

By Theorem 2.1, we have $I < \infty$. By Theorem 2.2, we have $J < \infty$. Then (2.1) holds. This completes the proof of the theorem.

Proof of Theorem 2.4.

The sufficient part can be obtained by Theorem 2.3. So we only need to show necessity.

Choose, for each $n \geq 1, a_{n1} = \cdots = a_{nn} = 1$. Then $\{a_{ni}\}$ satisfies (2.3). By (2.1), we obtain that

$$
\sum_{n=1}^{\infty} n^{p\alpha - 2} P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} X_i \right| > \varepsilon n^{\alpha}\right) < \infty, \quad \forall \varepsilon > 0,
\tag{3.22}
$$

which implies that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n} |X_j| > \varepsilon n^\alpha\right) \\
&\le \sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n} \left|\sum_{i=1}^{j} X_i\right| > \frac{\varepsilon n^\alpha}{2}\right) + \sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n} \left|\sum_{i=1}^{j-1} X_i\right| > \frac{\varepsilon n^\alpha}{2}\right) \\
&< \infty, \quad \forall \varepsilon > 0.
\end{aligned}
\tag{3.23}
$$

Observe that

$$
\begin{aligned}
\infty &> \sum_{i=1}^{\infty} \sum_{n=2^{i-1}+1}^{2^i} n^{p\alpha-2} P\left(\max_{1\le j\le n} |X_j| > \varepsilon n^\alpha\right) \\
&\ge \begin{cases} \sum_{i=1}^{\infty} (2^{i-1})^{p\alpha-2} 2^{i-1} P\left(\max_{1\le j\le 2^{i-1}} |X_j| > \varepsilon (2^i)^\alpha\right), & \text{if } p\alpha \ge 2, \\ \sum_{i=1}^{\infty} (2^i)^{p\alpha-2} 2^{i-1} P\left(\max_{1\le j\le 2^{i-1}} |X_j| > \varepsilon (2^i)^\alpha\right), & \text{if } 1 < p\alpha < 2, \end{cases} \\
&\ge \begin{cases} \sum_{i=1}^{\infty} P\left(\max_{1\le j\le 2^{i-1}} |X_j| > \varepsilon (2^i)^\alpha\right), & \text{if } p\alpha \ge 2, \\ 2^{p\alpha-2} \sum_{i=1}^{\infty} P\left(\max_{1\le j\le 2^{i-1}} |X_j| > \varepsilon (2^i)^\alpha\right), & \text{if } 1 < p\alpha < 2. \end{cases}
\end{aligned}
$$

By the convergence of series, we have that for any $\varepsilon > 0$,

$$
P\left(\max_{1\le j\le 2^{i-1}} |X_j| > \varepsilon (2^i)^\alpha\right) \to 0 \ \text{ as } i \to \infty.
$$

Hence, for any $n \ge 1$, there exists a i such that $2^{i-2} \le n < 2^{i-1}$, thus

$$
\begin{aligned}
P\left(\max_{1\le j\le n} |X_j| > n^\alpha\right) &\le P\left(\max_{1\le j\le 2^{i-1}} |X_j| > (2^{i-2})^\alpha\right) \\
&= P\left(\max_{1\le j\le 2^{i-1}} |X_j| > 2^{-2\alpha}(2^i)^\alpha\right) \to 0, \text{ as } n \to \infty.
\end{aligned}
\tag{3.24}
$$

Since

$$
P\left(\max_{1\le j\le n} |X_j| > n^\alpha\right) = \sum_{j=1}^{n} P\left(|X_j| > n^\alpha, \max_{1\le i\le j-1} |X_i| \le n^\alpha\right),
$$

it follows that

$$
nP(|X_1| > n^\alpha) = P\left(\max_{1\le j\le n} |X_j| > n^\alpha\right) + \sum_{j=1}^{n} P\left(|X_j| > n^\alpha, \max_{1\le i\le j-1} |X_i| > n^\alpha\right).
\tag{3.25}
$$

It is easy to check that

$$\sum_{j=1}^{n} P\left(|X_j| > n^\alpha, \max_{1 \le i \le j-1} |X_i| > n^\alpha\right)$$

$$= \sum_{j=1}^{n} EI\left(|X_j| > n^\alpha\right) I\left(\max_{1 \le i \le j-1} |X_i| > n^\alpha\right)$$

$$= \sum_{j=1}^{n} E\left((I(|X_j| > n^\alpha) - P(|X_j| > n^\alpha))I\left(\max_{1 \le i \le j-1} |X_i| > n^\alpha\right)\right)$$

$$+ \sum_{j=1}^{n} EP(|X_j| > n^\alpha)I\left(\max_{1 \le i \le j-1} |X_i| > n^\alpha\right)$$

$$\le E\left(\sum_{j=1}^{n} (I(|X_j| > n^\alpha) - P(|X_1| > n^\alpha))I\left(\max_{1 \le i \le n} |X_i| > n^\alpha\right)\right)$$

$$+ nP(|X_1| > n^\alpha)P\left(\max_{1 \le i \le n} |X_i| > n^\alpha\right)$$

$$\doteq I + II. \tag{3.26}$$

According to the Cauchy-Schwarz inequality, we have by (2.4) that

$$I^2 \le \left(E\left|\sum_{j=1}^{n} (I(|X_j| > n^\alpha) - P(|X_1| > n^\alpha))I\left(\max_{1 \le i \le n} |X_i| > n^\alpha\right)\right|\right)^2$$

$$\le E\left(\sum_{j=1}^{n} (I(|X_j| > n^\alpha) - P(|X_1| > n^\alpha))\right)^2 P\left(\max_{1 \le i \le n} |X_i| > n^\alpha\right)$$

$$\le C\left(\sum_{j=1}^{n} E(I(|X_j| > n^\alpha) - P(|X_1| > n^\alpha))^2\right) P\left(\max_{1 \le i \le n} |X_i| > n^\alpha\right)$$

$$= Cn\text{Var}(I(|X_1| > n^\alpha))P\left(\max_{1 \le i \le n} |X_i| > n^\alpha\right)$$

$$\le CnP(|X_1| > n^\alpha)P\left(\max_{1 \le i \le n} |X_i| > n^\alpha\right)$$

$$\le \left[\frac{1}{4}nP(|X_1| > n^\alpha) + CP\left(\max_{1 \le i \le n} |X_i| > n^\alpha\right)\right]^2. \tag{3.27}$$

Now we return the estimate from the relation (3.27) into the relation (3.26) and then into relation (3.25) and get

$$\frac{3}{4}nP(|X_1| > n^\alpha) \le 2(1 + C)P\left(\max_{1 \le i \le n} |X_i| > n^\alpha\right) + nP(|X_1| > n^\alpha)P\left(\max_{1 \le i \le n} |X_i| > n^\alpha\right). \tag{3.28}$$

By (3.24) and (3.28), for all n sufficiently large, we have

$$nP(|X_1| > n^\alpha) \le 4(1 + C)P\left(\max_{1 \le i \le n} |X_i| > n^\alpha\right). \tag{3.29}$$

Relations (3.23) and (3.29) finally yield that

$$\sum_{n=1}^{\infty} n^{p\alpha - 1} P(|X_1| > n^\alpha) < \infty.$$

Hence,

$$
\begin{aligned}
\infty \; > \; & \sum_{n=1}^{\infty} n^{p\alpha-1} P(|X_1| > n^\alpha) \\
= \; & \sum_{n=1}^{\infty} n^{p\alpha-1} \sum_{i=n}^{\infty} P(i^\alpha < |X_1| \le (i+1)^\alpha) \\
= \; & \sum_{i=1}^{\infty} P(i^\alpha < |X_1| \le (i+1)^\alpha) \sum_{n=1}^{i} n^{p\alpha-1} \\
\ge \; & C \sum_{i=1}^{\infty} P(i^\alpha < |X_1| \le (i+1)^\alpha) i^{p\alpha} \\
\ge \; & C \sum_{i=1}^{\infty} E\left(I(i^\alpha < |X_1| \le (i+1)^\alpha)(i+1)^{p\alpha}\right) \\
\ge \; & C \sum_{i=1}^{\infty} E\left(I(i^\alpha < |X_1| \le (i+1)^\alpha)|X_1|^p\right) \\
\ge \; & CE|X_1|^p,
\end{aligned}
$$

which implies that $E|X_1|^p < \infty$. This completes the proof of the theorem.

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