



A Discussion on “ α - ψ -Geraghty Contraction Type Mappings”

Erdal Karapınar^{a,b}

^aAtilim University, Department of Mathematics, 06836, Incek, Ankara, Turkey

^bNonlinear Analysis and Applied Mathematics Research Group (NAAM), King Abdulaziz University, Jeddah, Saudi Arabia

Abstract. In this paper we discuss the subadditivity property of some auxiliary functions which have been used to generalize the contractive conditions on maps. Our results show that this property is not required in many cases and therefore, can be removed. We emphasize that in several papers (see e.g.[1]-[11]) dealing with such contraction mappings, the hypotheses of the main theorems can be restated in the light of our results.

1. Introduction and Preliminaries

Throughout the paper, we shall follow the notions and definitions in [11]. Let \mathcal{F} be the collection of all functions $\beta : [0, \infty) \rightarrow [0, 1)$ which satisfies the condition:

$$\lim_{n \rightarrow \infty} \beta(t_n) = 1 \text{ implies } \lim_{n \rightarrow \infty} t_n = 0.$$

By using the auxiliary function $\beta \in \mathcal{F}$, Geraghty [15] introduced an interesting contraction and investigated the existence and uniqueness of such mappings.

Theorem 1.1. (Geraghty [15].) *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be an operator. If T satisfies the following inequality:*

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y), \text{ for any } x, y \in X, \tag{1}$$

where $\beta \in \mathcal{F}$, then T has a unique fixed point.

Let $T : X \rightarrow X$ be a map and $\alpha : X \times X \rightarrow \mathbb{R}$ be a function. Then, T is said to be α -admissible [16] if

$$\alpha(x, y) \geq 1 \text{ implies } \alpha(Tx, Ty) \geq 1.$$

An α -admissible map T is said to be *triangular α -admissible* [14] if

$$\alpha(x, z) \geq 1 \text{ and } \alpha(z, y) \geq 1 \text{ imply } \alpha(x, y) \geq 1. \tag{2}$$

Lemma 1.2. [14] *Let $T : X \rightarrow X$ be a triangular α -admissible map. Assume that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then, we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

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Email address: erdalkarapinar@yahoo.com, ekarapinar@atilim.edu.tr (Erdal Karapınar)

Definition 1.3. Let (X, d) be a complete metric space, $\alpha : X \times X \rightarrow \mathbb{R}$ be a function, and let $T : X \rightarrow X$ be a map. We say that the sequence $\{x_n\}$ is α -regular if the following condition is satisfied:

If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \rightarrow x \in X$ as $n \rightarrow +\infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k .

Let Ψ denote the class of the functions $\psi : [0, \infty) \rightarrow [0, \infty)$ which satisfy the following conditions:

- (ψ_1) ψ is nondecreasing;
- (ψ_2) ψ is subadditive, that is, $\psi(s + t) \leq \psi(s) + \psi(t)$;
- (ψ_3) ψ is continuous;
- (ψ_4) $\psi(t) = 0 \Leftrightarrow t = 0$.

Let Φ denote the class of the functions $\psi : [0, \infty) \rightarrow [0, \infty)$ which satisfies (ψ_1), (ψ_3) and (ψ_4).

Use of auxiliary functions to generalize the contractive conditions on maps have been a subject of interest in fixed point theory. Many papers studying such contractions via auxiliary function $\psi \in \Psi$ appeared in the literature recently, see e.g. [1]-[11]. Very recently, Karapınar and Samet [12] noticed that an auxiliary function ψ in the class Ψ has very strong properties.

Lemma 1.4. (See [12]) Let (X, d) be a metric space and $\psi \in \Psi$. Then, a function $d_\psi : X \times X \rightarrow [0, \infty)$ defined by $d_\psi(x, y) = \psi(d(x, y))$ forms a metric on X . Moreover, (X, d) is complete if and only if (X, d_ψ) is complete.

Notice also that in [17], the author get a similar results as in Lemma 1.4.

In [13], the authors introduced the following contraction types.

Definition 1.5. Let (X, d) be a metric space, and let $\alpha : X \times X \rightarrow \mathbb{R}$ be a function. A mapping $T : X \rightarrow X$ is said to be a generalized α -Geraghty contraction if there exists $\beta \in F$ such that

$$\alpha(x, y)d(Tx, Ty) \leq \beta(M(x, y))\psi(M(x, y)) \text{ for any } x, y \in X, \quad (3)$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

Definition 1.6. Let (X, d) be a metric space, and let $\alpha : X \times X \rightarrow \mathbb{R}$ be a function. A map $T : X \rightarrow X$ is called α - ψ -Geraghty contraction type map if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$

$$\alpha(x, y)d(Tx, Ty) \leq \beta(d(x, y))\psi(d(x, y)). \quad (4)$$

In [13], the authors proved the following theorems.

Theorem 1.7. Let (X, d) be a complete metric space, $\alpha : X \times X \rightarrow \mathbb{R}$ be a function, and let $T : X \rightarrow X$ be a map. Suppose that the following conditions are satisfied:

- (i) T is generalized α -Geraghty contraction type map;
- (ii) T is triangular α -admissible;
- (iii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;
- (iv) either, T is continuous, or $\{x_n\}$ is α -regular.

Then, T has a fixed point $x^* \in X$, and $\{T^n x_1\}$ converges to x^* .

Theorem 1.8. Let (X, d) be a complete metric space, $\alpha : X \times X \rightarrow \mathbb{R}$ be a function, and let $T : X \rightarrow X$ be a map. Suppose that the following conditions are satisfied:

- (1) T is α -Geraghty contraction type map;
- (2) T is triangular α -admissible;
- (3) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;
- (4) either, T is continuous, or $\{x_n\}$ is α -regular.

Then, T has a fixed point $x^* \in X$, and $\{T^n x_1\}$ converges to x^* .

For the uniqueness of a fixed point of Theorem 1.7, Theorem 1.8, the following notation is required.

(H1) For all $x, y \in \text{Fix}(T)$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.

Theorem 1.9. Adding condition (H1) to the hypotheses of Theorem 1.7 (resp. Theorem 1.8), we obtain that x^* is the unique fixed point of T .

In [11], the author defined the following notations.

Definition 1.10. Let (X, d) be a metric space, and let $\alpha : X \times X \rightarrow \mathbb{R}$ be a function. A mapping $T : X \rightarrow X$ is said to be a generalized α - ψ -Geraghty contraction if there exists $\beta \in F$ such that

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)) \text{ for any } x, y \in X, \quad (5)$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\},$$

and $\psi \in \Psi$.

Definition 1.11. Let (X, d) be a metric space, and let $\alpha : X \times X \rightarrow \mathbb{R}$ be a function. A map $T : X \rightarrow X$ is called α - ψ -Geraghty contraction type map if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq \beta(\psi(d(x, y)))\psi(d(x, y)), \quad (6)$$

where $\psi \in \Psi$.

Theorem 1.12. Let (X, d) be a complete metric space, $\alpha : X \times X \rightarrow \mathbb{R}$ be a function, and let $T : X \rightarrow X$ be a map. Suppose that the following conditions are satisfied:

- (i) T is generalized α - ψ -Geraghty contraction type map;
- (ii) T is triangular α -admissible;
- (iii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;
- (iv) either, T is continuous, or $\{x_n\}$ is α -regular.

Then, T has a fixed point $x^* \in X$, and $\{T^n x_1\}$ converges to x^* .

Theorem 1.13. Let (X, d) be a complete metric space, $\alpha : X \times X \rightarrow \mathbb{R}$ be a function, and let $T : X \rightarrow X$ be a map. Suppose that the following conditions are satisfied:

- (1) T is α - ψ -Geraghty contraction type map;
- (2) T is triangular α -admissible;
- (3) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;
- (4) either, T is continuous, or $\{x_n\}$ is α -regular.

Then, T has a fixed point $x^* \in X$, and $\{T^n x_1\}$ converges to x^* .

Theorem 1.14. Adding condition (H1) to the hypotheses of Theorem 1.12 (resp. Theorem 1.13), we obtain that x^* is the unique fixed point of T .

In this paper, we emphasize that subadditivity of ψ in the recent paper Karapınar [α - ψ -Geraghty contraction type mappings and some related fixed point results, Filomat, 28(2014),no:1, 37-48] can be removed. All results, examples and application in the mentioned paper are still valid although we omit the subadditivity condition for the auxiliary function ψ .

2. Auxiliary Results

We start this section with the following proposition.

Proposition 2.1. *Definition 1.5 (respectively, Definition 1.6) is equivalent to Definition 1.10 (respectively, Definition 1.11).*

Proof. By Lemma 1.4, $d_\psi(x, y) = \psi(d(x, y))$ is a metric whenever d is metric space. Hence, the contraction condition (5) turns into

$$\alpha(x, y)d_\psi(Tx, Ty) \leq \beta(M_\psi(x, y))M_\psi(x, y) \text{ for any } x, y \in X, \quad (7)$$

where

$$M_\psi(x, y) = \max\{d_\psi(x, y), d_\psi(x, Tx), d_\psi(y, Ty)\}.$$

Thus, Definition 1.5 is equivalent to Definition 1.10. Analogously, one can get that Definition 1.6 is equivalent to Definition 1.11. \square

The following is one of the main results of this note.

Theorem 2.2. *Theorem 1.12 is a consequence of Theorem 1.7.*

Proof. Due to Lemma 1.4, we derived that (X, d_ψ) is complete metric space By Proposition 2.1 the condition (5) turns into (5)

$$\alpha(x, y)d_\psi(Tx, Ty) \leq \beta(M_\psi(x, y))M_\psi(x, y) \text{ for any } x, y \in X, \quad (8)$$

Hence all conditions of Theorem 1.7 are satisfied. \square

By using the same argument above, we obtain the following two theorems.

Theorem 2.3. *Theorem 1.13 is a consequence of Theorem 1.8.*

Theorem 2.4. *Theorem 1.14 is a consequence of Theorem 1.9.*

Based on the observations above, we conclude that functions from the collection Ψ have a very strong condition (ψ_2), that is, subadditivity. Consequently, we notice that, in the literature, some of the published results supposed to be a generalizations of corresponding existing theorems via a auxiliary function $\psi \in \Psi$ may not be a real generalization.

3. Main Results

In this section, we improve the results given in [11] by replacing the auxiliary function $\psi \in \Psi$ with a function $\phi \in \Phi$. In other words, our auxiliary function do not need to satisfy subadditivity condition.

Definition 3.1. *Let (X, d) be a metric space, and let $\alpha : X \times X \rightarrow \mathbb{R}$ be a function. A mapping $T : X \rightarrow X$ is said to be a generalized α - ϕ -Geraghty contraction if there exists $\beta \in F$ such that*

$$\alpha(x, y)\phi(d(Tx, Ty)) \leq \beta(\phi(M(x, y)))\phi(M(x, y)) \text{ for any } x, y \in X, \quad (9)$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\},$$

and $\phi \in \Phi$.

Definition 3.2. Let (X, d) be a metric space, and let $\alpha : X \times X \rightarrow \mathbb{R}$ be a function. A map $T : X \rightarrow X$ is called α - ϕ -Geraghty contraction type map if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$

$$\alpha(x, y)\phi(d(Tx, Ty)) \leq \beta(\phi(d(x, y)))\phi(d(x, y)), \quad (10)$$

where $\phi \in \Phi$.

Theorem 3.3. Let (X, d) be a complete metric space, $\alpha : X \times X \rightarrow \mathbb{R}$ be a function, and let $T : X \rightarrow X$ be a map. Suppose that the following conditions are satisfied:

- (i) T is generalized α - ϕ -Geraghty contraction type map;
- (ii) T is triangular α -admissible;
- (iii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;
- (iv) either, T is continuous, or $\{x_n\}$ is α -regular.

Then, T has a fixed point $x^* \in X$, and $\{T^n x_1\}$ converges to x^* .

Proof. Let $x_1 \in X$ be such that $\alpha(x_1, Tx_1) \geq 1$. Define a sequence $\{x_n\} \subset X$ by $x_{n+1} = Tx_n$ for $n \in \mathbb{N}$. By following the lines in [[11],Theorem 2.4.], we get that

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \quad (11)$$

and

$$\lim_{m, n \rightarrow \infty} M(x_m, x_n) = \lim_{m, n \rightarrow \infty} d(x_m, x_n). \quad (12)$$

We assert that $\{x_n\}$ is a Cauchy sequence. Suppose, on the contrary, that we have

$$\varepsilon = \limsup_{m, n \rightarrow \infty} d(x_n, x_m) > 0. \quad (13)$$

By using the triangular inequality, we derive

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m) \\ &= d(x_n, x_{n+1}) + d(Tx_n, Tx_m) + d(x_{m+1}, x_m), \end{aligned} \quad (14)$$

which is equivalent to

$$d(x_n, x_m) - d(x_n, x_{n+1}) - d(x_{m+1}, x_m) \leq d(Tx_n, Tx_m) \quad (15)$$

Applying ϕ , we get that

$$\begin{aligned} \phi(d(x_n, x_m) - d(x_n, x_{n+1}) - d(x_{m+1}, x_m)) &\leq \phi(d(Tx_n, Tx_m)) \\ &\leq \alpha(x_n, x_m)\phi(d(Tx_n, Tx_m)) \\ &\leq \beta(\phi(M(x_n, x_m)))\phi(M(x_n, x_m)) \end{aligned} \quad (16)$$

Together with (12), (16) and (11), we deduce that

$$\begin{aligned} \lim_{m, n \rightarrow \infty} \phi(d(x_n, x_m) - d(x_n, x_{n+1}) - d(x_{m+1}, x_m)) &\leq \lim_{m, n \rightarrow \infty} \beta(\psi(M(x_n, x_m))) \lim_{m, n \rightarrow \infty} \psi(M(x_m, x_n)) \\ &\leq \lim_{m, n \rightarrow \infty} \beta(\psi(M(x_n, x_m))) \lim_{m, n \rightarrow \infty} \psi(d(x_m, x_n)). \end{aligned}$$

Employing(13), we get

$$1 \leq \lim_{m, n \rightarrow \infty} \beta(\psi(M(x_n, x_m))),$$

which implies $\lim_{m, n \rightarrow \infty} \beta(\psi(M(x_n, x_m))) = 1$. Consequently, we get $\lim_{m, n \rightarrow \infty} M(x_n, x_m) = 0$ and hence, $\lim_{m, n \rightarrow \infty} d(x_n, x_m) = 0$. It is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. The rest is exactly same as in the proof of [[11],Theorem 2.4. and Theorem 2.4]. \square

Theorem 3.4. Let (X, d) be a complete metric space, $\alpha : X \times X \rightarrow \mathbb{R}$ be a function, and let $T : X \rightarrow X$ be a map. Suppose that the following conditions are satisfied:

- (1) T is α - ϕ -Geraghty contraction type map;
- (2) T is triangular α -admissible;
- (3) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;
- (4) either, T is continuous, or $\{x_n\}$ is α -regular.

Then, T has a fixed point $x^* \in X$, and $\{T^n x_1\}$ converges to x^* .

Proof. It just follows from Theorem 3.3. \square

Theorem 3.5. Adding condition (H1) to the hypotheses of Theorem 3.3 (respectively, Theorem 3.4), we obtain that x^* is the unique fixed point of T .

The proof of theorem above is the same as in [[11],Theorem 2.7] (respectively, [[11],Theorem 3.4]).

Conclusion

We conclude first that the subadditivity condition (ψ_3) is superfluous. Hence, we get the main results of [11] by omitting subadditivity condition (ψ_3) . We also emphasize that all consequences, examples and applications in [11] are still valid for Theorem 3.3-Theorem 3.5.

Competing interests

The author declares that there is no conflict of interests regarding the publication of this article.

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