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Fixed Points and Stability in Nonlinear Neutral Integro-Differential Equations with Variable Delay

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Abstract. The nonlinear neutral integro-differential equation

$$x'(t) = -\int_{t-\tau(t)}^{t} a(t,s)\, g(x(s))\, ds + c(t)\, x'(t-\tau(t)),$$

with variable delay $\tau(t) \geq 0$ is investigated. We find suitable conditions for τ, a, c and g so that for a given continuous initial function ψ a mapping P for the above equation can be defined on a carefully chosen complete metric space S_ψ^0 in which P possesses a unique fixed point. The final result is an asymptotic stability theorem for the zero solution with a necessary and sufficient conditions. The obtained theorem improves and generalizes previous results due to Burton [6], Becker and Burton [5] and Jin and Luo [16].

1. Introduction

Lyapunov's direct method has been, for more than 100 years, the most efficient tool for investigating the stability properties of a wide variety of ordinary, functional, partial differential and integro-differential equations. Nevertheless, the application of this method to problems of stability in differential and integro-differential equations with delays has encountered serious obstacles if the delays are unbounded or if the equation has unbounded terms [7]–[9]. In recent years, several investigators have tried stability by using a new technique. Particularly, Burton, Furumochi, Becker and others began a study in which they noticed that some of these difficulties vanish or might be overcome by means of fixed point theory (see [1]–[18], [20]). The fixed point theory does not only solve the problem on stability but has other significant advantage over Lyapunov's. The conditions of the former are often averages but those of the latter are usually pointwise (see [7]).

In this paper we consider the nonlinear neutral integro-differential equation with variable delay

$$x'(t) = -\int_{t-\tau(t)}^{t} a(t,s)\, g(x(s))ds + c(t)\, x'(t-\tau(t)), \tag{1}$$

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with the initial condition

$$x(t) = \psi(t) \text{ for } t \in [m(0), 0],$$

such that $\psi \in C([m(0), 0], \mathbb{R})$ where $m(0) = \inf\{t - \tau(t),\ t \geq 0\}$.
Here $C(S_1, S_2)$ denotes the set of all continuous functions $\varphi : S_1 \to S_2$ with the supremum norm $\|\cdot\|$. Throughout this paper we assume that $a \in C(\mathbb{R}^+ \times [m(0), \infty), \mathbb{R})$, $c : \mathbb{R}^+ \to \mathbb{R}$ is differentiable and $\tau : \mathbb{R}^+ \to \mathbb{R}^+$ is differentiable with $t - \tau(t) \to \infty$ as $t \to \infty$.

Special cases of *(1)* have been considered and investigated by many authors. For example, Burton in [6], Becker and Burton in [5], Jin and Luo in [16] have studied the equation

$$x'(t) = -\int_{t-\tau(t)}^{t} a(t, s)\, g(x(s)) ds, \tag{2}$$

and have respectively proved the following theorems.

Theorem A (Burton [6]). *Suppose that $\tau(t) = r$ is constant and assume that there is an $l > 0$ such that g is an odd, Lipschitz, and increasing function on $[-l, l]$ such that $g'(x)$ is continuous, $g'(0) \neq 0$ and $0 \leqslant x - g(x)$ is non-decreasing on $[0, l]$. If there exists a constant $\alpha < 1$ such that*

$$\beta := \sup_{t\geq 0} \int_0^r \left| \int_w^r a(v + t - w, t - w)\, dv \right| dw < \frac{1}{2}, \tag{3}$$

and

$$\int_0^t A(s, s)\, ds \to \infty \text{ as } t \to \infty, \tag{4}$$

where

$$A(t, s) = \int_{t-s}^{r} a(u + s, s)\, du \text{ with } A(t, t) = \int_0^r a(u + t, t)\, du \geq 0.$$

Then the zero solution of (2) is asymptotically stable.

Theorem B (Becker and Burton [5]). *Suppose a constant $l > 0$ exists such that g satisfies a Lipschitz condition on $[-l, l]$ and $G(t, t) \geqslant 0$ for $t \geqslant 0$. Assume, further, that*
(i) g is odd and strictly increasing on $[-l, l]$;
(ii) $x - g(x)$ is non-decreasing on $[-l, l]$;
(iii) there is an $\alpha \in (0, 1)$ such that $2\int_{t-\tau(t)}^{t} |G(t, u)|\, du \leqslant \alpha$ for $t \geqslant 0$, with

$$G(t, s) = \int_t^{f(s)} a(u, s)\, du,\ G(t, t) = \int_t^{f(t)} a(u, t)\, du,$$

where f is the inverse function of $t - \tau(t)$ which is supposed to be strictly increasing. Then a $\delta \in (0, l)$ exists such that, for each continuous function $\psi : [m(0), 0] \to (-\delta, \delta)$, there is a unique continuous function $x : [m(0), \infty) \to \mathbb{R}$ with $x(t) = \psi(t)$ on $[m(0), 0]$ that satisfies (2) on $[0, \infty)$. Moreover, x is bounded by l on $[m(0), \infty)$. Furthermore, the zero solution of (2) is stable at $t = 0$. If, in addition, g is continuously differentiable, $g'(0) \neq 0$ and

$$\int_0^t G(s, s)\, ds \to \infty \text{ as } t \to \infty, \tag{5}$$

then the zero solution is asymptotically stable.

Theorem C (Jin and Luo [16]). *Suppose there exists a constant $l > 0$ such that g satisfies a Lipschitz condition on $[-l, l]$, and $H(t) \geqslant 0$ for $t \geqslant 0$ is a continuous function. Assume that there exists a continuous function q such that $|B(t,s)| \leqslant q(s)$ for $t - r(t) \leqslant s \leqslant t$. Suppose further that*

(i) g is odd and strictly increasing on $[-l, l]$;

(ii) $x - g(x)$ is non-decreasing on $[0, l]$;

(iii) there exists an $\alpha \in (0,1)$ such that, for $t \geqslant 0$

$$\int_{t-\tau(t)}^{t} |H(s) + B(t,s)|\, ds + \int_0^t e^{-\int_s^t H(v)dv} |H(s)| \left(\int_{s-\tau(s)}^{s} |H(u) + B(s,u)|\, du \right) ds$$
$$+ \int_0^t e^{-\int_s^t H(v)dv} |H(s-\tau(s)) + B(s, s-\tau(s))| \, |1 - \tau'(s)|\, ds \leq \alpha \ , \tag{6}$$

where

$$B(t,s) = \int_t^s a(u,s)\, du \text{ with } B(t, t-\tau(t)) = \int_t^{t-\tau(t)} a(u, t-\tau(t))\, du.$$

Then a $\delta \in (0, l)$ exits such that for each continuous initial function $\psi : [m(0), 0] \to (-\delta, \delta)$, there is a unique continuous function $x : [m(0), \infty) \to \mathbb{R}$ with $x(t) = \psi(t)$ on $[m(0), 0]$ that satisfies (2) on $[0, \infty)$. Moreover, x is bounded on $[m(0), \infty)$. Furthermore, the zero solution of (2) is stable at $t = 0$. If, in addition, g is continuously differentiable, $g'(0) \neq 0$ and

$$\int_0^t H(v)\, dv \to \infty \text{ as } t \to \infty, \tag{7}$$

then the zero solution is asymptotically stable.

Remark 1.1. *The result of Becker and Burton obtained in Theorems B requires that $t - \tau(t)$ is strictly increasing. However, in Theorem C, this condition is clearly removed. Also, the conditions of stability in Theorem C are less restrictive than Theorem B. That is, Theorem C improves Theorems A and B.*

Our hope is to improve Theorem C and extend it to support a wide class of nonlinear integro-differential equations with variable delay of neutral type presented as (1). Our results are obtained with no need of further assumptions on the inverse of delay $t - \tau(t)$, so that for a given continuous initial function ψ a mapping P for (1) is constructed in such a way to map, a suitably chosen, complete metric space S_ψ^0 into itself and on which P is a contraction mapping and so possesses a fixed point. This procedure will enable us to establish and prove an asymptotic stability theorem for the zero solution of (1) using less restrictive conditions. The theorem is concluded by means of the contraction mapping principle and the end results improve and generalize the main results in [5, 6, 16]. We provide an example to illustrate our claim.

2. Main results

For each $\psi \in C([m(0), 0], \mathbb{R})$, a solution of (1) through $(0, \psi)$ is a continuous function $x : [m(0), T) \to \mathbb{R}$ for some positive constant $T > 0$ such that x satisfies (1) on $[0, T)$ and $x = \psi$ on $[m(0), 0]$. We denote such a solution by $x(t) = x(t, 0, \psi)$. From the existence theory we can conclude that for each $\psi \in C([m(0), 0], \mathbb{R})$, there exists a unique solution $x(t) = x(t, 0, \psi)$ of (1) defined on $[0, \infty)$. We define $\left\|\psi\right\| = \max\left\{\left|\psi(t)\right| : m(0) \leq t \leq 0\right\}$. Stability definitions may be found in [7], for example. In our investigation we need that τ is twice differentiable and

$$\tau'(t) \neq 1 \text{ for all } t \in \mathbb{R}^+. \tag{8}$$

The absence of linear terms in (1) makes it difficult to obtain a fixed point mapping. So, to make equation (1) more tractable, we have to transform it.

Generally, the investigation of the stability of an equation using fixed point technic involves the construction of a suitable fixed point mapping. This can, in so many cases, be an arduous task. So, to construct our mapping, we begin by transforming (1) to a more tractable, but equivalent, equation, which we then invert to obtain an equivalent integral equation from which we derive a fixed point mapping. After then, we define prudently a suitable complete space, depending on the initial condition, so that the mapping is a contraction. Using Banach's contraction mapping principle, we obtain a solution for this mapping, and hence a solution for (1), which is asymptotically stable.

First, we have to transform (1) into an equivalent equation that possesses the same basic structure and properties to which we apply the variation of parameters to define a fixed point mapping.

Lemma 2.1. *Equation (1) is equivalent to*

$$\begin{aligned} x'(t) &= B(t, t-\tau(t))(1-\tau'(t))\, g(x(t-\tau(t))) \\ &\quad + \frac{d}{dt}\int_{t-\tau(t)}^{t} B(t,s)\, g(x(s))\, ds + c(t)\, x'(t-\tau(t)), \end{aligned} \tag{9}$$

where

$$B(t,s) := \int_t^s a(u,s)\, du \text{ with } B(t, t-\tau(t)) = \int_t^{t-\tau(t)} a(u, t-\tau(t))\, du.$$

Proof. Differentiating the integral term in (9), we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{t-\tau(t)}^{t} B(t,s)\, g(x(s))\, ds \\ &= B(t,t)\, g(x(t)) - B(t, t-\tau(t))(1-\tau'(t))\, g(x(t-\tau(t))) + \int_{t-\tau(t)}^{t} \frac{\partial}{\partial t} B(t,s)\, g(x(s))\, ds. \end{aligned}$$

Substituting this into (9), it follows that (9) is equivalent to (1) provided B satisfies the following conditions

$$B(t,t) = 0 \text{ and } \frac{\partial}{\partial t} B(t,s) = -a(t,s). \tag{10}$$

Now (10) implies

$$B(t,s) = -\int_0^t a(u,s)\, du + \phi(s), \tag{11}$$

for some function ϕ, and $B(t,s)$ must satisfy

$$B(t,t) = -\int_0^t a(u,t)\, du + \phi(t) = 0.$$

Consequently,

$$\phi(t) = \int_0^t a(u,t)\, du.$$

Substituting this into (11), we obtain

$$\begin{aligned} B(t,s) &= -\int_0^t a(u,s)\, du + \int_0^s a(u,s)\, du \\ &= \int_t^s a(u,s)\, du. \end{aligned}$$

This definition of B satisfies (10). Consequently, (1) is equivalent to (9). $\square$

Lemma 2.2. *Let $\psi : [m(0), 0] \to \mathbb{R}$ be a given continuous initial function. If x is a solution of (1) on an interval $[0, T)$ with $x = \psi$ on $[m(0), 0]$, then x is a solution of the integral equation*

$$\begin{aligned} x(t) &= \left\{\psi(0) - \frac{c(0)}{1-\tau'(0)}\psi(-\tau(0)) - \int_{-\tau(0)}^{0}[H(s) + B(0,s)]\, g(\psi(s))ds\right\} e^{-\int_0^t H(v)dv} \\ &\quad + \frac{c(t)}{1-\tau'(t)} x(t-\tau(t)) + \int_{t-\tau(t)}^{t} [H(s) + B(t,s)]\, g(x(s))ds \\ &\quad - \int_0^t e^{-\int_s^t H(v)dv} H(s) \int_{s-\tau(s)}^{s} [H(u) + B(s,u)]\, g(x(u))du\, ds \\ &\quad + \int_0^t e^{-\int_s^t H(v)dv} \{[H(s-\tau(s)) + B(s, s-\tau(s))]\,(1-\tau'(s)) g(x(s-\tau(s))) - \mu(s) x(s-\tau(s))\}\, ds \\ &\quad + \int_0^t e^{-\int_s^t H(v)dv} H(s)\, [x(s) - g(x(s))]\, ds, \end{aligned} \tag{12}$$

where

$$\mu(s) = \frac{[c'(s) + H(s)c(s)](1-\tau'(s)) + \tau''(s)c(s)}{(1-\tau'(s))^2},$$

and $H : [m(0), \infty) \to \mathbb{R}$ is an arbitrary continuous function. Conversely, if a continuous function x is equal to ψ on $[m(0), 0]$ and is a solution of (12) on an interval $[0, \sigma)$, then x is a solution of (1) on $[0, \sigma)$.

Proof. *Multiplying both sides of (9) by the factor $e^{\int_0^t H(v)dv}$ and integrating from 0 to any $t \in [0, T)$, we obtain*

$$\begin{aligned} x(t) &= e^{-\int_0^t H(v)dv}\psi(0) + \int_0^t e^{-\int_s^t H(v)dv} H(s)x(s)ds + \int_0^t e^{-\int_s^t H(v)dv} c(s)x'(s-\tau(s))ds \\ &\quad + \int_0^t e^{-\int_s^t H(v)dv} \frac{d}{ds}\int_{s-\tau(s)}^{s} B(s,u)g(x(u))duds \\ &\quad + \int_0^t e^{-\int_s^t H(v)dv} B(s, s-\tau(s))(1-\tau'(s))g(x(s-\tau(s)))ds \\ &= e^{-\int_0^t H(v)dv}\psi(0) + \int_0^t e^{-\int_s^t H(v)dv} H(s)x(s)ds \\ &\quad + \int_0^t e^{-\int_s^t H(v)dv} \frac{d}{ds}\int_{s-\tau(s)}^{s} [H(u) + B(s,u)]\, g(x(u))duds \\ &\quad + \int_0^t e^{-\int_s^t H(v)dv} c(s)x'(s-\tau(s))ds + \int_0^t e^{-\int_s^t H(v)dv} B(s, s-\tau(s))(1-\tau'(s))g(x(s-\tau(s)))ds \\ &\quad - \int_0^t e^{-\int_s^t H(v)dv} \frac{d}{ds}\int_{s-\tau(s)}^{s} H(u)g(x(u))duds. \end{aligned}$$

Then an integration by parts yields (12). Conversely, suppose that a continuous function x is equal to ψ on $[m(0), 0]$ and satisfies (12) on an interval $[0, \sigma)$. Then, x is differentiable on $[0, \sigma)$. Differentiating x with the aid of Leibniz's rule, we obtain (9). □

From equation (12) we shall derive a fixed point mapping P for (1). But the challenge here is to choose a suitable metric space of functions on which the map P can be defined. Below a weighted metric on a specific space is defined. Let C be the set of real-valued bounded continuous functions on $[m(0), \infty)$ with the supremum norm $\|\cdot\|$, that is, for $\phi \in C$,

$$\|\phi\| := \sup\left\{|\phi(t)| : t \in [m(0), \infty)\right\}.$$

In other words, we carry out our investigations in the complete metric space (C, d) where d denotes the supremum metric $d\left(\phi_1, \phi_2\right) = \left\|\phi_1 - \phi_2\right\|$ for $\phi_1, \phi_2 \in C$. For a given initial function $\psi : [m(0), 0] \to [-l, l]$, $l > 0$, define the set

$$S_\psi := \{\phi : [m(0), \infty) \to \mathbb{R} \mid \phi \in C,\ \phi(t) = \psi(t)\ for\ t \in [m(0), 0], \left|\phi(t)\right| \leq l\}.$$

Since S_ψ is a closed subset of C, the metric space $\left(S_\psi, d\right)$ is complete.

Theorem 2.3. *Let $H : [m(0), \infty) \to \mathbb{R}$ be a continuous function and define a mapping P on S_ψ as follows, for $\phi \in S_\psi$ $\left(P\phi\right)(t) = \psi(t)$ if $t \in [m(0), 0]$; while, for $t > 0$*

$$\begin{aligned}\left(P\phi\right)(t) &= \left\{\psi(0) - \frac{c(0)}{1-\tau'(0)}\psi(-\tau(0)) - \int_{-\tau(0)}^{0} [H(s) + B(0,s)]\, g(\psi(s))ds\right\} e^{-\int_0^t H(v)dv} \\ &+ \frac{c(t)}{1-\tau'(t)}\phi(t-\tau(t)) + \int_{t-\tau(t)}^{t} [H(s) + B(t,s)]\, g(\phi(s))ds \\ &- \int_0^t e^{-\int_s^t H(v)dv} H(s) \int_{s-\tau(s)}^{s} [H(u) + B(s,u)]\, g(\phi(u))duds \\ &+ \int_0^t e^{-\int_s^t H(v)dv} \left\{[H(s-\tau(s)) + B(s, s-\tau(s))]\,(1-\tau'(s))g(\phi(s-\tau(s))) - \mu(s)\phi(s-\tau(s))\right\} ds \\ &+ \int_0^t e^{-\int_s^t H(v)dv} H(s)\left[\phi(s) - g(\phi(s))\right] ds. \end{aligned} \tag{13}$$

Suppose that the following conditions are satisfied,

(i) there exists a constant $l > 0$ such that g satisfies a Lipschitz condition on $[-l, l]$ and let L be the Lipschitz constant for both $g(x)$ and $x - g(x)$ on $[-l, l]$;

(ii) $H(t) \geqslant 0$ for $t \geqslant 0$;

(iii) there exists a continuous function q such that $|B(t, u)| \leqslant q(u)$ for $t - \tau(t) \leqslant u \leqslant t$.

Assume, further, that the following condition is satisfied for some constant $k > 5$

$$k\left|\frac{c(t)}{1-\tau'(t)}\right| \leq 1. \tag{14}$$

Then there is a metric d_h for S_ψ such that $\left(S_\psi, d_h\right)$ is complete and P is a contraction on $\left(S_\psi, d_h\right)$ if P maps S_ψ into itself.

Proof. It is clear that $P\phi$ is continuous when ϕ is such. Now, for $t \in [m(0), \infty)$ and a constant $k > 5$, define

$$h(t) = kL\int_0^t [H(v) + q(v) + \omega(v)]dv,$$

where

$$\omega(v) = \begin{cases} 0, & if\ v \in [m(0), 0] \\ |[H(v-\tau(v)) + B(v, v-\tau(v))]\,(1-\tau'(v))| + \frac{|\mu(v)|}{L}, & if\ v \in [0, \infty). \end{cases} \tag{15}$$

Let S be the space of all continuous functions $\phi : [m(0), \infty) \to \mathbb{R}$ such that

$$\left|\phi\right|_h := \sup\left\{\left|\phi(t)\right| e^{-h(t)} : t \in [m(0), \infty)\right\} < \infty.$$

Then $(S, |\cdot|_h)$ is a Banach space, which can be checked with Cauchy criterion for uniform convergence. Thus, (S, d_h) is a complete metric space where d_h denotes the induced metric $d_h(\phi, \eta) = \left|\phi - \eta\right|_h$ for $\phi, \eta \in S$. Being closed in S with this metric, the space $(S_\psi, |\cdot|_h)$ is also complete. Suppose now, that $P : S_\psi \to S_\psi$. We need to show that P defined by (13) is a contraction. Toward this end. Let $\phi, \eta \in S_\psi$, since, by hypothesis (i), g satisfies a Lipschitz condition on $[-l, l]$, it follows that

$$\begin{aligned}
&\left|(P\phi)(t) - (P\eta)(t)\right| e^{-h(t)} \\
&\leqslant \left|\frac{c(t)}{1-\tau'(t)}\right| \left|\phi(t-\tau(t)) - \eta(t-\tau(t))\right| e^{-h(t)-h(t-\tau(t))+h(t-\tau(t))} \\
&+ \int_{t-\tau(t)}^{t} |H(s) + B(t,s)|\, L \left|\phi(s) - \eta(s)\right| e^{-h(t)+h(s)-h(s)} ds \\
&+ \int_{o}^{t} e^{-\int_s^t H(v)dv} H(s) \int_{s-\tau(s)}^{s} |H(u) + B(s,u)|\, L \left|\phi(u) - \eta(u)\right| e^{-h(t)+h(u)-h(u)} duds \\
&+ \int_{0}^{t} e^{-\int_s^t H(v)dv} \left\{L\, |H(s-\tau(s)) + B(s, s-\tau(s))|\, |1-\tau'(s)| + \left|\mu(s)\right|\right\} \\
&\times \left|\phi(s-\tau(s)) - \eta(s-\tau(s))\right| e^{-h(t)+h(s-\tau(s))-h(s-\tau(s))} ds \\
&+ \int_{0}^{t} e^{-\int_s^t H(v)dv} H(s) L \left|\phi(s) - \eta(s)\right| e^{-h(t)+h(s)-h(s)} ds, \qquad (16)
\end{aligned}$$

for $t > 0$. There are five terms on the right hand side of inequality (16), denote them respectively by I_i, $i = 1, 2, ..., 5$. For $t - \tau(t) \le s \le v \le t$, we have

$$-h(t) + h(s) = -kL \int_s^t [H(v) + q(v) + \omega(v)]dv \le -kL \int_s^t [H(v) + q(v)]dv.$$

Consequently,

$$I_2 \le \int_{t-\tau(t)}^{t} e^{-kL \int_s^t [H(v)+q(v)]dv} |H(s) + B(t,s)|\, L \left|\phi(s) - \eta(s)\right| e^{-h(s)} ds.$$

For $s - \tau(s) \le u \le v \le s$, we have

$$-h(t) + h(u) = -kL \int_u^t [H(v) + q(v) + \omega(v)]dv \le -kL \int_u^s [H(v) + q(v)]dv.$$

So

$$I_3 \le \int_0^t e^{-\int_s^t H(v)dv} H(s) \int_{s-\tau(s)}^{s} e^{-kL \int_u^s [H(v)+q(v)]dv} |H(u) + B(s,u)|\, L \left|\phi(u) - \eta(u)\right| e^{-h(u)} duds$$

Similarly, using (15), we obtain for $s - \tau(s) \le v \le t$

$$-h(t) + h(s - \tau(s)) = -kL \int_{s-\tau(s)}^{t} [H(v) + q(v) + \omega(v)]dv \le -kL \int_s^t \omega(v)dv$$

Thus

$$I_4 \le \int_0^t e^{-kL \int_s^t \omega(v)dv} \omega(s)\, L \left|\phi(s-\tau(s)) - \eta(s-\tau(s))\right| e^{-h(s-\tau(s))} ds$$

Consequently, inequality (16), became

$$
\begin{aligned}
&\left|(P\phi)(t)-(P\eta)(t)\right|e^{-h(t)} \\
&\leq e^{-kL\int_{t-\tau(t)}^{t}[H(v)+q(v)+\omega(v)]dv}\left|\frac{c(t)}{1-\tau'(t)}\right|\left|\phi(t-\tau(t))-\eta(t-\tau(t))\right|e^{-h(t-\tau(t))} \\
&+\int_{t-\tau(t)}^{t}e^{-kL\int_{s}^{t}[H(v)+q(v)]dv}\left|H(s)+B(t,s)\right|L\left|\phi(s)-\eta(s)\right|e^{-h(s)}ds \\
&+\int_{0}^{t}e^{-\int_{s}^{t}H(v)dv}H(s)\int_{s-\tau(s)}^{s}e^{-kL\int_{u}^{s}[H(v)+q(v)]dv}\left|H(u)+B(s,u)\right|L\left|\phi(u)-\eta(u)\right|e^{-h(u)}duds \\
&+\int_{0}^{t}e^{-kL\int_{s}^{t}\omega(v)dv}\omega(s)L\left|\phi(s-\tau(s))-\eta(s-\tau(s))\right|e^{-h(s-\tau(s))}ds \\
&+\int_{0}^{t}e^{-(kL+1)\int_{s}^{t}H(v)dv}H(s)L\left|\phi(s)-\eta(s)\right|e^{-h(s)}ds.
\end{aligned}
$$

By condition (iii), we have

$$|H(s)+B(t,s)|\leq H(s)+q(s) \text{ for } t-r(t)\leq s\leq t.$$

Consequently, by using (14), we obtain

$$\left|(P\phi)(t)-(P\eta)(t)\right|e^{-h(t)}\leq\left[\frac{1}{k}+\left(\frac{1}{kL}+\frac{1}{kL}+\frac{1}{kL}+\frac{1}{kL+1}\right)L\right]\left|\phi-\eta\right|_{h},$$

for all $t>0$. Since $P\phi$ and $P\eta$ agree on $[m(0),0]$, the last inequality holds for all $t\geq m(0)$.

Thus

$$\left|P\phi-P\eta\right|_{h}\leq\frac{5}{k}\left|\phi-\eta\right|_{h}.$$

Since $k>5$, we conclude that P is a contraction on (S_{ψ},d_{h}). □

Now, by choosing $\left\|\psi\right\|$ sufficiently small, we establish the existence and uniqueness of solutions by showing that $P:S_{\psi}\to S_{\psi}$.

Definition 2.4. *The zero solution of (1) is said to be stable at $t=0$ if, for every $\varepsilon>0$, there exists a $\delta>0$ such that $\psi:[m(0),0]\to(-\delta,\delta)$ implies that $|x(t)|<\varepsilon$ for $t\geq m(0)$.*

Theorem 2.5. *Suppose g, H and B satisfy conditions (i)–(iii) in Theorem 2.3, (14) holds and suppose further that*

(i) g is odd and strictly increasing on $[-l,l]$;

(ii) $x-g(x)$ is non-decreasing on $[0,l]$;

(iii) there exists an $\alpha\in(0,1)$ such that, for $t\geq0$

$$
\begin{aligned}
&l\left(\left|\frac{c(t)}{1-\tau'(t)}\right|+\int_{0}^{t}e^{-\int_{s}^{t}H(v)dv}\left|\mu(s)\right|ds\right) \\
&+g(l)\left(\int_{t-\tau(t)}^{t}\left|H(s)+B(t,s)\right|ds+\int_{0}^{t}e^{-\int_{s}^{t}H(v)dv}H(s)\int_{s-\tau(s)}^{s}\left|H(u)+B(s,u)\right|duds\right. \\
&\left.+\int_{0}^{t}e^{-\int_{s}^{t}H(v)dv}\left|H(s-\tau(s))+B(s,s-\tau(s))\right|\left|1-\tau'(s)\right|ds\right) \\
&\leqslant\alpha g(l).
\end{aligned}
$$

Then a $\delta\in(0,l)$ exists such that for each initial continuous function $\psi:[m(0),0]\to(-\delta,\delta)$, there is a unique continuous function $x:[m(0),\infty)\to\mathbb{R}$ with $x=\psi$ on $[m(0),0]$, which is a solution of (1) on $[0,\infty)$. Moreover, x is bounded by l on $[m(0),\infty)$. Furthermore, the zero solution of (1) is stable at $t=0$.

Proof. Since g is odd and satisfies the Lipschitz condition on $[-l,l]$, $g(0)=0$ and g is uniformly continuous on $[-l,l]$. So we can choose a δ that satisfies

$$\delta\left(1+\left|\frac{c(0)}{1-\tau'(0)}\right|\right)+g(\delta)\int_{-\tau(0)}^{0}|H(s)+B(0,s)|\,ds\leqslant(1-\alpha)g\,(l)\,. \tag{17}$$

Let $\psi:[m(0),0]\to(-\delta,\delta)$ be a continuous function. Note that (17) implies $\delta<l$ since $g(l)\leq l$ by condition (ii). Thus, $\left|\psi\,(t)\right|\leq l$ for $m(0)\leq t\leq 0$. Now we show that for such a ψ the mapping $P:S_\psi\to S_\psi$. Indeed, consider (13). For an arbitrary $\phi\in S_\psi$, it follows from conditions (i) and (ii) that

$$\begin{aligned}
&\left|(P\phi)(t)\right|\\
&\leq\delta\left(1+\left|\frac{c(0)}{1-\tau'(0)}\right|\right)+g(\delta)\int_{-\tau(0)}^{0}|H(s)+B(0,s)|\,ds+l\left|\frac{c(t)}{1-\tau'(t)}\right|\\
&+g(l)\int_{t-\tau(t)}^{t}|H(s)+B(t,s)|\,ds+g(l)\int_{o}^{t}e^{-\int_s^t H(v)dv}H(s)\int_{s-\tau(s)}^{s}|H(u)+B(s,u)|\,duds\\
&+\int_0^t e^{-\int_s^t H(v)dv}\left\{g(l)\,|H(s-\tau(s))+B(s,s-\tau(s))|\,|1-\tau'(s)|+l\left|\mu(s)\right|\right\}ds\\
&+(l-g(l))\int_0^t e^{-\int_s^t H(v)dv}H(s)ds,
\end{aligned}$$

for $t>0$. By applying (iii) and (17), we see that

$$\begin{aligned}
\left|(P\phi)(t)\right| &\leq \delta\left(1+\left|\frac{c(0)}{1-\tau'(0)}\right|\right)+g(\delta)\int_{-\tau(0)}^{0}|H(s)+B(0,s)|\,ds+\alpha g\,(l)+l-g(l)\\
&\leq (1-\alpha)g\,(l)+\alpha g\,(l)+l-g(l)=(1-\alpha)g\,(l)+(\alpha-1)g\,(l)+l=l.
\end{aligned}$$

Hence, $\left|(P\phi)(t)\right|\leq l$ for $t\in[m(0),\infty)$ because $\left|(P\phi)(t)\right|=\left|\psi\,(t)\right|\leq l$ for $t\in[m\,(0)\,,0]$. Therefore, $P\phi\in S_\psi$. By Theorem 2.3, P is a contraction on the complete metric space (S_ψ,d_h). Then P has a unique fixed point $x\in S_\psi$. Thus $|x\,(t)|\leq l$ for all $t\geq m\,(0)$ and is a solution of (1) on $[0,\infty)$ by Lemmas 2.1 and 2.2. Hence x is the only continuous function satisfying (1) such that $x(t)=\psi(t)$ for $t\in[m(0),0]$.
To prove the stability at $t=0$, let $\varepsilon>0$ be given and choose $r>0$ such that $r<\min\{\varepsilon,l\}$. Replacing l with r beginning with (17), we see that there is a $\delta>0$ such that $\left\|\psi\right\|<\delta$ implies that the unique continuous solution x agreeing on $[m\,(0)\,,0]$ with ψ, satisfies $|x(t)|\leqslant r<\varepsilon$ for all $t\geqslant m(0)$. □

Supposing that the conditions in Theorem 2.3 and Theorem 2.5 hold for some $l>0$, we investigate asymptotic stability with a necessary and sufficient condition by shifting our attention to the subset of functions in S_ψ that tend to zero as $t\to\infty$, namely,

$$S_\psi^0=\left\{\phi\in S_\psi\mid\phi\,(t)\to 0\text{ as }t\to\infty\right\}.$$

Since S_ψ^0 is a closed subset of S_ψ, the metric space $\left(S_\psi^0,d\right)$ is complete. Under the conditions of the next theorem, the zero solution of (1) is asymptotically stable.

Definition 2.6. *The zero solution of (1) is asymptotically stable if it is stable at $t=0$ and a $\delta>0$ exists such that for any continuous functions $\psi:[m\,(0)\,,0]\to(-\delta,\delta)$, the solution x with $x=\psi$ on $[m\,(0)\,,0]$ tends to zero as $t\to\infty$.*

Theorem 2.7. *Suppose all of the conditions in Theorem 2.3 and Theorem 2.5 hold. Furthermore, suppose g is continuously differentiable on $[-l,l]$ and $g'\,(0)\neq 0$. Then the zero solution of (1) is asymptotically stable if and only if*

$$\int_0^t H\,(v)\,dv\to\infty\text{ as }t\to\infty. \tag{18}$$

Proof. First, suppose that (18) holds. We set

$$K = \sup_{t\geq 0}\left\{e^{-\int_0^t H(v)dv}\right\}. \tag{19}$$

We argue that $P : S_\psi^0 \to S_\psi^0$, the conditions of Theorems 2.3 and 2.5 hold, and $\|\psi\|$ is sufficiently small. For $\delta > 0$ satisfying (17), let $\psi : [m(0), 0] \to (-\delta, \delta)$ be a continuous function. Let $\phi \in S_\psi^0$. The proof of Theorem 2.5 shows that $\delta < l$ and $\left|\left(P\phi\right)(t)\right| \leq l$ for $t \in [m(0), \infty)$. Hence, $\left(P\phi\right)(t) \to 0$ would imply that P maps S_ψ^0 into itself. To show that this is the case, consider $\left|\left(P\phi\right)(t)\right|$. But first note for any $\phi \in S_\psi^0$ that

$$\left|g\left(\phi(t)\right)\right| \leq L\left|\phi(t)\right| \text{ and } \left|\phi(t) - g\left(\phi(t)\right)\right| \leq L\left|\phi(t)\right|,$$

since $g(x)$ and $x - g(x)$ satisfy a Lipschitz condition on $[-l, l]$ with a common Lipschitz constant L and $g(0) = 0$. Because of this and (iii) of Theorem 2.5, it follows from (13) that

$$\begin{aligned}
&\left|\left(P\phi\right)(t)\right| \\
&\leq \left\{\left|\psi(0)\right| + \left|\frac{c(0)}{1-\tau'(0)}\right|\left|\psi(-\tau(0))\right| + \int_{-\tau(0)}^{0}|H(s)+B(0,s)|\left|g(\psi(s))\right|ds\right\}e^{-\int_0^t H(v)dv} \\
&+ \left|\frac{c(t)}{1-\tau'(t)}\right|\left|\phi(t-\tau(t))\right| + \int_{t-\tau(t)}^{t}|H(s)+B(t,s)|\left|g(\phi(s))\right|ds \\
&+ \int_0^t e^{-\int_s^t H(v)dv}H(s)\int_{s-\tau(s)}^{s}|H(u)+B(s,u)|\left|g(\phi(u))\right|duds \\
&+ \int_0^t e^{-\int_s^t H(v)dv}\left\{|H(s-\tau(s))+B(s,s-\tau(s))||1-\tau'(s)|\left|g(\phi(s-\tau(s)))\right| + \left|\mu(s)\right|\left|\phi(s-\tau(s))\right|\right\}ds \\
&+ \int_0^t e^{-\int_s^t H(v)dv}H(s)\left|\phi(s)-g(\phi(s))\right|ds.
\end{aligned} \tag{20}$$

Denote the six terms on the right hand side of (20) by $I_1, I_2, ..., I_6$, respectively. It is obvious that the first term I_1 tends to zero as $t \to \infty$, by condition (18). Also, due to the facts that $\phi(t) \to 0$ and $t - \tau(t) \to \infty$ as $t \to \infty$, the second term I_2 in (20) tends to zero as $t \to \infty$. What is left to show is that each of the remaining terms in (20) tends to zero as $t \to \infty$.

Let $\phi \in S_\psi^0$ be fixed. For a given $\epsilon > 0$, we choose $T_0 > 0$ large enough such that $t - \tau(t) \geq T_0$, implies $\left|\phi(s)\right| < \epsilon$ if $s \geq t - \tau(t)$. Therefore, the third term I_3 in (20) satisfies

$$\begin{aligned}
I_3 &\leq \int_{t-\tau(t)}^{t}|H(s)+B(t,s)|L\left|\phi(s))\right|ds \\
&\leq L\epsilon\int_{t-\tau(t)}^{t}|H(s)+B(t,s)|ds \leq L\alpha\epsilon.
\end{aligned}$$

Thus, $I_3 \to 0$ as $t \to \infty$. Now consider I_4. For the given $\epsilon > 0$, there exists a $T_1 > 0$ such that $s \geq T_1$ implies

$\left|\phi\left(s-\tau\left(s\right)\right)\right|<\epsilon$. Thus, for $t \geq T_1$, the term I_4 in (20) satisfies

$$\begin{aligned}
I_4 &\leq \int_o^{T_1} e^{-\int_s^t H(v)dv} H(s) \int_{s-\tau(s)}^{s} |H(u)+B(s,u)|\, L \left|\phi(s))\right| duds \\
&+ \int_{T_1}^{t} e^{-\int_s^t H(v)dv} H(s) \int_{s-\tau(s)}^{s} |H(u)+B(s,u)|\, L \left|\phi(s))\right| duds \\
&\leq \sup_{\sigma\geq m(0)} \left|\phi(\sigma)\right| L \int_o^{T_1} e^{-\int_s^t H(v)dv} H(s) \int_{s-\tau(s)}^{s} |H(u)+B(s,u)|\, duds \\
&+ L\epsilon \int_{T_1}^{t} e^{-\int_s^t H(v)dv} H(s) \int_{s-\tau(s)}^{s} |H(u)+B(s,u)|\, duds.
\end{aligned}$$

By (18), there exists $T_2 > T_1$ such that $t \geq T_2$ implies

$$\begin{aligned}
&\sup_{\sigma\geq m(0)} \left|\phi(\sigma)\right| L \int_o^{T_1} e^{-\int_s^t H(v)dv} H(s) \int_{s-\tau(s)}^{s} |H(u)+B(s,u)|\, duds \\
&= \sup_{\sigma\geq m(0)} \left|\phi(\sigma)\right| L e^{-\int_{T_2}^{t} H(v)dv} \int_o^{T_1} e^{-\int_s^{T_2} H(v)dv} H(s) \int_{s-\tau(s)}^{s} |H(u)+B(s,u)|\, duds \\
&< L\epsilon.
\end{aligned}$$

Now, apply condition (iii) in Theorem 2.5 to have $I_4 < L\epsilon + L\alpha\epsilon < 2L\epsilon$. Thus, $I_4 \to 0$ as $t \to \infty$. Similarly, by using condition (iii) in Theorem 2.5, then, if $t \geq T_2$ then the terms I_5 and I_6 in (20) satisfy

$$\begin{aligned}
I_5 \quad &\leq \quad \sup_{\sigma\geq m(0)} \left|\phi(\sigma)\right| e^{-\int_{T_2}^{t} H(v)dv} \int_o^{T_1} e^{-\int_s^{T_2} H(v)dv} \\
&\quad \times \left\{L\left|H(s-\tau(s))+B(s,s-\tau(s))\right|\left|1-\tau'(s)\right| + \left|\mu(s)\right|\right\} ds \\
&\quad +\epsilon \int_{T_1}^{t} e^{-\int_s^t H(v)dv} \left\{L\left|H(s-\tau(s))+B(s,s-\tau(s))\right|\left|1-\tau'(s)\right| + \left|\mu(s)\right|\right\} ds \\
&< \quad L\epsilon + L\alpha\epsilon + \frac{\alpha g\left(l\right)}{l} < \left(2L + \frac{g\left(l\right)}{l}\right)\epsilon,
\end{aligned}$$

and

$$\begin{aligned}
I_6 &\leq \sup_{\sigma\geq m(0)} \left|\phi(\sigma)\right| L e^{-\int_{T_2}^{t} H(v)dv} \int_o^{T_1} e^{-\int_s^{T_2} H(v)dv} H(s) ds + L\epsilon \int_{T_1}^{t} e^{-\int_s^t H(v)dv} H(s) ds \\
&< L\epsilon + L\epsilon = 2L\epsilon.
\end{aligned}$$

Thus, $I_5, I_6 \to 0$ as $t \to \infty$. In conclusion $\left(P\phi\right)(t) \to 0$ as $t \to \infty$, as required. Hence P maps S_ψ^0 into S_ψ^0.

We verify that P is a contraction on $\left(S_\psi^0, d\right)$. In a similar way as in [5, Theorem 3.13], let $\zeta \in [0, l]$, and define

$$q\left(\zeta\right) := \min\left\{g'\left(x\right) : |x| \leq \zeta\right\} \text{ and } Q\left(\zeta\right) := \max\left\{g'\left(x\right) : |x| \leq \zeta\right\}.$$

We prove that the mapping P defined by (13) is a contraction on $\left(S_\psi^0, d\right)$ provided

$$\alpha Q\left(l\right) < q\left(l\right), \tag{21}$$

where α is given by condition (iii) in Theorem 2.5. To see this, first notice from (iii) in Theorem 2.5 that $\alpha < 1$, thus,

$$\alpha = 1 - \epsilon, \tag{22}$$

for some $\epsilon \in (0,1)$. Clearly, $\lim_{\zeta\to 0} q(\zeta) = \lim_{\zeta\to 0} Q(\zeta) = g'(0)$. Since g is strictly increasing, $g'(0) \neq 0$, and g' is continuous, a neighborhood of $x = 0$ exists in which $g'(x) > 0$. Consequently, $Q(\zeta) > 0$ for $0 \le \zeta \le l$. This and $\lim_{\zeta\to 0} Q(\zeta) \neq 0$ imply

$$\lim_{\zeta\to 0}\frac{q(\zeta)}{Q(\zeta)} = \frac{\lim_{\zeta\to 0} q(\zeta)}{\lim_{\zeta\to 0} Q(\zeta)} = 1.$$

Hence, there is a $\varsigma \in (0, l]$ such that

$$\left|\frac{q(\zeta)}{Q(\zeta)} - 1\right| < \epsilon,$$

for $0 < \zeta < \varsigma$. Choosing a value for ζ from $(0, \varsigma)$, we have $(1-\epsilon) Q(\zeta) < q(\zeta)$. This, along with (22), yields

$$\alpha Q(\zeta) = (1-\epsilon) Q(\zeta) < q(\zeta).$$

Replacing the original value of l with $l = \zeta$, we obtain (21). Note that the conditions in Theorems 2.3 and 2.5 will still hold with this smaller l.

In the ensuing argument, bounds on the derivatives of $g(x)$ and $b(x) := x - g(x)$ on the interval $[-l, l]$ are used. By (*i*) in Theorem 2.5 and the definition of Q, $0 \le g'(x) \le Q(l)$. By (*i*) and (*ii*) in Theorem 2.5, b is non-decreasing on $[-l, l]$. This and $g'(x) \ge q(l)$ imply $0 \le b'(x) \le 1 - q(l)$. Let $\phi, \eta \in S^0_\psi$. By (13) and the Mean Value Theorem, we have

$$\begin{aligned}
&\left|\left(P\phi\right)(t) - (P\eta)(t)\right| \\
&\le \left|\frac{c(t)}{1-\tau'(t)}\right| \left|\phi(t-\tau(t)) - \eta(t-\tau(t))\right| \\
&+ \int_{t-\tau(t)}^{t} |H(s) + B(t,s)| \left|g(\phi(s)) - g(\eta(s))\right| ds \\
&+ \int_0^t e^{-\int_s^t H(v)dv} H(s) \int_{s-\tau(s)}^{s} |H(u) + B(s,u)| \left|g(\phi(u)) - g(\eta(u))\right| duds \\
&+ \int_0^t e^{-\int_s^t H(v)dv} \Big\{ |H(s-\tau(s)) + B(s, s-\tau(s))| \, |1-\tau'(s)| \left|g\left(\phi(s-\tau(s))\right) - g(\eta(s-\tau(s)))\right| \\
&+ \left|\mu(s)\right| \left|\phi(s-\tau(s)) - \eta(s-\tau(s))\right| \Big\} ds \\
&+ \int_0^t e^{-\int_s^t H(v)dv} H(s) \left|b\left(\phi(s)\right) - b\left(\eta(s)\right)\right| ds \\
&\le \left[\alpha Q(l) + (1-q(l))\right] \left\|\phi - \eta\right\| = \rho \left\|\phi - \eta\right\|,
\end{aligned}$$

for all $t > 0$, where $\rho = \alpha Q(l) + (1-q(l))$. This implies $\left\|P\phi - P\eta\right\| \le \rho \left\|\phi - \eta\right\|$. Note $\rho \in (0,1)$. Consequently, for a continuous $\psi : [m(0), 0] \to (-\delta, \delta)$, P has a unique fixed point $x \in S^0_\psi$. By Lemma 2.2, x is the unique continuous solution of (1) with $x(t) = \psi(t)$ on $[m(0), 0]$. By virtue of $x \in S^0_\psi$, x tends to 0 as $t \to \infty$. By Theorem 2.5, the zero solution is stable at $t = 0$. This shows that the zero solution of (1) is asymptotically stable if (18) holds.

Conversely, suppose (18) fails. Then, there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \int_0^{t_n} H(v)\,dv = \beta$ for some $\beta \in \mathbb{R}^+$. We may also choose a positive constant J satisfying

$$-J \le \int_0^{t_n} H(v)\,dv \le J,$$

for all $n \geq 1$. To simplify our expressions, we define

$$\omega(s) = l\left|\mu(s)\right| + g(l)\{|H(s-\tau(s)) + B(s, s-\tau(s))|\,|1-\tau'(s)|$$
$$+H(s)\int_{s-\tau(s)}^{s} |H(u) + B(s,u)|\,du\Big\} + LH(s),$$

for all $s \geq 0$. By by condition (iii) in Theorem 2.5, we have

$$\int_0^{t_n} e^{-\int_s^{t_n} H(v)dv}\omega(s)\,ds \leq \alpha g(l) + L.$$

This yields

$$\int_0^{t_n} e^{\int_0^s H(v)dv}\omega(s)\,ds \leq (\alpha g(l) + L)\, e^{\int_0^{t_n} H(v)dv} \leq (g(l) + L)\, e^{J}.$$

The sequence $\left\{\int_0^{t_n} e^{\int_0^s H(v)dv}\omega(s)\,ds\right\}$ is bounded, so there exists a convergent subsequence. For brevity of notation, we may assume that

$$\lim_{n\to\infty}\int_0^{t_n} e^{\int_0^s H(v)dv}\omega(s)\,ds = \gamma,$$

for some $\gamma \in \mathbb{R}^+$ and choose a positive integer m so large that

$$\int_{t_m}^{t_n} e^{\int_0^s H(v)dv}\omega(s)\,ds < \delta_0/4K,$$

for all $n \geq m$, where $\delta_0 > 0$ satisfies

$$\left\{\delta_0\left(1 + \left|\frac{c(t_m)}{1-\tau'(t_m)}\right|\right) + g(\delta_0)\int_{t_m-\tau(t_m)}^{t_m} |H(s) + B(t_m, s)|\,ds\right\} Ke^{J} \leq (1-\alpha)\, g(l).$$

If we replace l by 1 in the proof of Theorem 2.5, we have $|x(t)| \leq 1$ for $t \geq t_m$. Now we consider the solution $x(t) = x(t, t_m, \psi)$ of (1) with $\psi(t_m) = \delta_0$ and $\left|\psi(s)\right| \leq \delta_0$ for $s \leq t_m$. We may choose ψ so that

$$\psi(t_m) - \frac{c(t_m)}{1-\tau'(t_m)}\psi(t_m - \tau(t_m)) - \int_{t_m-\tau(t_m)}^{t_m} [H(s) + B(t_m, s)]\, g(\psi(s))\,ds \geq \frac{1}{2}\delta_0.$$

It follows from (13) with $x(t) = (Px)(t)$ that for $n \geq m$

$$\begin{aligned}
&\left|x(t_n) - \frac{c(t_n)}{1-\tau'(t_n)}x(t_n - \tau(t_n)) - \int_{t_n-\tau(t_n)}^{t_n} [H(s) + B(t_n, s)]\, g(x(s))\,ds\right| \\
&\geq \frac{1}{2}\delta_0 e^{-\int_{t_m}^{t_n} H(v)dv} - \int_{t_m}^{t_n} e^{-\int_s^{t_n} H(v)dv}\omega(s)\,ds \\
&= \frac{1}{2}\delta_0 e^{-\int_{t_m}^{t_n} H(v)dv} - e^{-\int_0^{t_n} H(v)dv}\int_{t_m}^{t_n} e^{\int_0^s H(v)dv}\omega(s)\,ds \\
&= e^{-\int_{t_m}^{t_n} H(v)dv}\left(\frac{1}{2}\delta_0 - e^{-\int_0^{t_m} H(v)dv}\int_{t_m}^{t_n} e^{\int_0^s H(v)dv}\omega(s)\,ds\right) \\
&\geq e^{-\int_{t_m}^{t_n} H(v)dv}\left(\frac{1}{2}\delta_0 - K\int_{t_m}^{t_n} e^{\int_0^s H(v)dv}\omega(s)\,ds\right) \\
&\geq \frac{1}{4}\delta_0 e^{-\int_{t_m}^{t_n} H(v)dv} \geq \frac{1}{4}\delta_0 e^{-2J} > 0.
\end{aligned} \tag{23}$$

On the other hand, if the zero solution of (1) is asymptotically stable, then $x(t) = x(t, t_m, \psi) \to 0$ as $t \to \infty$. Since $t_n - \tau(t_n) \to \infty$ as $n \to \infty$ and condition (iii) in Theorem 2.5 holds, we have

$$x(t_n) - \frac{c(t_n)}{1-\tau'(t_n)} x(t_n - \tau(t_n)) - \int_{t_n-\tau(t_n)}^{t_n} [H(s) + B(t_n, s)]\, g(x(s))\, ds \to 0 \text{ as } n \to \infty,$$

which contradicts (23). Hence condition (18) is necessary for the asymptotic stability of the zero solution of (1). The proof is complete. □

Remark 2.8. *Obviously, if $c(t) = 0$, Theorem 2.7 extends Theorem C.*

3. An example

In this section, we give an example to illustrate the application of Theorem 2.7.

Example 3.1. *Consider the following nonlinear neutral integro-differential equation with variable delay*

$$x'(t) = -\int_{t-\tau(t)}^{t} a(t,s)\, g(x(s))\, ds + c(t)\, x'(t - \tau(t)), \tag{24}$$

where $\tau(t) = 0.482t$, $a(t,s) = 1/\left(s^2+1\right)$, $c(t) = 0.031$, $g(x) = \sin x$. Then the zero solution of (24) is asymptotically stable.

Proof. We have

$$B(t,s) = \int_t^s \frac{1}{s^2+1} du = \frac{(s-t)}{s^2+1}.$$

Choosing $l = \pi/4$, $k = 6$ and $H(t) = t/\left(t^2+1\right)$, clearly, condition (18) holds. Furthermore, we have $L = 2$, $g(0) = 0$, $g(l) = \sqrt{2}/2$, $g'(0) = 1$, g is odd and strictly increasing on $[-\pi/4, \pi/4]$, $x - g(x)$ is non-decreasing on $[0, \pi/4]$,

$$\begin{aligned}
|B(t,s)| &\le \frac{s}{s^2+1} \text{ for } 0.518t \le s \le t, \\
\left|\frac{c(t)}{1-\tau'(t)}\right| &= \left|\frac{0.031}{1-0.482}\right| < 0.06, \\
\int_0^t e^{-\int_s^t H(v)dv} \left|\mu(s)\right| ds &= \int_0^t e^{-\int_s^t \frac{v}{v^2+1}dv} \frac{0.031s}{0.518\,(s^2+1)} ds < 0.06,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t-\tau(t)}^{t} |H(s) + B(t,s)|\, ds \\
&= \int_{0.518t}^{t} \left|\frac{2s-t}{s^2+1}\right| ds = \int_{0.518t}^{t} \frac{2s-t}{s^2+1} ds \\
&= t\left[\arctan 0.518t - \arctan t\right] + \ln\left(t^2+1\right) - \ln\left(0.518^2 t^2 + 1\right) \\
&= \omega(t).
\end{aligned}$$

Since the function ω is increasing in $[0, \infty)$ and

$$\lim_{t\to\infty} \omega(t) = 1 - 1/0.518 - 2\ln(0.518) \simeq 0.385,$$

then

$$\int_{t-\tau(t)}^{t} |H(s) + B(t,s)|\, ds < 0.386,$$

$$\int_{0}^{t} e^{-\int_s^t H(v)dv} |H(s)| \left(\int_{s-\tau(s)}^{s} |H(u) + B(s,u)|\, du \right) ds < 0.386,$$

and

$$\begin{aligned}
&\int_{0}^{t} e^{-\int_s^t H(v)dv} |H(s-\tau(s)) + B(s, s-\tau(s))|\, |1-\tau'(s)|\, ds \\
&= (2 - 1/0.518) \int_{0}^{t} e^{-\int_s^t \frac{v}{v^2+1} dv} \frac{s}{s^2 + 1/0.518^2} ds \\
&< 2 - 1/0.518 < 0.07.
\end{aligned}$$

It is easy to see that all the conditions of Theorems 2.3, 2.5 and 2.7 hold for $\alpha = \left[(\pi/4) / \left(\sqrt{2}/2\right)\right](0.06 + 0.06) + 0.386 + 0.386 + 0.07 \simeq 0.976 < 1$. Thus, Theorem 2.7 implies that the zero solution of (24) is asymptotically stable. □

References

[1] A. Ardjouni, A. Djoudi, Fixed points and stability in linear neutral differential equations with variable delays. *Nonlinear Analysis 74 (2011) 2062-2070.*

[2] A. Ardjouni, A. Djoudi, Stability in nonlinear neutral differential with variable delays using fixed point theory, *Electronic Journal of Qualitative Theory of Differential Equations 2011, No. 43, 1-11.*

[3] A. Ardjouni, A. Djoudi, Fixed point and stability in neutral nonlinear differential equations with variable delays, *Opuscula Mathematica, Vol. 32, No. 1, 2012, pp. 5-19.*

[4] A. Ardjouni, A. Djoudi, I. Soualhia, Stability for linear neutral integro-differential equations with variable delays, *Electronic journal of Differential Equations, 2012(2012), No. 172, 1-14.*

[5] L. C. Becker, T. A. Burton, Stability, fixed points and inverse of delays, *Proc. Roy. Soc. Edinburgh 136A (2006) 245-275.*

[6] T. A. Burton, Fixed points and stability of a nonconvolution equation, *Proceedings of the American Mathematical Society 132 (2004) 3679–3687.*

[7] T. A. Burton, Stability by Fixed Point Theory for Functional Differential Equations, *Dover Publications, New York, 2006.*

[8] T. A. Burton, Liapunov functionals, fixed points, and stability by Krasnoselskii's theorem, *Nonlinear Studies 9 (2001) 181–190.*

[9] T. A. Burton, Stability by fixed point theory or Liapunov's theory: A comparison, *Fixed Point Theory 4 (2003) 15–32.*

[10] T. A. Burton, T. Furumochi, A note on stability by Schauder's theorem, *Funkcialaj Ekvacioj 44 (2001) 73–82.*

[11] T. A. Burton, T. Furumochi, Fixed points and problems in stability theory, *Dynamical Systems and Applications 10 (2001) 89–116.*

[12] T. A. Burton, T. Furumochi, Asymptotic behavior of solutions of functional differential equations by fixed point theorems, *Dynamic Systems and Applications 11 (2002) 499–519.*

[13] T. A. Burton, T. Furumochi, Krasnoselskii's fixed point theorem and stability, *Nonlinear Analysis 49 (2002) 445–454.*

[14] Y. M. Dib, M. R. Maroun, Y. N. Raffoul, Periodicity and stability in neutral nonlinear differential equations with functional delay, *Electronic Journal of Differential Equations, Vol. 2005(2005), No. 142, pp. 1-11.*

[15] A. Djoudi, R. Khemis, Fixed point techniques and stability for neutral nonlinear differential equations with unbounded delays, *Georgian Mathematical Journal, Vol. 13 (2006), No. 1, 25-34.*

[16] C. H. Jin, J. W. Luo, Stability of an integro-differential equation, *Computers and Mathematics with Applications 57 (2009) 1080-1088.*

[17] C. H. Jin, J. W. Luo, Stability in functional differential equations established using fixed point theory, *Nonlinear Anal. 68 (2008) 3307-3315.*

[18] C. H. Jin, J. W. Luo, Fixed points and stability in neutral differential equations with variable delays, *Proceedings of the American Mathematical Society, Vol. 136, Nu. 3 (2008) 909-918.*

[19] D. R. Smart, Fixed point theorems, *Cambridge Tracts in Mathematics, No. 66. Cambridge University Press, London-New York, 1974.*

[20] B. Zhang, Fixed points and stability in differential equations with variable delays, *Nonlinear Anal. 63 (2005) e233–e242.*