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Tauberian Theorems for Statistically $(C, 1)$-Convergent Sequences of Fuzzy Numbers

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Abstract. In this paper, we have determined necessary and sufficient Tauberian conditions under which statistically convergence follows from statistically $(C, 1)$-convergence of sequences of fuzzy numbers. Our conditions are satisfied if a sequence of fuzzy numbers is statistically slowly oscillating. Also, under additional conditions it is proved that a bounded sequence of fuzzy numbers which is $(C, 1)$-level-convergent to its statistical limit superior is statistically convergent .

1. Introduction and Preliminaries

Let K be a subset of natural numbers $\mathbb{N}$ and $K_n = \{k \le n : k \in K\}$, The natural density of K is given by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n}|K_n|$$

if this limit exists, where $|A|$ denotes the number of elements in A. The concept of statistical convergence was introduced by Fast [9]. A sequence $(x_k)_{k\in\mathbb{N}}$ of (real or complex) numbers is said to be *statistically convergent* to some number l if for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \frac{1}{n+1} |\{k \le n : |x_k - l| \ge \varepsilon\}| = 0.$$

In this case, we write st$-\lim_{k\to\infty} x_k = l$.

A sequence (x_k) is $(C, 1)$ convergent to ℓ if $\lim \sigma_n = \ell$, where

$$\sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} x_k \quad (n = 0, 1, 2...), \tag{1}$$

is the first arithmetic mean, also called Cesàro mean (of first order). In this case we write (C, 1)$-\lim_{k\to\infty} x_k = \ell$.

The idea of statistical $(C, 1)$-convergence was introduced in [15] by Móricz. A sequence (x_k) is statistically $(C, 1)$-convergent to ℓ if $st - \lim \sigma_n = \ell$. In addition to, necessary and sufficient conditions, under which st$-\lim_{k\to\infty} x_k = \ell$ follows from $st - \lim \sigma_n = \ell$ are presented in [15] by Moricz.

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In this paper, our primary interest is to obtain the results in [15] for sequences of fuzzy numbers.

We recall the basic definitions dealing with fuzzy numbers.

A *fuzzy number* is a fuzzy set on the real axis, i.e. a mapping $u : \mathbb{R} \to [0,1]$ which satisfies the following four conditions:

(i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$.

(ii) u is fuzzy convex, i.e. $u[\lambda x + (1-\lambda)y] \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0,1]$.

(iii) u is upper semi-continuous.

(iv) The set $[u]_0 := \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact,

where $\overline{\{x \in \mathbb{R} : u(x) > 0\}}$ denotes the closure of the set $\{x \in \mathbb{R} : u(x) > 0\}$ in the usual topology of $\mathbb{R}$. We denote the set of all fuzzy numbers on $\mathbb{R}$ by E^1 and called it as *the space of fuzzy numbers*. *α-level set* $[u]_\alpha$ of $u \in E^1$ is defined by

$$[u]_\alpha := \begin{cases} \{x \in \mathbb{R} : u(x) \geq \alpha\} & , \quad \text{if } 0 < \alpha \leq 1, \\ \overline{\{x \in \mathbb{R} : u(x) > \alpha\}} & , \quad \text{if } \alpha = 0. \end{cases}$$

The set $[u]_\alpha$ is closed, bounded and non-empty interval for each $\alpha \in [0,1]$ which is defined by $[u]_\alpha := [u^-(\alpha), u^+(\alpha)]$. $\mathbb{R}$ can be embedded in E^1, since each $r \in \mathbb{R}$ can be regarded as a fuzzy number $\overline{r}$ defined by

$$\overline{r}(x) := \begin{cases} 1 & , \quad \text{if } x = r, \\ 0 & , \quad \text{if } x \neq r. \end{cases}$$

Let $u, v, w \in E^1$ and $k \in \mathbb{R}$. Then the operations addition and scalar multiplication are defined on E^1 as

$$\begin{aligned} u + v = w \quad &\Longleftrightarrow \quad [w]_\alpha = [u]_\alpha + [v]_\alpha \text{ for all } \alpha \in [0,1] \\ &\Longleftrightarrow \quad w^-(\alpha) = u^-(\alpha) + v^-(\alpha) \text{ and } w^+(\alpha) = u^+(\alpha) + v^+(\alpha) \text{ for all } \alpha \in [0,1], \end{aligned}$$

$$[ku]_\alpha = k[u]_\alpha \text{ for all } \alpha \in [0,1].$$

Lemma 1.1. *[6]*

(i) If $\overline{0} \in E^1$ is neutral element with respect to +,i.e., $u + \overline{0} = \overline{0} + u = u$, for all $u \in E^1$.

(ii) With respect to $\overline{0}$, none of $u \neq \overline{r}$, $r \in \mathbb{R}$ has opposite in E^1.

(iii) For any $a, b \in \mathbb{R}$ with $a, b \geq 0$ or $a, b \leq 0$, and any $u \in E^1$, we have $(a + b)u = au + bu$. For general $a, b \in \mathbb{R}$, the above property does not hold.

(iv) For any $a \in \mathbb{R}$ and any $u, v \in E^1$, we have $a(u + v) = au + av$.

(v) For any $a, b \in \mathbb{R}$ and any $u \in E^1$, we have $a(bu) = (ab)u$.

Let W be the set of all closed bounded intervals A of real numbers with endpoints $\underline{A}$ and $\overline{A}$, i.e., $A := [\underline{A}, \overline{A}]$. Define the relation d on W by

$$d(A, B) := \max\{|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|\}.$$

Then it can be easily observed that d is a metric on W and (W, d) is a complete metric space, (cf. Nanda [16]). Now, we may define the metric D on E^1 by means of the Hausdorff metric d as follows

$$D(u, v) := \sup_{\alpha \in [0,1]} d([u]_\alpha, [v]_\alpha) := \sup_{\alpha \in [0,1]} \max\{|u^-(\alpha) - v^-(\alpha)|, |u^+(\alpha) - v^+(\alpha)|\}.$$

One can see that

$$D(u, \overline{0}) = \sup_{\alpha \in [0,1]} \max\{|u^-(\alpha)|, |u^+(\alpha)|\} = \max\{|u^-(0)|, |u^+(0)|\}. \tag{2}$$

Now, we may give:

Proposition 1.2. *[6] Let $u, v, w, z \in E^1$ and $k \in \mathbb{R}$. Then,*

(i) (E^1, D) *is a complete metric space.*

(ii) $D(ku, kv) = |k|D(u, v)$.

(iii) $D(u + v, w + v) = D(u, w)$.

(iv) $D(u + v, w + z) \leq D(u, w) + D(v, z)$.

(v) $|D(u, \overline{0}) - D(v, \overline{0})| \leq D(u, v) \leq D(u, \overline{0}) + D(v, \overline{0})$.

One can extend the natural order relation on the real line to intervals as follows:

$$A \leq B \text{ if and only if } \underline{A} \leq \underline{B} \text{ and } \overline{A} \leq \overline{B}.$$

Also, the partial ordering relation on E^1 is defined as follows:

$$u \leq v \Longleftrightarrow [u]_\alpha \leq [v]_\alpha \Longleftrightarrow u^-(\alpha) \leq v^-(\alpha) \text{ and } u^+(\alpha) \leq v^+(\alpha) \text{ for all } \alpha \in [0, 1].$$

We say that $u \prec v$ if $u \leq v$ and there exists $\alpha_0 \in [0, 1]$ such that $u^-(\alpha_0) < v^-(\alpha_0)$ or $u^+(\alpha_0) < v^+(\alpha_0)$. Two fuzzy numbers u and v are said to be incomparable if neither $u \leq v$ nor $v \leq u$ holds. In this case we use the notation $u \nsim v$.

Following Matloka [14], we give some definitions concerning with the sequences of fuzzy numbers. Nanda [16] introduced Cauchy sequences of fuzzy numbers and showed that every Cauchy sequence of fuzzy numbers is convergent.

A sequence $u = (u_k)$ of fuzzy numbers is a function u from the set $\mathbb{N}$, the set of natural numbers, into the set E^1. The fuzzy number u_k denotes the value of the function at $k \in \mathbb{N}$ and is called as the k^{th} term of the sequence. By $w(F)$, we denote the set of all sequences of fuzzy numbers.

A sequence $(u_n) \in w(F)$ is said to be convergent to $u \in E^1$, if for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$D(u_n, u) < \varepsilon \text{ for all } n \geq n_0.$$

By $c(F)$, we denote the set of all convergent sequences of fuzzy numbers.

A sequence $u = (u_k)$ of fuzzy numbers is said to be Cauchy if for every $\varepsilon > 0$ there exists a positive integer n_0 such that

$$D(u_k, u_m) < \varepsilon \quad \text{for all} \quad k, m \geq n_0.$$

By $C(F)$, we denote the set of all Cauchy sequences of fuzzy numbers.

A sequence $(u_n) \in \omega(F)$ is said to be bounded if the set of its terms is a bounded set. That is to say that a sequence $(u_n) \in \omega(F)$ is said to be bounded if there exists $M > 0$ such that $D(u_n, \overline{0}) \leq M$ for all $n \in \mathbb{N}$. By $\ell_\infty(F)$, we denote the set of all bounded sequences of fuzzy numbers.

The level convergence of a sequence of fuzzy numbers given by Fang and Huang [8] , as follows: A sequence (u_n) is level-converges to $\mu \in E^1$, written as $(l) - \lim_{n\to\infty} u_n = \mu$ if

$$\lim_{n\to\infty} u_n^-(\alpha) = \mu^-(\alpha) \text{ and } \lim_{n\to\infty} u_n^+(\alpha) = \mu^+(\alpha)$$

for all $\alpha \in [0, 1]$. Obviously, $u_n \to \mu$ implies that $(l) - \lim_{n\to\infty} u_n = \mu$ and the converse is not true in general.

Statistical convergence of sequences of fuzzy numbers was introduced by Nuray and Savaş [17]. A sequence $(u_k : k = 0, 1, 2, \ldots)$ of fuzzy numbers is said to be *statistically convergent* to some fuzzy number μ_0 if for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \frac{1}{n+1} \Big|\{k \leq n : D(u_k, \mu_0) \geq \varepsilon\}\Big| = 0.$$

In this case we write

$$\text{st-}\lim_{k\to\infty} u_k = \mu_0. \tag{3}$$

The statistical boundedness of a sequence of fuzzy numbers was introduced and studied by Aytar and Pehlivan [4]. The sequence $u = (u_k)$ is said to be statistically bounded if there exists a real number M such that the set

$$\{k \in \mathbb{N} : D(u_k, \overline{0}) > M\}$$

has natural density zero.

Aytar et al.[2] defined the concept of statistical limit superior and limit inferior of statistically bounded sequences of fuzzy numbers. Given $u = (u_k) \in w(F)$, define the following sets:

$$A_u = \left\{\mu \in E^1 : \delta\left(\{k \in \mathbb{N} : u_k \prec \mu\}\right) \neq 0\right\},$$

$$\overline{A}_u = \left\{\mu \in E^1 : \delta\left(\{k \in \mathbb{N} : u_k \succ \mu\}\right) = 1\right\},$$

$$B_u = \left\{\mu \in E^1 : \delta(\{k \in \mathbb{N} : u_k \succ \mu\}) \neq 0\right\},$$

$$\overline{B}_u = \left\{\mu \in E^1 : \delta(\{k \in \mathbb{N} : u_k \prec \mu\}) = 1\right\}.$$

If $u = (u_k)$ is a statistically bounded sequence of fuzzy numbers, then, the notions $st - \liminf$ and $st - \limsup$ is defined as follows:

$$\begin{aligned} \text{st-}\liminf u_k &= \inf A_u = \sup \overline{A}_u, \\ \text{st-}\limsup u_k &= \sup B_u = \inf \overline{B}_u. \end{aligned}$$

Lemma 1.3. *[2] Let $u = (u_k)$ be a statistically bounded sequence of fuzzy numbers, $\nu = st - \liminf u_k$ and $\mu = st - \limsup u_k$. Then, for every $\varepsilon > 0$*

$$\delta\left(\{k \in \mathbb{N} : u_k \prec \nu - \overline{\varepsilon}\}\right) = 0 \quad \text{and} \quad \delta\left(\{k \in \mathbb{N} : u_k \succ \mu + \overline{\varepsilon}\}\right) = 0.$$

Basic results on statistical convergence of sequences of fuzzy numbers may be found in [3–5, 13, 18].

The statistical Cesàro convergence of a sequence of fuzzy numbers has been defined in [1]. We say that (u_k) is statistically Cesàro convergent (written statistically (C,1)-convergent) to a fuzzy number μ_0 if

$$\text{st-}\lim_{n\to\infty} \sigma_n = \mu_0. \tag{4}$$

Kwon [13] proved that if a sequence (u_k) is bounded, then

$$\text{st-}\lim_{k\to\infty} u_k = \mu_0 \quad \text{implies} \quad \text{st-}\lim_{n\to\infty} \sigma_n = \mu_0.$$

Example 1.4. *Let $(u_k) = (u_0, v_0, u_0, v_0, ...)$ where*

$$u_0(t) = \begin{cases} \frac{2-t}{2} & , \text{ if } t \in [0,2] \\ 0 & , \text{ otherwise} \end{cases}$$

and

$$v_0(t) = \begin{cases} \frac{2+t}{2} & , \text{ if } t \in [-2,0] \\ 0 & , \text{ otherwise} \end{cases}$$

Then $\alpha-$level set of arithmetic means σ_n of (u_k)

$$[\sigma_{2n}]_\alpha = \left[\frac{2n}{2n+1}(\alpha - 1), \frac{2(n+1)}{2n+1}(1-\alpha)\right] \text{ and } [\sigma_{2n-1}]_\alpha = [\alpha - 1, 1 - \alpha].$$

So, (σ_n) is convergent to $w_0 = (u_0 + v_0)/2$ and hence it is statistically $(C, 1)$-convergent to w_0. But (u_n) is not statistically convergent.

Our goal is to find (so-called Tauberian) conditions under which the converse implication holds.

2. The Main Results

Firstly we need two lemmas.

Lemma 2.1. *Let (u_k) be a sequence of fuzzy numbers which is statistically $(C,1)$-convergent to a fuzzy number μ_0. Then for every $\lambda > 0$,*

$$\text{st}-\lim_n \sigma_{\lambda_n} = \mu_0 \tag{5}$$

where by λ_n we denote the integral part of the product λn, in symbol $\lambda_n := [\lambda n]$.

Proof. Case $\lambda > 1$. For all $\varepsilon > 0$,

$$\{n \le N : D(\sigma_{\lambda_n}, \mu_0) \ge \varepsilon\} \subseteq \{n \le \lambda_N : D(\sigma_n, \mu_0) \ge \varepsilon\},$$

whence we find

$$\frac{1}{N+1}\left|\{n \le N : D(\sigma_{\lambda_n}, \mu_0) \ge \varepsilon\}\right| \le \frac{\lambda}{\lambda_N + 1}\left|\{n \le \lambda_N : D(\sigma_n, \mu_0) \ge \varepsilon\}\right|.$$

Now, (5) follows from the statistical convergence of (σ_n) to μ_0.

Case $0 < \lambda < 1$. We claim that the same term σ_m can not occur more than $1 + \lambda^{-1}$ times in the sequence $\{\sigma_{\lambda_n} : n = 0, 1, 2, ...\}$. In fact, if for some integers k and t, we have

$$m = \lambda_k = \lambda_{k+1} = ... = \lambda_{k+t-1} < \lambda_{k+t}$$

or equivalently

$$m \le \lambda k < \lambda(k+1) < ... < \lambda(k+t-1) < m+1 \le \lambda(k+t)$$

then

$$m + \lambda(t-1) \le \lambda(k+t-1) < m+1$$

whence $\lambda(t-1) < 1$, that is $t < 1 + \lambda^{-1}$. Consequently,

$$\begin{aligned}\frac{1}{N+1}\left|\left\{n \le N : D(\sigma_{\lambda_n}, \mu_0) \ge \varepsilon\right\}\right| &\le \left(1 + \frac{1}{\lambda}\right)\frac{\lambda_N + 1}{N+1}\frac{1}{\lambda_N + 1}\left|\left\{n \le \lambda_N : D(\sigma_n, \mu_0) \ge \varepsilon\right\}\right| \\ &\le \frac{2(\lambda+1)}{\lambda_N + 1}\left|\left\{n \le \lambda_N : D(\sigma_n, \mu_0) \ge \varepsilon\right\}\right|\end{aligned}$$

provided $(\lambda_N + 1)/(N+1) \le 2\lambda$, which is the case if N is large enough. So, (5) follows from the statistical convergence of (σ_n) to μ_0. □

Lemma 2.2. *Let (u_k) be a sequence of fuzzy numbers which is statistically $(C,1)$-convergent to a fuzzy number μ_0. Then, for every $\lambda > 1$,*

$$\text{st}-\lim_n \frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n} u_k = \mu_0 \tag{6}$$

and for every $0 < \lambda < 1$,

$$\text{st}-\lim_n \frac{1}{n - \lambda_n}\sum_{k=\lambda_n+1}^{n} u_k = \mu_0.$$

Proof. Case $\lambda > 1$. If $\lambda > 1$ and n is large enough in the sense that $\lambda_n > n$, then

$$D\left(\frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n} u_k, \mu_0\right) \le \frac{\lambda_n + 1}{\lambda_n - n} D\left(\sigma_{\lambda_n}, \sigma_n\right) + D(\sigma_n, \mu_0) \tag{7}$$

Now (6) follows from (7), Lemma 2.1, the statistical convergence of (σ_n) and the fact that for large enough n

$$\frac{\lambda}{\lambda - 1} = \frac{\lambda n}{\lambda n - n} < \frac{\lambda_n + 1}{\lambda_n - n} < \frac{\lambda n + 1}{\lambda n - n - 1} \le \frac{2\lambda}{\lambda - 1}. \tag{8}$$

Case $0 < \lambda < 1$. This time, we use the following inequality:

$$D\left(\frac{1}{n - \lambda_n}\sum_{k=\lambda_n+1}^{n} u_k, \mu_0\right) \le \frac{\lambda_n + 1}{n - \lambda_n} D\left(\sigma_{\lambda_n}, \sigma_n\right) + D(\sigma_n, \mu_0)$$

provided n is large enough in the sense that $\lambda_n < n$; and the following inequality for large enough n,

$$\frac{\lambda_n + 1}{n - \lambda_n} \le \frac{2\lambda}{1 - \lambda}. \tag{9}$$

□

Now we are ready to give our main results.

Theorem 2.3. *Let (u_k) be a sequence of fuzzy numbers which is statistically $(C,1)$-convergent to a fuzzy number μ_0. Then (u_k) is statistically convergent to μ_0 if and only if one of the following two conditions holds: For every $\varepsilon > 0$,*

$$\inf_{\lambda>1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : D\left(\frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n} u_k, u_n\right) \ge \varepsilon \right\} \right| = 0 \tag{10}$$

or

$$\inf_{0<\lambda<1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : D\left(\frac{1}{n - \lambda_n}\sum_{k=\lambda_n+1}^{n} u_k, u_n\right) \ge \varepsilon \right\} \right| = 0. \tag{11}$$

Proof. The necessity follows from Lemma 2.2.

Sufficiency. Assume that conditions (4) and one of (10) and (11) are satisfied. In order to prove (3), it is enough to prove that

$$\text{st}-\lim_{n} D(u_n, \sigma_n) = 0.$$

First, we consider the case $\lambda > 1$. Since

$$D(\sigma_n, u_n) \le D\left(\frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n} u_k, u_n\right) + \frac{\lambda_n + 1}{\lambda_n - n} D\left(\sigma_{\lambda_n}, \sigma_n\right)$$

for any $\varepsilon > 0$ we have

$$\begin{aligned}\left\{ n \le N : D(\sigma_n, u_n) \ge \varepsilon \right\} \quad \subseteq \quad & \left\{ n \le N : \frac{\lambda_n + 1}{\lambda_n - n} D\left(\sigma_{\lambda_n}, \sigma_n\right) \ge \frac{\varepsilon}{2} \right\} \\ & \cup \left\{ n \le N : D\left(\frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n} u_k, u_n\right) \ge \frac{\varepsilon}{2} \right\}.\end{aligned} \tag{12}$$

Given any $\delta > 0$, by (10) there exists some $\lambda > 1$ such that

$$\limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : D\left(\frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n} u_k, u_n\right) \ge \frac{\varepsilon}{2} \right\} \right| \le \delta. \tag{13}$$

On the other hand, by Lemma 2.1 and (8), we have

$$\limsup_{N\to\infty} \frac{1}{N+1} \left|\left\{n \le N : \frac{\lambda_n + 1}{\lambda_n - n} D\left(\sigma_{\lambda_n}, \sigma_n\right) \ge \frac{\varepsilon}{2}\right\}\right| = 0. \tag{14}$$

Combining (12) with (14) we get that

$$\limsup_{N\to\infty} \frac{1}{N+1} \left|\left\{n \le N : D\left(u_n, \sigma_n\right) \ge \varepsilon\right\}\right| \le \delta$$

Since $\delta > 0$ is arbitrary, we conclude that for every $\varepsilon > 0$,

$$\lim_{N\to\infty} \frac{1}{N+1} \left|\left\{n \le N : D\left(u_n, \sigma_n\right) \ge \varepsilon\right\}\right| = 0.$$

Secondly, we consider the case $0 < \lambda < 1$. Since

$$D(\sigma_n, u_n) \le D\left(\frac{1}{n-\lambda_n} \sum_{k=\lambda_n+1}^{n} u_k, u_n\right) + \frac{\lambda_n + 1}{n - \lambda_n} D\left(\sigma_{\lambda_n}, \sigma_n\right)$$

for any $\varepsilon > 0$, we have

$$\begin{aligned}\left\{n \le N : D(\sigma_n, u_n) \ge \varepsilon\right\} \quad \subseteq \quad & \left\{n \le N : \frac{\lambda_n + 1}{n - \lambda_n} D\left(\sigma_{\lambda_n}, \sigma_n\right) \ge \frac{\varepsilon}{2}\right\} \\ & \cup \left\{n \le N : D\left(\frac{1}{n-\lambda_n} \sum_{k=\lambda_n+1}^{n} u_k, u_n\right) \ge \frac{\varepsilon}{2}\right\}.\end{aligned}$$

Given any $\delta > 0$, by (11) there exist some $0 < \lambda < 1$ such that

$$\limsup_{N\to\infty} \frac{1}{N+1} \left|\left\{n \le N : D\left(\frac{1}{n-\lambda_n} \sum_{k=\lambda_n+1}^{n} u_k, u_n\right) \ge \frac{\varepsilon}{2}\right\}\right| \le \delta.$$

Using a similar argument as in the case $\lambda > 1$, by Lemma 2.1 and condition (9), we conclude that

$$\lim_{N\to\infty} \frac{1}{N+1} \left|\left\{n \le N : D\left(u_n, \sigma_n\right) \ge \varepsilon\right\}\right| = 0.$$

□

A sequence (u_k) of fuzzy numbers is said to be *statistically slowly oscillating* if for every $\varepsilon > 0$

$$\inf_{\lambda>1} \limsup_{N\to\infty} \frac{1}{N+1} \left|\left\{n \le N : \max_{n<k\le\lambda_n} D(u_k, u_n) \ge \varepsilon\right\}\right| = 0 \tag{15}$$

or equivalently,

$$\inf_{0<\lambda<1} \limsup_{N\to\infty} \frac{1}{N+1} \left|\left\{n \le N : \max_{\lambda_n<k\le n} D(u_k, u_n) \ge \varepsilon\right\}\right| = 0. \tag{16}$$

Example 2.4. *The sequence (u_k) where*

$$u_k(t) = \begin{cases} 1 - \frac{t}{1+\log(k+1)} & , \quad \text{if } 0 \le t \le 1 + \log(k+1), \\ 0 & , \quad \text{otherwise,} \end{cases}$$

is statistically slowly oscillating. Because for every $\varepsilon > 0$ there exist $n_0 = n_0(\varepsilon)$ and $\lambda = 10^{\varepsilon}$ such that for all $n_0 < n < k \le \lambda_n$

$$D(u_k, u_n) = \left|\log(k+1) - log(n+1)\right| = \log\frac{k+1}{n+1} < \log\lambda = \varepsilon.$$

Therefore, for $n_0 < N$ the set

$$\left\{n_0 < n \le N : \max_{n<k\le\lambda_n} D(u_k, u_n) \ge \varepsilon\right\}$$

is empty. Consequently, condition (15) is satisfied.

The conditions (15) and (16) are clearly imply the conditions (10) and (11), respectively. This gives rise to the following corollary of Theorem 2.3.

Corollary 2.5. *Let (u_k) be a statistically slowly oscillating sequence of fuzzy numbers. Then*

$$\text{st}-\lim_n \sigma_n = \mu_0 \ \textit{implies}\ \text{st}-\lim_n u_n = \mu_0. \tag{17}$$

Condition (15) is satisfied if there exists a constant H such that

$$kD(u_k, u_{k-1}) \le H \tag{18}$$

for all large enough k, say $k > N_1$. In fact, given $\varepsilon > 0$, chose $1 < \lambda < 1 + \varepsilon/H$. Since $N_1 < n < k \le \lambda_n$, by (18) we have

$$D(u_k, u_n) \le \sum_{j=n+1}^{k} D(u_j, u_{j-1}) \le \sum_{j=n+1}^{k} \frac{H}{j} \le H\left(\frac{k-n}{n}\right) = H\left(\frac{k}{n} - 1\right) < H(\lambda - 1) < \varepsilon.$$

But, for $N_1 < N$, the set

$$\left\{N_1 < n \le N : \max_{n<k\le\lambda_n} D(u_k, u_n) \ge \varepsilon\right\}$$

is empty. Consequently, condition (15) is satisfied.

Remark 2.6. *Kwon [13] proved that if condition (18) is satisfied, then implication (17) holds as well as*

$$\text{st}-\lim_n u_n = \mu_0 \ \textit{implies}\ \lim_n u_n = \mu_0.$$

Definition 2.7. *(u_k) is Cesàro level-convergent (written $(C,1)$-level-convergent) to a fuzzy number μ_0, if $(l) - \lim_{n\to\infty} \sigma_n = \mu_0$ i.e.,*

$$\lim_{n\to\infty} \sigma_n^-(\alpha) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} u_k^-(\alpha) = \mu_0^-(\alpha), \quad \lim_{n\to\infty} \sigma_n^+(\alpha) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} u_k^+(\alpha) = \mu_0^+(\alpha)$$

for all $\alpha \in [0, 1]$.

Note that level-convergence implies (C,1)-level-convergence. But the converse is not true in general.

Example 2.8. *[21] Let*

$$v_n^+(\alpha) = 1, \quad v_n^-(\alpha) = \begin{cases} (\alpha - \frac{1}{2})^{1/(n+1)}, & \text{if } \frac{1}{2} < \alpha \le 1, \\ 0, & \text{if } 0 \le \alpha \le \frac{1}{2}, \end{cases}$$

$$\mu_0^+(\alpha) = \frac{1}{2}, \quad \mu_0^-(\alpha) = \begin{cases} \frac{1}{2}, & \text{if } \frac{1}{2} < \alpha \le 1, \\ 0, & \text{if } 0 \le \alpha \le \frac{1}{2}. \end{cases}$$

There exists a unique fuzzy number $v_n \in E^1$ and a unique fuzzy number $u_0 \in E^1$ such that $[v_n]_\alpha = [v_n^-(\alpha), v_n^+(\alpha)]$ and $[\mu_0]_\alpha = [\mu_0^-(\alpha), \mu_0^+(\alpha)]$ for all $\alpha \in [0, 1]$. Let $u = (u_n) = (v_0, \overline{0}, v_1, \overline{0}, v_2, ...)$. Since

$$\lim_{n\to\infty} \sigma_n^+(\alpha) = \mu_0^+(\alpha), \ \lim_{n\to\infty} \sigma_n^-(\alpha) = \mu_0^-(\alpha)$$

(u_n) is (C,1)-level-convergent to μ_0 but (u_n) is not level-convergent to μ_0. Furthermore, since $D(\sigma_n, \mu_0) = \frac{1}{2}$, (u_n) is not (C,1) convergent to μ_0.

In [11, Theorem 5] Fridy and Orhan proved that a sequence of real numbers which is $(C,1)$-convergent to its statistical limit superior is statistically convergent. The next theorem is an analogue of that result for sequences of fuzzy numbers.

Theorem 2.9. *Let (u_k) be a bounded sequence of fuzzy numbers. Assume that (u_k) is $(C,1)$-level-convergent to $\mu = st - \limsup u_k$ and there is a number $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0,\varepsilon_0)$,*

$$\delta(\{k \in \mathbb{N} : u_k \nsim \mu - \overline{\varepsilon}\}) = 0, \quad \delta(\{k \in \mathbb{N} : u_k \nsim \mu + \overline{\varepsilon}\}) = 0.$$

Then (u_k) is statistically convergent to μ.

Proof. Since $\mu = st - \limsup u_k$, then

$$\delta(\{k \in \mathbb{N} : u_k \succ \mu + \overline{\varepsilon}\}) = 0$$

for each $\varepsilon > 0$. Suppose that (u_n) is not statistically convergent to μ. Then, there is a $\varepsilon_1 \in (0,\varepsilon_0)$ such that

$$\delta(\{k \in \mathbb{N} : u_k \prec \mu - \overline{\varepsilon}_1\}) \neq 0.$$

Define $m = \mu - \overline{\varepsilon}_1$, $B = \sup_n u_n$ and

$$\begin{aligned} K' &= \{k \in \mathbb{N} : u_k \prec m\} \\ K'' &= \{k \in \mathbb{N} : m \preceq u_k \preceq \mu + \overline{\varepsilon}\} \\ K''' &= \{k \in \mathbb{N} : u_k \succ \mu + \overline{\varepsilon}\} \cup \{k \in \mathbb{N} : u_k \nsim m\} \cup \{k \in \mathbb{N} : u_k \nsim \mu + \overline{\varepsilon}\}. \end{aligned}$$

Since $\delta(K''') = 0$, $\delta(K') \neq 0$ and $\delta(K'') = 1 - \delta(K')$ there are infinitely many n such that

$$\frac{1}{n+1}|K_n'| \geq d > 0$$

and for each such n we have

$$\begin{aligned} \sigma_n &= \frac{1}{n+1}\sum_{k\in K_n'} u_k + \frac{1}{n+1}\sum_{k\in K_n''} u_k + \frac{1}{n+1}\sum_{k\in K_n'''} u_k \\ &\prec \frac{m}{n+1}|K_n'| + \frac{\mu+\overline{\varepsilon}}{n+1}|K_n''| + \frac{B}{n+1}|K_n'''|. \end{aligned}$$

So there is an $\alpha_0 \in [0,1]$ such that

$$\begin{aligned} \sigma_n^-(\alpha_0) &< \frac{m^-(\alpha_0)}{n+1}|K_n'| + \frac{\mu^-(\alpha_0)+\varepsilon}{n+1}|K_n''| + \frac{B^-(\alpha_0)}{n+1}|K_n'''| \\ &= m^-(\alpha_0)\frac{|K_n'|}{n+1} + (\mu^-(\alpha_0)+\varepsilon)\left(1 - \frac{|K_n'|}{n+1}\right) + o(1) \\ &\leq \mu^-(\alpha_0) - d\left(\mu^-(\alpha_0) - m^-(\alpha_0)\right) + \varepsilon(1-d) + o(1). \end{aligned}$$

Since $\varepsilon \in (0,\varepsilon_0)$ is arbitrary it follows that

$$\liminf \sigma_n^-(\alpha_0) \leq \mu^-(\alpha_0) - d\left(\mu^-(\alpha_0) - m^-(\alpha_0)\right) < \mu^-(\alpha_0).$$

Hence, (u_n) is not $(C,1)$-level-convergent to μ and this completes the proof. Note that the result can be found also by using the following inequality

$$\sigma_n^+(\alpha_0) < \frac{m^+(\alpha_0)}{n+1}|K_n'| + \frac{\mu^+(\alpha_0)+\varepsilon}{n+1}|K_n''| + \frac{B^+(\alpha_0)}{n+1}|K_n'''|.$$

□

The following is a dual result for $st - \liminf$.

Theorem 2.10. *Let (u_k) be a bounded sequence of fuzzy numbers. Assume that (u_k) is $(C, 1)$-level-convergent to $\nu = st - \liminf u_k$ and there is a number $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$,*

$$\delta(\{k \in \mathbb{N} : u_k \nsim \nu - \overline{\varepsilon}\}) = 0, \;\; \delta(\{k \in \mathbb{N} : u_k \nsim \nu + \overline{\varepsilon}\}) = 0.$$

Then, (u_k) is statistically convergent to ν.

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