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Squashing Maximum Packings of K_n with 8-Cycles into Maximum Packings of K_n with 4-Cycles

Charles Curtis Lindner[a], Giovanni Lo Faro[b], Mariusz Meszka[c], Antoinette Tripodi[b]

[a] *Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849-5307 U.S.A.*
[b] *Dipartimento di Matematica e Informatica, Università di Messina, 98166 Messina, Italia*
[c] *AGH University of Science and Technology, Kraków, 30059 Poland*

Abstract. An 8-cycle is said to be squashed if we identify a pair of opposite vertices and name one of them with the other (and thereby turning the 8-cycle into a pair of 4-cycles with exactly one vertex in common). The resulting pair of 4-cycles is called a bowtie. We say that we have *squashed* the 8-cycle into a bowtie. Evidently an 8-cycle can be squashed into a bowtie in eight different ways. The object of this paper is the construction, for *every* $n \geqslant 8$, of a maximum packing of K_n with 8-cycles which can be squashed in a maximum packing of K_n with 4-cycles.

1. Introduction

Let G be a graph. A *G-design of order n* is a pair (X, B) where B is a collection of subgraphs (*blocks*), each isomorphic to G, which partitions the edge set of the complete undirected graph K_n with vertex set X. After determining the spectrum for G-designs for different graphs G, many problems have been studied also recently (for example, see [2]-[6]). A triple (X, B, L), where B is a collection of edge disjoint copies of G with vertices in X, L is the set (*leave*) of all edges of K_n not belonging to any subgraph of B and $|L|$ is as small as possible, is a *maximum packing* of K_n with copies of G; a G-design of order n is a maximum packing of K_n with copies of G and $L = \emptyset$.

An m-cycle system of order n is a G-design of order n where G is an m-cycle. The necessary and sufficient conditions for the existence of an m-cycle system are [1, 10]:

(1) $n \geqslant m$, if $n > 1$;
(2) n is odd; and
(3) $n(n-1)/2m$ is an integer.

If $c = (x_1, x_2, \ldots, x_m)$ is an m-cycle, we will denote by $c(2)$ the collection of edges $\{x_i, x_{i+2}\}$, $i = 1, 2, \ldots, m$, modulo m. The graph $c(2)$ is called the distance 2 graph of c. For example the distance 2 graphs of the 6-cycle $(1, 2, 3, 4, 5, 6)$ and the 8-cycle $(1, 2, 3, 4, 5, 6, 7, 8)$ look like:

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Email addresses: lindncc@auburn.edu (Charles Curtis Lindner), lofaro@unime.it (Giovanni Lo Faro), meszka@agh.edu.pl (Mariusz Meszka), atripodi@unime.it (Antoinette Tripodi)

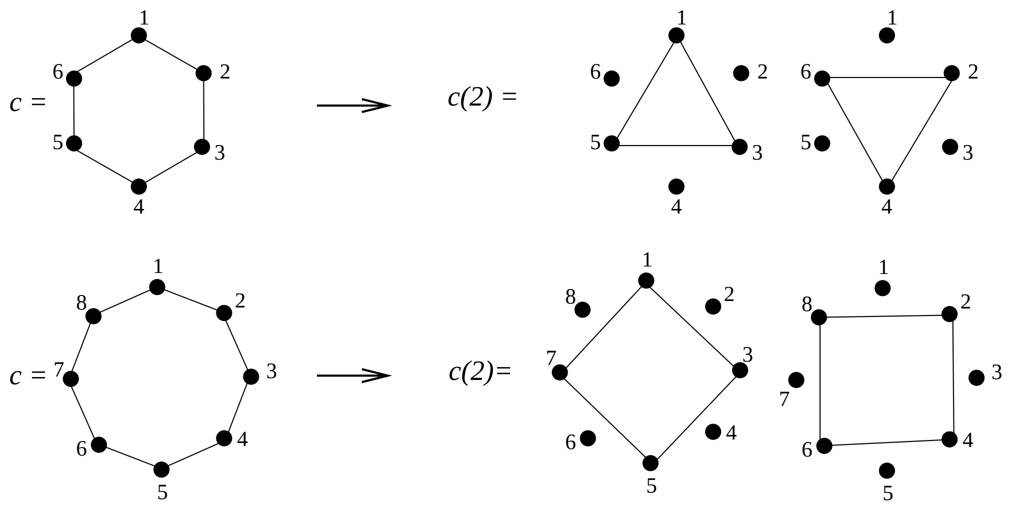


Figure 1

An m-cycle system (X, C) of order n is said to be 2-*perfect* provided the collection of graphs $C(2) = \{c(2) \mid c \in C\}$ covers the edges of K_n. For 6-cycle systems this says that $(X, C(2))$ is a Steiner triple system and for 8-cycle systems a 4-cycle system. A lot of work has been done on 2-perfect m-cycle systems; and rather than going into a detailed history of the problem of constructing 2-perfect m-cycle systems the reader is referred to [7].

Quite recently a *new connection* between 6-cycle systems and Steiner triple systems was introduced: the *squashing* of a 6-cycle system into a Steiner triple system [9]. A definition is in order. Let c be a 6-cycle and x and y opposite vertices in c. If we rename y with x we obtain a bowtie B; i.e., two 3-cycles with the common vertex x. We say that we have *squashed* c into B; see Figure 2.

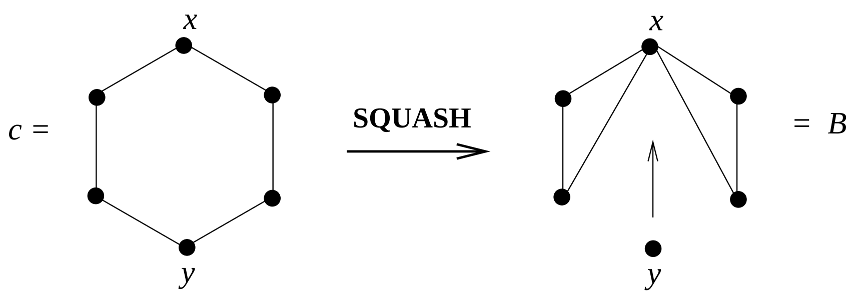


Figure 2

Clearly a 6-cycle can be squashed into a bowtie in six different ways.

If it is possible to squash each of the 6-cycles of the 6-cycle system (X, C) into a bowtie so that the resulting collection $S(C)$ of bowties is a Steiner triple system, we will say that (X, C) is *squashed* into $(X, S(C))$. In [9] it is shown that for every $n \equiv 1$ or $9 \pmod{12}$ (the spectrum for 6-cycle systems), there exists a 6-cycle system that can be squashed into a Steiner triple system. This result has been generalized to maximum packings. Rather than go into details, the reader is referred to [8].

This paper gives a *complete solution* of the problem of squashing maximum packings of 8-cycle systems into maximum packings of 4-cycle systems. We begin with some preliminaries.

2. Preliminaries

The following tables give leaves for maximum packings of K_n for both 8-cycles and 4-cycles.

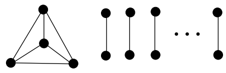
n even		
K_n	8-cycle leave	4-cycle leave
$n \equiv 0, 2, 8, 10 \pmod{16}$	1-factor	1-factor
$n \equiv 4, 6, 12, 14 \pmod{16}$		1-factor

Table 1: n even

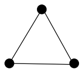
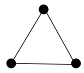
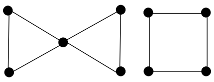
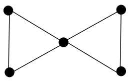
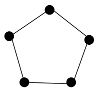
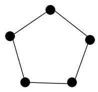
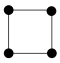
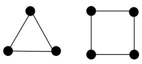
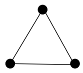
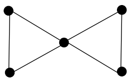
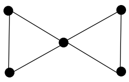
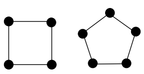
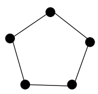
n odd		
K_n	8-cycle leave	4-cycle leave
$n \equiv 1 \pmod{16}$	$\emptyset$	$\emptyset$
$n \equiv 3 \pmod{16}$		
$n \equiv 5 \pmod{16}$		
$n \equiv 7 \pmod{16}$		
$n \equiv 9 \pmod{16}$		$\emptyset$
$n \equiv 11 \pmod{16}$		
$n \equiv 13 \pmod{16}$		
$n \equiv 15 \pmod{16}$		

Table 2: n odd

Now let c be an 8-cycle and x and y a pair of opposite vertices in c. If we rename y with x we have two 4-cycles B with the common vertex x. We remark (and this is *important*) that the two 4-cycles in B have only the vertex x in common. We will call a pair of 4-cycles with exactly one vertex in common a *bowtie*.

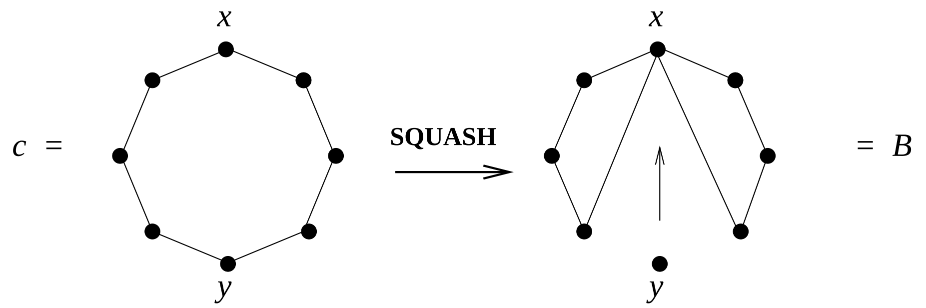


Figure 3

Just as with 6-cycles, we say that we have *squashed* c into B (and also that we have squashed y onto x). Clearly an 8-cycle can be squashed into a bowtie in eight different ways.

Now let (X, C, L) be a maximum packing of K_n with 8-cycles with leave L *exactly* as in Tables 1 and 2. We remark (and this is *important*) that the leaves for maximum packings are *not necessarily unique*.

Let $S(C)$ be a squashing of the 8-cycles in C which covers *exactly* the same edges as C. If L contains no 4-cycles, then $(X, S(C), L)$ is a maximum packing of K_n with 4-cycles. If L contains a 4-cycle ($n \equiv 4, 5, 6, 9, 11, 12, 14$ or $15 \pmod{16}$) and we remove a 4-cycle (a, b, c, d) from L, then $(X, S(C) \cup \{(a, b, c, d)\}, L \setminus \{(a, b, c, d)\})$ is a maximum packing of K_n with 4-cycles. In the cases $n \equiv 5, 9, 11$ and $15 \pmod{16}$, there is only one 4-cycle in L. However, in the cases $n \equiv 4, 6, 12, 14$ there are three 4-cycles that can be removed (all from K_4). We remove just one (any one) to obtain a 1-factor.

Example 2.1. (The squashing of a maximum packing of K_{11} with 8-cycles into a maximum packing of K_{11} with 4-cycles.)

$$X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

$$C = \left\{ \begin{array}{lcl} (0,1,2,3,4,5,6,7) & & (3,0,1,2)(3,4,5,6) \\ (0,2,4,1,3,5,7,8) & & (8,0,2,4)(8,3,5,7) \\ (0,3,6,1,8,9,2,10) & \mathit{SQUASH} & (9,6,1,8)(9,0,10,2) \\ (0,4,10,7,9,6,2,5) & \longrightarrow & (0,4,10,7)(0,5,2,6) \\ (0,6,8,2,7,3,10,9) & & (7,2,8,6)(7,3,10,9) \\ (1,5,9,3,8,4,6,10) & & (1,3,9,5)(1,4,6,10) \end{array} \right\} = S(C).$$

$$L = \left\{ \begin{array}{lcl} (1,7,4,9) & & (1,7,4,9)\} = L^* \\ (5,8,10) & & (5,8,10)\} = L \setminus L^*. \end{array} \right.$$

Then (X, C, L) is a maximum packing of K_{11} with 8-cycles which has been squashed into the maximum packing of K_{11} with 4-cycles $(X,\ S(C) \cup L^*,\ L \setminus L^*)$. (See Tables 1 and 2.)

In what follows we will write $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)\, x_i$ to indicate that the vertex *opposite* x_i (namely, x_{i+4}) modulo 8 has been squashed onto x_i.

3. $n \equiv 0, 2, 8$ and $10 \pmod{16}$

We begin with an important lemma.

Lemma 3.1. *There exists a decomposition of $K_{4m,4n}$ into 8-cycles that can be squashed into a decomposition of $K_{4m,4n}$ into 4-cycles.*

Proof. Let $K_{4,4}$ have parts $\{a, b, c, d\}$ and $\{a', b', c', d'\}$. Then the following two 8-cycles can be squashed into 4-cycles.

$$\begin{array}{lcl} (a, d', d, b', b, a', c, c')\, a & \text{SQUASH} & (a, d', d, b')(a, a', c, c') \\ (b, d', c, b', a, a', d, c')\, b & \longrightarrow & (b, b', c, d')(b, a', d, c'). \end{array}$$

Let $K_{4m,4n}$ have parts $\{a,b,c,d\}\times\{1,2,\dots,m\}$ and $\{a',b',c',d'\}\times\{1,2,3,\dots,n\}$. For each $i\in\{1,2,\dots,m\}$ and $j\in\{1,2,\dots,n\}$, define a copy of $K_{4,4}$ (above) with parts $\{a,b,c,d\}\times\{i\}$ and $\{a',b',c',d'\}\times\{j\}$. □

$n \equiv 0$ or 8 (mod 16)
We need one example.

Example 3.2. ($n = 8$)

$$X=\{0,1,2,3,4,5,6,7\}$$
$$C=\begin{cases}(0,1,2,7,4,5,6,3)\,0\\(4,2,3,5,1,7,6,0)\,4\\(1,6,2,5,0,7,3,4)\,1\end{cases}$$
$$L=\{(0,2),(1,3),(4,6),(5,7)\}.$$

Then (X,C,L) is a maximum packing of K_8 with 8-cycles which can be squashed into the maximum packing of K_8 with 4-cycles $(X,S(C),L)$.

Now let Z be a set of size 8 and set $X = Z\times\{1,2,3,\dots,k\}$ and define a collection of 8-cycles C as follows:

(1) For each $i\in\{1,2,3,\dots,k\}$ define a copy of Example 3.2 on $Z\times\{i\}$ and place these 8-cycles in C.

(2) For each $i\neq j\in\{1,2,3,\dots,k\}$ place a copy of $K_{8,8}$ (Lemma 3.1) with parts $Z\times\{i\}$ and $Z\times\{j\}$ in C.

If we denote by L the union of the leaves in (1), the result is a maximum packing of K_{8k} (X,C,L) with 8-cycles, which can be squashed into the maximum packing of K_{8k} $(X,S(C),L)$ with 4-cycles.

$n \equiv 2$ or 10 (mod 16)
We begin with an example.

Example 3.3. ($n = 10$)

$$X=Z_{10}$$
$$C=\begin{cases}(0,2,1,3,4,6,5,7)\,3\\(0,3,5,1,4,8,6,9)\,8\\(0,4,2,7,1,9,5,8)\,7\\(0,5,2,8,7,9,3,6)\,2\\(1,6,2,9,4,7,3,8)\,3\end{cases}$$
$$L=\{\{0,1\},\{2,3\},\{4,5\},\{6,7\},\{8,9\}\}.$$

Then (X,C,L) is a maximum packing of K_{10} with 8-cycles which can be squashed into the maximum packing of K_{10} with 4-cycles $(X,S(C),L)$.

Let Z be a set of size 8 and set $X=\{\infty_1,\infty_2\}\cup(Z\times\{1,2,3,\dots,k\})$ and define a collection of 8-cycles C as follows:

(1) For each $i\in\{1,2,3,\dots,k\}$ define a copy of Example 3.3 on $\{\infty_1,\infty_2\}\cup(Z\times\{i\})$ and place these 8-cycles in C. Be sure that $\{\infty_1,\infty_2\}$ is part of the leave.
(2) For each $i\neq j\in\{1,2,3,\dots,k\}$ place a copy of $K_{8,8}$ (Lemma 3.1) with parts $Z\times\{i\}$ and $Z\times\{j\}$ in C.

Denote by L the union of the leaves in (1). Then (X,C,L) is a maximum packing of K_{8k+2} with 8-cycles which can be squashed into the maximum packing of K_{8k+2} $(X,S(C),L)$ with 4-cycles.

Lemma 3.4. *There is a maximum packing of K_n with 8-cycles that can be squashed into a maximum packing of K_n with 4-cycles for all $n\equiv 0,2,8$ and 10 (mod 16).* □

4. $n \equiv 4, 6, 12, 14 \pmod{16}$

We will need three examples.

Example 4.1. ($n = 12$)

$$X = Z_{12},$$
$$C = \begin{cases} (0,5,1,4,2,6,3,7)\,4 \\ (0,4,3,5,2,7,1,6)\,7 \\ (0,8,1,9,2,10,3,11)\,9 \\ (0,9,3,8,2,11,1,10)\,11 \\ (4,6,5,7,8,10,9,11)\,7 \\ (4,7,10,5,11,8,6,9)\,11 \\ (4,8,5,9,7,11,6,10)\,4 \end{cases}$$
$$L = \begin{cases} K_4 \text{ on } \{0,1,2,3\} \\ \{4,5\}\{6,7\}\{8,9\}\{10,11\} \end{cases}$$

Then (X, C, L) is a maximum packing of K_{12} with 8-cycles which can be squashed into the maximum packing of K_{12} with 4-cycles $(X, S(C) \cup \{(0,1,2,3)\}, L \setminus \{(0,1,2,3)\})$. We remark that $L \setminus \{(0,1,2,3)\}$ is a 1-factor.

Example 4.2. ($n = 14$)

$$X = Z_{14},$$
$$C = \begin{cases} (0,5,1,4,2,6,3,7)\,4 & (0,4,3,5,2,7,1,6)\,7 \\ (0,8,1,9,2,10,3,11)\,9 & (0,9,3,8,2,11,1,10)\,11 \\ (0,12,4,6,5,7,8,13)\,4 & (1,12,5,8,4,9,6,13)\,5 \\ (2,12,6,10,4,11,5,13)\,6 & (3,12,11,7,10,5,9,13)\,11 \\ (4,7,9,11,8,12,10,13)\,8 & (6,8,10,9,12,7,13,11)\,9 \end{cases}$$
$$L = \begin{cases} K_4 \text{ on } \{0,1,2,3\} \\ \{4,5\}\{6,7\}\{8,9\}\{10,11\}\{12,13\} \end{cases}$$

Then (X, C, L) is a maximum packing of K_{14} with 8-cycles which can be squashed into the maximum packing of K_{14} with 4-cycles $(X, S(C) \cup \{(0,1,2,3)\}, L \setminus \{(0,1,2,3)\})$. We remark that $L \setminus \{(0,1,2,3)\}$ is a 1-factor.

Example 4.3. (Maximum packing of $K_{14} \setminus K_6$ with 8-cycles with leave a 1-factor consisting of four edges whose vertices are contained in $V(K_{14}) \setminus V(K_6)$ which can be squashed into a maximum packing of $K_{14} \setminus K_6$ with 4-cycles (the same leave)).

Let Y and Z be sets of size 4 and 8 and set $X = \{\infty_1, \infty_2\} \cup Y \cup Z$. Define a collection C of 8-cycles as follows:

(1) Place a copy of Example 3.3 on $\{\infty_1, \infty_2\} \cup Z$ and make sure that the leave L contains $\{\infty_1, \infty_2\}$. (L is a 1-factor.)
(2) Partition $K_{4,8}$ with parts Y and Z into four 8-cycles (which can be squashed into eight 4-cycles (Lemma 3.1)) and place these 8-cycles in C.

Then $(K_{14} \setminus K_6, C, L \setminus \{\infty_1, \infty_2\})$ is a maximum packing of $K_{14} \setminus K_6$ with 8-cycles (K_6 is defined on $\{\infty_1, \infty_2\} \cup Y$) that can be squashed into the maximum packing $(K_{14} \setminus K_6, S(C), L \setminus \{\infty_1, \infty_2\})$ of $K_{14} \setminus K_6$ with 4-cycles. (We remark that $V(L \setminus \{\infty_1, \infty_2\})$ is contained in $V(K_{14}) \setminus V(K_6)$.)

With the above three examples in hand we can proceed to the general construction for $n \equiv 4, 6, 12, 14 \pmod{16}$.

$n \equiv 4$ or $12 \pmod{16}$

Let Z be a set of size 8, $\infty = \{\infty_1, \infty_2, \infty_3, \infty_4\}$, set $X = \infty \cup (Z \times \{1, 2, 3, \ldots, k\})$, and define a collection of 8-cycles C as follows:

(1) For each $i \in \{1, 2, 3, \ldots, k\}$ define a copy of Example 4.1 on $\infty \cup \{Z \times \{i\}\}$. (Make sure that K_4 is defined on ∞.)
(2) For each $i \neq j \in \{1, 2, 3, \ldots, k\}$ place a copy of $K_{8,8}$ (Lemma 3.1) with parts $Z \times \{i\}$ and $Z \times \{j\}$ in C.

Denote by L the union of the leaves in (1) by considering the K_4 on ∞ only once. The result is a maximum packing of K_{8k+4} (X, C, L) with 8-cycles that can be squashed into the maximum packing of K_{8k+4} $(X, S(C) \cup \{(\infty_1, \infty_2, \infty_3, \infty_4)\}), L \setminus \{(\infty_1, \infty_2, \infty_3, \infty_4)\})$ with 4-cycles. $(L \setminus \{(\infty_1, \infty_2, \infty_3, \infty_4)\})$ is a 1-factor.)

$n \equiv 6$ or $14 \pmod{16}$

Let Z be a set of size 8, $\infty = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6\}$, and set $X = \infty \cup (Z \times \{1, 2, 3, \ldots, k\})$. Define a collection of 8-cycles C as follows:

(1) Define a copy of Example 4.2 on $\infty \cup (Z \times \{1\})$.
(2) For each $i \in \{2, 3, \ldots, k\}$ define a copy of Example 4.3 on $\infty \cup (Z \times \{i\})$. Make sure that K_6 is defined on ∞.
(3) For each $i \neq j \in \{1, 2, 3, \ldots, k\}$ take a copy of $K_{8,8}$ (Lemma 3.1) with parts $Z \times \{i\}$ and $Z \times \{j\}$ and place these 8-cycles in C.

Then (X, C, L) is a maximum packing of K_{8k+6} with 8-cycles where the leave L is the union of the leaves in (1) and (2). Removing a 4-cycle from the leave in (1) squashes (X, C, L) into a maximum packing $(X, S(C) \cup (\text{4-cycle}), L \setminus (\text{4-cycle}))$ of K_{8k+6} with 4-cycles.

Lemma 4.4. *There is a maximum packing of K_n with 8-cycles that can be squashed into a maxmum packing of K_n with 4-cycles for all $n \equiv 4, 6, 12, 14 \pmod{16}$.* □

5. $n \equiv 1$ or $9 \pmod{16}$

We begin with an example.

Example 5.1. (The squashing of a maximum packing of K_9 with 8-cycles into a maximum packing of K_9 with 4-cycles.)

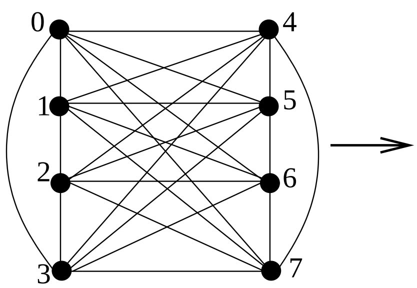
$X = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$.

$$C = \begin{cases} (0, 1, 2, 3, 4, 5, 6, 7)\,3 \\ (0, 2, 4, 6, 1, 3, 7, 8)\,6 \\ (0, 3, 8, 6, 2, 7, 1, 5)\,7 \\ (0, 4, 1, 8, 2, 5, 3, 6)\,8 \end{cases}$$

$L = \{(4, 7, 5, 8)\}$.

Then (X, C, L) is a maximum packing of K_9 with 8-cycles which is squashed into the 4-cycle system $(X, C \cup \{(4, 7, 5, 8)\})$.

We will need the following interpretation of Example 3.2. It is best done with a diagram.



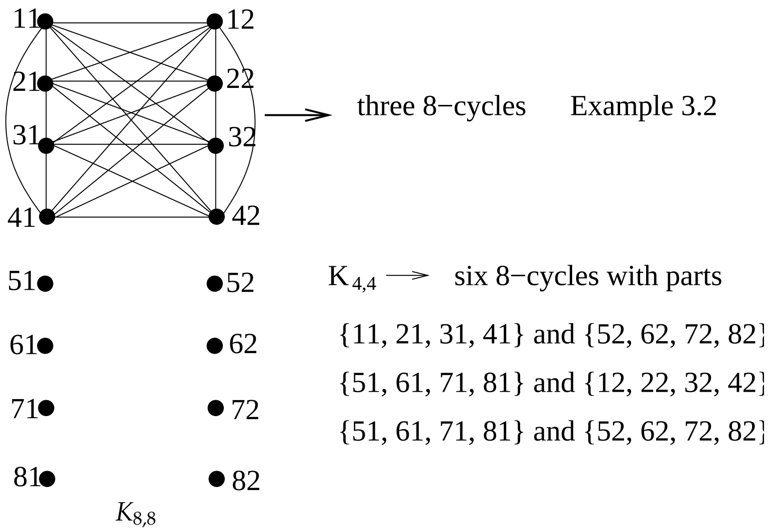
$(0, 1, 2, 7, 4, 5, 6, 3)\,0$
$(4, 2, 3, 5, 1, 7, 6, 0)\,4$
$(1, 6, 2, 5, 0, 7, 3, 4)\,1$

Figure 5

Let Z be a set of size 8 and set $X = \{\infty\} \cup (Z \times \{1, 2, 3, \ldots, k\})$ and define a collection of 8-cycles C as follows:

(1) For each $i \in \{1, 2, 3, \ldots, k\}$ define a copy of Example 5.1 on $\{\infty\} \cup (Z \times \{i\})$ and make sure the leave does not contain ∞.

(2) If k is even, pair up the copies of $Z \times \{i\}$: $Z \times \{1\}, Z \times \{2\}; Z \times \{3\}, Z \times \{4\}; \ldots ; Z \times \{k-1\}, Z \times \{k\}$, and use Lemma 3.1 and Example 3.2 to partition $K_{8,8}$ with parts $Z \times \{i\}$ and $Z \times \{i+1\}$ union the two 4-cycle leaves into nine 8-cycles.



If k is odd, pair up $Z \times \{i\}$s with one left.

(3) For all other pairs $i \neq j \in \{1, 2, 3, \ldots, k\}$, place a copy of $K_{8,8}$ (Lemma 3.1) with parts $Z \times \{i\}$ and $Z \times \{j\}$ in C.

Then $(X, C, \emptyset)$ is an 8-cycle system that can be squashed into a 4-cycle system for all $n \equiv 1 \pmod{16}$. If k is odd we have a 4-cycle leave left over in (2). Then (X, C, L) can be squashed into the 4-cycle system $(X, S(C) \cup L, \emptyset)$ for all $n \equiv 9 \pmod{16}$.

We have the following lemma.

Lemma 5.2. *There is a maximum packing of K_n with 8-cycles that can be squashed into a maximum packing of K_n with 4-cycles for all $n \equiv 1$ or $9 \pmod{16}$.* (We remark that the result is always a 4-cycle system.) □

6. $n \equiv 3$ or 11 (mod 16)

Let Z be a set of size 8 and set $X = \{\infty_1, \infty_2, \infty_3\} \cup (Z \times \{1, 2, 3, \ldots, k\})$. Define a collection of 8-cycles C as follows:

(1) For each $i \in \{1, 2, 3, 4, \ldots, k\}$, define a copy of Example 2.1 on $\{\infty_1, \infty_2, \infty_3\} \cup (Z \times \{i\})$ where the leave consists of the two disjoint cycles $(\infty_1, \infty_2, \infty_3)$ and $(a, b, c, d) \times \{i\}$ and place these 8-cycles in C.

(2) and (3) Exactly the same as (2) and (3) in the cases 1 or 9 (mod 16).

Let (X, C, L) be the resulting maximum packing of K_{8k+3} with 8-cycles. If k is odd, L consists of a disjoint 3-cycle and 4-cycle. Squashing C into 4-cycles and adding the 4-cycle from L to $S(C)$ gives a maximum packing of K_{8k+3} with 4-cycles with leave a 3-cycle. If k is even, L consists of a 3-cycle only, and $(X, S(C), L)$ is a maximum packing of K_{8k+3} with 4-cycles. We have the following lemma.

Lemma 6.1. *There is a maximum packing of K_n with 8-cycles that can be squashed into a maximum packing of K_n with 4-cycles for all $n \equiv 3$ or $11 \pmod{16}$.* □

7. $n \equiv 5$ or 13 (mod 16)

Here is an example for $n = 13$.

Example 7.1. ($n = 13$)

Let Y and Z be sets of size 4 and 8 and set $X = \{\infty\} \cup Y \cup Z$. Define a collection C of 8-cycles as follows:

(1) Place a copy of Example 5.1 on $\{\infty\} \cup Z$ and make sure the leave does not contain ∞. (The leave is a 4-cycle, say (a, b, c, d).)
(2) Let (y_1, y_2, y_3, y_4) be a 4-cycle in Y and partition $K_{4,8}$ with parts Y and Z union the 4-cycles (y_1, y_2, y_3, y_4) and (a, b, c, d) into five 8-cycles.
(3) The leave consists of the *bowtie* $(\infty, y_1, y_3)(\infty, y_2, y_4)$.

Then (X, C, L) is a maximum packing of K_{13} with 8-cycles which can be squashed into the maximum packing $(X, S(C), L)$ (same leave) of K_{13} with 4-cycles.

Example 7.2. (Maximum packing of $K_{13} \setminus K_5$ with 8-cycles with leave a 4-cycle which can be squashed into a maximum packing of $K_{13} \setminus K_5$ with 4-cycles (no leave).)

(1) Exactly the same as (1) in Example 7.1.
(2) Partition $K_{4,8}$ with parts Y and Z into four 8-cycles.

Then $(K_{13} \setminus K_5, C, (a, b, c, d))$ is a maximum packing of $K_{13} \setminus K_5$ with 8-cycles with leave the 4-cycle (a, b, c, d) which can be squashed into the maximum packing $(K_{13} \setminus K_5, S(C) \cup (a, b, c, d), \emptyset)$ of $K_{13} \setminus K_5$ with 4-cycles.

With these two examples we can now give the general construction.

Let $\infty = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$ and let Z be a set of size 8. Let $X = \infty \cup (Z \times \{1, 2, 3, \ldots, k\})$ and define a collection of 8-cycles C as follows:

(1) Place a maximum packing of K_{13} with 8-cycles on $\infty \cup (Z \times \{1\})$ with leave a bowtie defined on ∞.
(2) For each $i \in \{2, 3, \ldots, k\}$ place a copy of Example 7.2 on $\infty \cup (Z \times \{i\})$ with leave a 4-cycle contained in $Z \times \{i\}$.
(3) If $k - 1$ is *even*, consecutive K_8s can be partitioned into nine 8-cycles and all other pairings into eight 8-cycles, giving a maximum packing of K_{8k+5} into 8-cycles with leave a bowtie in ∞. These 8-cycles can be squashed into 4-cycles with leave the bowtie in ∞.
(4) If $k - 1$ is odd, pair up the K_8s with one left over. This gives a maximum packing of K_{8k+5} into 8-cycles with leave a bowtie in ∞ and a 4-cycle. These 8-cycles can be squashed into 4-cycles with leave the bowtie in ∞.

Lemma 7.3. *There is a maximum packing of K_n with 8-cycles that can be squashed into a maximum packing of K_n with 4-cycles for all $n \equiv 5$ or 13 (mod 16).* □

8. $n \equiv 7$ or 15 (mod 16)

We will need two examples.

Example 8.1. ($n = 15$)

Define a collection C of 8-cycles on Z_{15} as follows:

$$C = \begin{cases} (0,2,9,5,7,10,8,11)\,9 & (0,3,5,12,7,14,11,13)\,12 \\ (0,5,1,6,2,7,3,8)\,6 & (0,6,4,5,2,8,1,7)\,8 \\ (0,9,1,10,3,11,4,12)\,10 & (0,10,4,9,3,13,2,14)\,13 \\ (1,3,6,12,9,13,8,14)\,8 & (1,4,7,13,14,10,11,12)\,13 \\ (1,11,2,12,3,14,4,13)\,12 & (2,4,8,12,14,9,6,10)\,6 \\ (5,10,13,6,8,9,7,11)\,8 & (5,13,12,10,9,11,6,14)\,5 \end{cases}$$

$L = \{(0, 1, 2, 3, 4),\ (5, 6, 7, 8)\}$.

This can be squashed into a maximum packing of K_{15} with 4-cycles with leave the 5-cycle $(0, 1, 2, 3, 4)$.

Example 8.2. (Maximum packing of $K_{15} \setminus K_7$ with 8-cycles with leave a 4-cycle (contained in $K_{15} \setminus K_7$) which can be squashed into a maximum packing of $K_{15} \setminus K_7$ with 4-cycles (no leave).)

Let $\infty = \{\infty_1, \infty_2, \infty_3\}$ and Y and Z sets of size 4 and 8. Set $X = \infty \cup Y \cup Z$ and define a collectiion C of 8-cycles as follows:

(1) Place a copy of Example 2.1 on $\infty \cup Z$ where the leave consists of the two disjoint cycles $(\infty_1, \infty_2, \infty_3)$ and $(a, b, c, d) \subseteq Z$.
(2) Partition $K_{4,8}$ with parts Y and Z into four 8-cycles.

Then $(K_{15} \setminus K_7, C, (a, b, c, d))$ is a maximum packing of $K_{15} \setminus K_7$ with 8-cycles with leave the 4-cycle (a, b, c, d) which can be squashed into the maximum packing $(K_{15} \setminus K_7, C \cup (a, b, c, d), \emptyset)$ of $K_{15} \setminus K_7$ with 4-cycles.

We can now give the general construction for 7 or 15 (mod 16). Let $\infty = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7\}$ and let Z be a set of size 8. Set $X = \infty \cup (Z \times \{1, 2, 3, \ldots, k\})$ and define a collection of 8-cycles C as follows:

(1) Place a maximum packing of K_{15} with 8-cycles on $\infty \cup (Z \times \{1\})$ with leave a disjoint 5-cycle and 4-cycle, where the 5-cycle is contained in ∞ and the 4-cycle is contained in $Z \times \{1\}$.
(2) For each $i \in \{2, 3, 4, \ldots, k\}$ place a copy of Example 8.2 on $\infty \cup (Z \times \{i\})$ with leave a 4-cycle contained in $Z \times \{i\}$.
(3) If $k - 1$ is *even* proceed as in 5 or 13 (mod 16) with leave a disjoint 5-cycle and 4-cycle which can be squashed into a maximum packing of K_{8k+7} with leave a 5-cycle.
If $k - 1$ is *odd* we have a maximum packing of K_{8k+7} with 8-cycles with leave a 5-cycle which can be squashed into 4-cycles with the same leave.

Lemma 8.3. *There is a maximum packing of K_n with 8-cycles that can be squashed into a maximum packing of K_n with 4-cycles for all $n \equiv 7$ or 15 (mod 16).* □

9. Summary

Putting together Lemmas 3.4, 4.4, 5.2, 6.1, 7.3 and 8.3 we have the following theorem (a complete solution agreeing with Tables 1 and 2 in Section 2).

Theorem 9.1. *There exists a maximum packing of K_n with 8-cycles that can be squashed into a maximmum packing of K_n with 4-cycles for every $n \geqslant 8$.* (See Tables 1 and 2 in Section 2.) □

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