Filomat 28:4 (2014), 735–743
DOI 10.2298/FIL1404735J

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Common Fixed Point Theorem in Strictly Convex Menger PM-spaces

Siniša N. Ješić[a], Rale M. Nikolić[b], Nataša A. Babačev[a]

[a]*Department of Mathematics, Faculty of Electrical Engineering, Bulevar Kralja Aleksandra 73, P.O. Box 35-54, 11120 Beograd, Serbia*
[b]*Belgrade Metropolitan University, Tadeuša Košćuška 63, 11000 Beograd, Serbia*

Abstract. In this paper we will define a notions of strictly convex and normal structure in Menger PM-space. Also, existence of a common fixed point for two self-mappings defined on strictly convex Menger PM-spaces will be proved. As a consequence of main result we will give probabilistic variant of Browder's result [3].

1. Introduction

The notion of statistical metric spaces, as a generalization of metric spaces, with non-deterministic distance, was introduced by K. Menger [14] in 1942. Schweizer and Sklar in [16] and [17] studied the properties of spaces introduced by K. Menger and gave some basic results on these spaces. They studied topology, convergence of sequences, continuity of mappings, defined the completeness of these spaces, etc.

On the other hand, fixed point theory is one of the most famous mathematical theories with application in several branches of science, especially in chaos theory, game theory, theory of differential equations etc. The first theorem of fixed point theory for non-expansive mappings was proved independently by Browder [3] and Göhde [6], and by Kirk [13] in a more general form, than form stated in [3] and [6].

The concept of normal structure was introduced in 1948 by Brodskii and Milman [2]. In 1970 Takahashi [20] has defined convex and normal structures for sets in metric spaces. General set-theoretic convexity in the study of the fixed point property of non-expansive mappings made its first explicit appearance in the work of Penot [15]. In 1987 Hadžić [7] has defined convex structure for sets in probabilistic metric spaces.

The first result from the fixed point theory in probabilistic metric spaces was obtain by Sehgal and Bharucha–Reid [18]. Hadžić [7] has proved fixed point theorem for mappings in probabilistic metric spaces with a convex structure. For more details about convexity and fixed point results for mappings defined on metric and probabilistic metric spaces see [1], [4], [5], [7], [8], [9], [10], [12], [13] and [16]. Recently, Ješić [11] has observed a wide class of non-expansive mappings defined on intuitionistic fuzzy metric spaces with convex, strictly convex and normal structure and proved existence of a fixed point for that class of non-expansive mappings in strictly convex intuitionistic fuzzy metric spaces.

2010 *Mathematics Subject Classification.* Primary 47H10; Secondary 54H25
Keywords. PM-spaces; Fixed point; Non-expansive mapping; Strictly convex structure
Received: 23 October 2013; Accepted: 25 November 2013
Communicated by Vladimir Rakočević
This paper is supported by the Ministry of Science of Republic of Serbia, Grant No. 174032.
Email addresses: jesha@eunet.rs (Siniša N. Ješić), ralevb@open.telekom.rs (Rale M. Nikolić), natasa@etf.rs (Nataša A. Babačev)

The purpose of this paper is to define a notions of strictly convex and normal structure in Menger PM-space. Also, a theorem that provides existence of a common fixed point for two self-mappings defined on strictly convex Menger PM-spaces will be proved. In the proof of the main result topological methods for characterization spaces with nondeterministic distances will be used. As a consequence of main result we will give probabilistic variant of Browder's result [3].

2. Preliminaries

In the standard notation, let D^+ be the set of all distribution functions $F : \mathbb{R} \to [0,1]$, such that F is a nondecreasing, left-continuous mapping, which satisfies $F(0) = 0$ and $\sup_{x\in\mathbb{R}} F(x) = 1$. The space D^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \le G$ if and only if $F(t) \le G(t)$ for all $t \in \mathbb{R}$. The maximal element for D^+ in this order is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0, & t \le 0, \\ 1, & t > 0. \end{cases}$$

Definition 2.1 [17] *A binary operation $T : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if T satisfies the following conditions:*

(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a,1) = a$ for all $a \in [0,1]$;
(d) $T(a,b) \le T(c,d)$ whenever $a \le c$ and $b \le d$, and $a,b,c,d \in [0,1]$.

Examples of t-norm are $T(a,b) = \min\{a,b\}$ and $T(a,b) = ab$.

The t-norms are defined recursively by $T^1 = T$ and

$$T^n(x_1,\ldots,x_{n+1}) = T(T^{n-1}(x_1,\ldots,x_n), x_{n+1})$$

for $n \ge 2$ and $x_i \in [0,1]$ for all $i \in \{1,\ldots,n+1\}$.

Definition 2.2 *A Menger probabilistic metric space (briefly, Menger PM-space) is a triple $(X,\mathcal{F},T)$ where X is a nonempty set, T is a continuous t-norm, and $\mathcal{F}$ is a mapping from $X \times X$ into D^+ such that, if $F_{x,y}$ denotes the value of $\mathcal{F}$ at the pair (x,y), the following conditions hold:*

(PM1) $F_{x,y}(t) = \varepsilon_0(t)$ if and only if $x = y$;
(PM2) $F_{x,y}(t) = F_{y,x}(t)$;
(PM3) $F_{x,z}(t+s) \ge T(F_{x,y}(t), F_{y,z}(s))$ for all $x,y,z \in X$ and $s,t \ge 0$.

Remark 2.3 [18] *Every metric space is a PM-space. Let (X,d) be a metric space and $T(a,b) = \min\{a,b\}$ is a continuous t-norm. Define $F_{x,y}(t) = \varepsilon_0(t - d(x,y))$ for all $x,y \in X$ and $t > 0$. The triple $(X,\mathcal{F},T)$ is a PM-space induced by the metric d.*

Definition 2.4 *Let $(X,\mathcal{F},T)$ be a Menger PM-space.*

(1) A sequence $\{x_n\}_n$ in X is said to be convergent to x in X if, for every $\varepsilon > 0$ and $\lambda > 0$ there exists positive integer N such that $F_{x_n,x}(\varepsilon) > 1 - \lambda$ whenever $n \ge N$.

(2) A sequence $\{x_n\}_n$ in X is called Cauchy sequence if, for every $\varepsilon > 0$ and $\lambda > 0$ there exists positive integer N such that $F_{x_n,x_m}(\varepsilon) > 1 - \lambda$ whenever $n,m \ge N$.

(3) A Menger PM-space is said to be complete if every Cauchy sequence in X is convergent to a point in X.

The (ε,λ)-topology ([17]) in a Menger PM-space $(X,\mathcal{F},T)$ is introduced by the family of neighbourhoods $\mathcal{N}_x$ of a point $x \in X$ given by

$$\mathcal{N}_x = \{N_x(\varepsilon,\lambda) : \ \varepsilon > 0, \lambda \in (0,1)\}$$

where

$$N_x(\varepsilon, \lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1 - \lambda\}.$$

The (ε, λ)-topology is a Hausdorff topology. In this topology the function f is continuous in $x_0 \in X$ if and only if for every sequence $x_n \to x_0$ it holds that $f(x_n) \to f(x_0)$.

The following Lemma is proved by B. Schweizer and A. Sklar.

Lemma 2.5 [17] *Let $(X, \mathcal{F}, T)$ be a Menger PM-space. Then the function $\mathcal{F}$ is lower semi-continuous for every fixed $t > 0$, i.e. for every fixed $t > 0$ and every two convergent sequences $\{x_n\}, \{y_n\} \subseteq X$ such that $x_n \to x, y_n \to y$ it follows that*

$$\liminf_{n\to\infty} F_{x_n,y_n}(t) = F_{x,y}(t).$$

Definition 2.6 *Let $(X, \mathcal{F}, T)$ be a Menger PM-space and $A \subseteq X$. The closure of the set A is the smallest closed set containing A, denoted by $\overline{A}$.*

Definition 2.7 *Let $(X, \mathcal{F}, T)$ be a Menger PM-space, $r \in (0, 1)$, $t > 0$ and $x \in X$. The set $N_x[\varepsilon, \lambda] = \{y \in X : F_{x,y}(\varepsilon) \geq 1 - \lambda\}$ is called closed (ε, λ)-neighbourhood of a point $x \in X$.*

Definition 2.8 *A subset K of a Menger PM-space is called compact if following statement holds*

$$K \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha \quad \Longrightarrow \quad K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i} \quad \text{for some} \quad \alpha_1, \dots, \alpha_n \in \Lambda$$

for every collection $\{U_\alpha : \alpha \in \Lambda\}$ of open sets $U_\alpha \subset X$.

Lemma 2.9 *Let $(X, \mathcal{F}, T)$ be a Menger PM-space and let $K \subseteq X$. Then, K is compact if and only if for every collection of closed sets $\{F_\alpha\}_{\alpha\in\Lambda}$ such that $F_\alpha \subseteq K$ it holds that*

$$\bigcap_{\alpha\in\Lambda} F_\alpha = \emptyset \quad \Longrightarrow \quad \bigcap_{i=1}^{n} F_{\alpha_i} = \emptyset \quad \text{for some} \quad \alpha_1, \dots, \alpha_n \in \Lambda.$$

Proof. The proof follows from Definition 2.8 and De-Morgan's laws. □

Obviously, keeping in mind the Hausdorff topology, and the definition of converging sequences we note that the next remark holds.

Remark 2.10 $x \in \overline{A}$ *if and only if there exists a sequence $\{x_n\}$ in A such that $x_n \to x$.*

The concept of probabilistic boundedness was introduced by H. Sherwood [19]. A version on this definition follows.

Definition 2.11 2 *Let $(X, \mathcal{F}, T)$ be a Menger PM-space and $A \subseteq X$. The probabilistic diameter of set A is given by*

$$\delta_A(t) = \inf_{x,y\in A} \sup_{\varepsilon< t} F_{x,y}(\varepsilon).$$

The diameter of the set A is defined by

$$\delta_A = \sup_{t>0} \inf_{x,y\in A} \sup_{\varepsilon< t} F_{x,y}(\varepsilon).$$

If there exists $\lambda \in (0,1)$ such that $\delta_A = 1-\lambda$ the set A will be called probabilistic semi-bounded. If $\delta_A = 1$ the set A will be called probabilistic bounded.

Lemma 2.12 *Let $(X, \mathcal{F}, T)$ be a Menger PM-space. A set $A \subseteq X$ is probabilistic bounded if and only if for each $\lambda \in (0,1)$ there exists $t > 0$ such that $F_{x,y}(t) > 1-\lambda$ for all $x, y \in A$.*

Proof. The proof follows from the definitions of $supA$ and $infA$ of nonempty sets. □

It is not difficult to see that every metrically bounded set is also probabilistic bounded if it is considered in the induced PM-space.

Theorem 2.13 *Every compact subset A of a Menger PM-space $(X, \mathcal{F}, T)$ is probabilistic semi-bounded.*

Proof. Let A be a compact subset of a Menger PM space X. Let fix $\varepsilon > 0$ and $\lambda \in (0,1)$. Now, we will consider an (ε, λ)-cover $\{N_x(\varepsilon, \lambda) : x \in A\}$. Since, A is compact, there exist $x_1, x_2, \ldots x_n \in A$ such that $A \subseteq \bigcup_{i=1}^n N_{x_i}(\varepsilon, \lambda)$. Let $x, y \in A$. Then exists $i \in \{1, \ldots, n\}$ such that $x \in N_{x_i}(\varepsilon, \lambda)$ and exists $j \in \{1, \ldots, n\}$ such that $y \in N_{x_j}(\varepsilon, \lambda)$. Thus we have $F_{x,x_i}(\varepsilon) > 1-\lambda$ and $F_{y,x_j}(\varepsilon) > 1-\lambda$. Now, let $m = min\{F_{x_i,x_j}(\varepsilon) : 1 \le i, j \le n\}$. It is obvious that $m > 0$ and we have

$$F_{x,y}(\varepsilon) \ge T(F_{x,x_i}(\varepsilon), F_{x_i,x_j}(\varepsilon), F_{x_j,y}(\varepsilon)) \ge T(1-\lambda, 1-\lambda, m) > 1-\delta,$$

for some $0 < \delta < 1$. If we take that $\varepsilon_1 = 3\varepsilon$ we have $F_{x,y}(\varepsilon_1) > 1-\delta$ for all $x, y \in A$. Hence, we obtain that A is probabilistic semi-bounded set. □

Remark 2.14 *In a Menger PM-space every compact set is closed and bounded.*

3. Convex structure, normal structure and strictly convex structure on Menger PM-spaces

Takahashi [20] introduced the notion of metric spaces with a convex structure. This class of metric spaces includes normed linear spaces and metric spaces of the hyperbolic type.

Definition 3.1 *Let (X, δ) be a metric space. We say that a metric space possesses a Takahashi's convex structure if there exists a function $W : X \times X \times [0,1] \to X$ which satisfies*

$$\delta(z, W(x, y, \theta)) \le \theta\delta(z, x) + (1-\theta)\delta(z, y),$$

for all $x, y, z \in X$ and arbitrary $\theta \in [0,1]$. A metric space (X, δ) with Takahashi's convex structure is called convex metric space.

Hadžić [7] introduced a generalization of the Takahashi's definition to the case of a Menger PM-space.

Definition 3.2 *Let $(X, \mathcal{F}, T)$ be a Menger PM-space. A mapping $S : X \times X \times [0,1] \to X$, is said to be a convex structure on X if for every $(x, y) \in X \times X$ holds $S(x, y, 0) = y$, $S(x, y, 1) = x$ and for all $x, y, z \in X, \theta \in (0,1)$ and $t > 0$*

$$F_{S(x,y,\theta),z}(2t) \ge T\left(F_{x,z}\left(\frac{t}{\theta}\right), F_{y,z}\left(\frac{t}{1-\theta}\right)\right). \tag{1}$$

It is easy to see that every metric space (X, d) with a convex structure S can be consider as a Menger PM-space $(X, \mathcal{F}, T_{min})$ (the associated Menger PM-space) with the same function S. For nontrivial example of a Menger PM-space with a convex structure see [7]. A Menger PM-space $(X, \mathcal{F}, T)$ with a convex structure is called a convex Menger PM-space.

Firstly, we will define the condition for a point to be diametral.

Definition 3.3 *A point $x \in A$ will be called diametral if*

$$\inf_{y\in A}\sup_{\varepsilon< t} F_{x,y}(\varepsilon) = \delta_A(t)$$

holds for all $t > 0$.

Definition 3.4 *Let $(X, \mathcal{F}, T)$ be a Menger PM-space with a convex structure $S(x, y, \theta)$. A subset $A \subseteq X$ is said to be a convex set if for every $x, y \in A$ and $\theta \in [0, 1]$ it follows that $S(x, y, \theta) \in A$.*

Lemma 3.5 *Let $(X, \mathcal{F}, T)$ be a Menger PM-space and $\{K_\alpha\}$ for $\alpha \in \Delta$ be a family of convex subsets of X. Then the intersection $K = \cap_{\alpha\in\Delta} K_\alpha$ is a convex set.*

Proof. If $x, y \in K$, then $x, y \in K_\alpha$ for every $\alpha \in \Delta$. It follows that $S(x, y, \theta) \in K_\alpha$ for every $\alpha \in \Delta$, i.e. $S(x, y, \theta) \in K$, which means that the set K is convex. □

Definition 3.6 *A convex Menger PM-space $(X, \mathcal{F}, T)$ with a convex structure $S : X \times X \times [0, 1] \to X$ will be called strictly convex if, for arbitrary $x, y \in X$ and $\theta \in (0, 1)$ the element $z = S(x, y, \theta)$ is the unique element which satisfies*

$$F_{x,y}\left(\frac{t}{\theta}\right) = F_{z,y}(t), \quad F_{x,y}\left(\frac{t}{1-\theta}\right) = F_{z,y}(t), \tag{2}$$

for all $t > 0$.

Lemma 3.7 *Let $(X, \mathcal{F}, T)$ be a Menger PM-space with a convex structure $S(x, y, \theta)$. Suppose that for every $\theta \in (0, 1)$, $t > 0$ and $x, y, z \in X$ hold*

$$F_{S(x,y,\theta),z}(t) > \min\{F_{z,x}(t), F_{z,y}(t)\}. \tag{3}$$

If there exists $z \in X$ such that

$$F_{S(x,y,\theta),z}(t) = \min\{F_{z,x}(t), F_{z,y}(t)\} \tag{4}$$

is satisfied, for all $t > 0$, then $S(x, y, \theta) \in \{x, y\}$.

Proof. Let us assume that (4) holds for some $z \in X$ and for all $t > 0$. Since (3) holds, it follows that $\theta = 0$ or $\theta = 1$ and, consequently we have that $S(x, y, 0) = y$ or $S(x, y, 1) = x$, which proves the statement of the lemma. □

Lemma 3.8 *Let $(X, \mathcal{F}, T)$ be a strictly convex Menger PM-space with a convex structure $S(x, y, \theta)$. Then for arbitrary $x, y \in X, x \neq y$ there exists $\theta \in (0, 1)$ such that $S(x, y, \theta) \notin \{x, y\}$.*

Proof. Suppose that for every $\theta \in [0, 1]$ it holds that $S(x, y, \theta) \in \{x, y\}$. From (2) it follows that $F_{x,y}(t) = 1$ for all $t > 0$ which means that $x = y$. This completes the proof. □

Definition 3.9 *A Menger PM-space $(X, \mathcal{F}, T)$ possesses a normal structure if, for every closed, probabilistic semi-bounded and convex set $Y \subset X$, which consists of at least two different points, there exists a point $x \in Y$ which is non-diametral, i.e. there exists $t_0 > 0$ such that*

$$\inf_{y\in Y}\sup_{\varepsilon< t_0} F_{x,y}(\varepsilon) > \delta_Y(t_0)$$

holds.

It is obvious that compact and convex sets in convex metric space possess a normal structure (see [20]).

Definition 3.10 *Let* $(X, \mathcal{F}, T)$ *be a convex Menger PM-space and* $Y \subseteq X$. *The closed convex shell of set* Y, *denoted by* $conv(Y)$, *is the intersection of all closed, convex sets that contain* Y.

It is easy to see that the set $conv(Y)$ exists, since the collection of closed, convex sets that contain Y is nonempty, because of the fact that X belongs to this collection. From the Lemma 3.5 it follows that this intersection is convex set. Also, this intersection is closed as an intersection of closed sets.

Definition 3.11 *Let* $(X, \mathcal{F}, T)$ *be a Menger PM-space and let* f *be a self-mapping on* X. *We say that* f *is a non-expansive mapping if*

$$F_{fx,fy}(t) \geq F_{x,y}(t) \tag{5}$$

holds for all $x, y \in X$ *and* $t > 0$.

4. Main results

Lemma 4.1 *Let* $(X, \mathcal{F}, T)$ *be a strictly convex Menger PM-space with a convex structure* $S(x, y, \theta)$ *satisfying* (3) *and let* $K \subseteq X$ *be nonempty, convex and compact subset of* X. *Then* K *possesses a normal structure.*

Proof. Suppose that K does not possess a normal structure. Then there exists a closed, probabilistic semi-bounded and convex subset $Y \subset K$, which contains at least two different points such that Y does not contain a non-diametral point i.e.

$$\inf_{y \in Y} \sup_{\varepsilon < t} F_{x,y}(\varepsilon) = \delta_Y(t)$$

for every $x \in Y$. Since X is strictly convex and condition (3) is satisfied, then the statements of Lemma 3.7 and Lemma 3.8 hold. Let x_1 and x_2 be arbitrary points in Y. From the statement of Lemma 3.8 there exists $\theta_0 \in (0, 1)$ such that $S(x_1, x_2, \theta_0) \notin \{x_1, x_2\}$. Since Y is a convex set, it follows that $S(x_1, x_2, \theta_0) \in Y$. Y is a closed subset of the compact set K, so Y is compact, too. Since $\delta_Y(t) = \inf_{y \in Y} \sup_{\varepsilon < t} F_{y, S(x_1, x_2, \theta)}(t)$ is left continuous function on the compact set Y for arbitrary $t > 0$ there exist $x_3, x_4 \in Y$ such that $\sup_{\varepsilon < t} F_{x_3, S(x_1, x_2, \theta_0)}(\varepsilon) = \delta_Y(t)$ holds. From Lemma 3.7 and the fact that $F_{x,y}(\cdot)$ is non-decreasing left continuous function it follows that

$$\begin{aligned} \delta_Y(t) &= \sup_{\varepsilon < t} F_{x_3, S(x_1, x_2, \theta_0)}(\varepsilon) = F_{x_3, S(x_1, x_2, \theta_0)}(t) \\ &> \min \{F_{x_3, x_1}(t), F_{x_3, x_2}(t)\} \\ &= \min \{\sup_{\varepsilon < t} F_{x_3, x_1}(\varepsilon), \sup_{\varepsilon < t} F_{x_3, x_2}(\varepsilon)\} \geq \delta_Y(t) \end{aligned} \tag{6}$$

From the last it follows that $\delta_Y(t) > \delta_Y(t)$ which is a contradiction. This proves the statement of the lemma. □

Lemma 4.2 *Let* $(X, \mathcal{F}, T)$ *be a convex Menger PM-space with a convex structure* $S(x, y, \theta)$ *satisfying* (3). *Then closed* (ε, λ)*-neighbourhoods* $N_x[\varepsilon, \lambda]$ *are convex sets.*

Proof. Let $a, b \in N_x[\varepsilon, \lambda]$ be arbitrary points. This implies that $F_{a,x}(\varepsilon) \geq 1 - \lambda$ and $F_{b,x}(\varepsilon) \geq 1 - \lambda$ for all $\varepsilon > 0$. We shall prove that $F_{S(a,b,\theta),x}(\varepsilon) \geq 1 - \lambda$ for all $\varepsilon > 0$, i.e. $S(a, b, \theta) \in N_x[\varepsilon, \lambda]$. Indeed, for $\theta \in (0, 1)$, from (3) we have that

$$F_{S(a,b,\theta),x}(\varepsilon) > \min \{F_{a,x}(\varepsilon), F_{b,x}(\varepsilon)\} \geq \min\{1 - \lambda, 1 - \lambda\} = 1 - \lambda.$$

For $\theta = 0$ or $\theta = 1$ it follows that $S(a, b, 0) = b$ and $S(a, b, 1) = a$ belong to $N_x[\varepsilon, \lambda]$. □

Lemma 4.3 [Zorn's lemma] *Let X be a nonempty partially ordered set in which every chain has a lower (upper) bound. Then X has a minimal (maximal) element.*

Next we shall give the main result of this paper.

Theorem 4.4 *Let $(X, \mathcal{F}, T)$ be a strictly convex Menger PM-space with a convex structure $S(x, y, \theta)$ satisfying* (3) *and let $K \subseteq X$ be a nonempty, convex and compact subset of X. Let f and g be a self mappings on K, $g(K) \cap K \subseteq f(K)$, satisfying the conditions*

$$F_{f(x),g(y)}(t) \geq F_{x,y}(t) \tag{7}$$

for all $x, y \in K$, $x \neq y$ and for every $t > 0$. Then f and g have at least one common fixed point on K.

Proof. Now, let us notice a collection Υ of all nonempty, closed, convex sets $K_\alpha \subseteq K$ such that $g(K_\alpha) \cap f(K_\alpha) \subseteq K_\alpha$. This collection is nonempty, because $K \subseteq \Upsilon$. Indeed, set K is closed set because of the fact that it is compact set in Hausdorff's space and it is satisfied that $g(K) \cap f(K) \subseteq K$. If we order this collection with inclusion, then $(\Upsilon, \subseteq)$ is a partially ordered set. Let $\{K_\alpha : \alpha \in \Lambda\}$ be an arbitrary chain of this family. Then the set $\cap_{\alpha\in\Lambda} K_\alpha$ is nonempty, closed, convex subset of K, which is a lower bound of this chain. Indeed, let us assume that $\cap_{\alpha\in\Lambda} K_\alpha = \emptyset$. Then, from Lemma 2.9 it follows that there exists a finite sub-collection $K_{\alpha_1} \supseteq \ldots \supseteq K_{\alpha_n}$ of the chain $\{K_\alpha : \alpha \in \Lambda\}$ which has an empty intersection, which is impossible, since this intersection is $K_{\alpha_n} \neq \emptyset$. From Zorn's Lemma it follows that there exists a minimal element K_0 of the collection Υ such that $g(K_0) \cap f(K_0) \subseteq K_0$. We will prove that K_0 consists of only one point and since $g \cap f : K_0 \to K_0$ this will mean that mappings g and f have a common fixed point.

Let us assume that K_0 contains at least two different points. From Lemma 4.1 it follows that K possesses a normal structure. From Theorem 2.13 it follows that K_0 is probabilistic semi-bounded set. Since K_0 is closed and convex set it follows that there exists some non-diametral point $x_0 \in K_0$, i.e. there exists $t_0 > 0$ such that the following inequality holds:

$$\inf_{y\in K_0} \sup_{\varepsilon < t_0} F_{x_0,y}(\varepsilon) \quad > \quad \delta_{K_0}(t_0) \tag{8}$$

Denote $1 - \xi := \inf_{y\in K_0} \sup_{\varepsilon < t_0} F_{x_0,y}(\varepsilon)$.

Let us denote with K_1 the closed convex shell of the set $g(K_0) \cap f(K_0)$. Since $g(K_0) \cap f(K_0) \subseteq K_0$ it holds that $K_1 = conv(g(K_0) \cap f(K_0)) = \overline{conv(g(K_0) \cap f(K_0))} \subseteq \overline{conv(K_0)} = \overline{K_0} = K_0$. Therefore, $K_1 \subseteq K_0$ and it follows that $g(K_1) \cap f(K_1) \subseteq g(K_0) \cap f(K_0) \subseteq \overline{(conv(g(K_0) \cap f(K_0))} = K_1$, i.e. $g(K_1) \cap f(K_1) \subseteq K_1$. This means that $K_1 \in \Upsilon$, and since K_0 is the minimal element we have that $K_1 = K_0$.

If inequality (8) holds, i.e. if $1 - \xi > \delta_{K_0}(t_0)$, let us define sets

$$C := \Big(\bigcap_{y\in K_0} N_y[\xi, t_0]\Big) \bigcap K_0 \qquad \text{and} \qquad C_1 := \Big(\bigcap_{y\in g(K_0)\cap f(K_0)} N_y[\xi, t_0]\Big) \bigcap K_0.$$

The set C is nonempty since $x_0 \in C$. Indeed, from inequality stated in (8) it follows that $F_{x_0,y}(t_0) \geq 1-\xi$. From the previous we conclude that x_0 belongs to $N_y[\xi, t_0]$ for all $y \in K_0$. Consequently, x_0 belongs to C. We will show that $C = C_1$. Since $g(K_0) \cap f(K_0) \subseteq K_0$ then $C_1 \supseteq C$.

Now, let $z \in C_1$. We will prove that $z \in C$. Since $z \in C_1$, for arbitrary $y \in g(K_0) \cap f(K_0)$ it holds that $F_{y,z}(t_0) \geq 1 - \xi$ i.e. $y \in N_z[\xi, t_0]$. Since y is arbitrary point from $g(K_0) \cap f(K_0)$ it follows that $g(K_0) \cap f(K_0) \subseteq N_z[\xi, t_0]$. Because of the fact that $N_z[\xi, t_0]$ is a closed and convex set which contains $g(K_0) \cap f(K_0)$, we conclude that

$$K_1 = \overline{conv(g(K_0) \cap f(K_0))} \subseteq N_z[\xi, t_0]$$

holds. Since $K_0 = K_1$ it follows that $K_0 \subseteq N_z[\xi, t_0]$. ¿From last we have that for every $y \in K_0$ it holds that $z \in N_y[\xi, t_0]$, which means that $C_1 \subseteq C$, i.e. $C = C_1$.

We will show that $C \in \Upsilon$. Set C is closed as an intersection of closed sets. From Lemma 3.5 and Lemma 4.2 it follows that C is a convex set. Let us prove that $g(C) \cap f(C) \subseteq C$. Let $z \in C$ and $y \in g(K_0) \cap f(K_0)$. Then there exists $x \in K_0$ such that $y = f(x)$ and $y = g(x)$. Applying inequality (7) for $t = t_0$, we have

$$F_{f(z),y}(t_0) = F_{f(z),g(x)}(t_0) \geq F_{z,x}(t_0) \geq 1 - \xi.$$

This means that $f(z) \in C_1$. Since z is arbitrary point from $z \in C$ we obtain $f(C) \subseteq C_1$ and because $C_1 = C$, we have that $f(C) \subseteq C$.

On the other hand, we have

$$F_{g(z),y}(t_0) = F_{g(z),f(x)}(t_0) \geq F_{z,x}(t_0) \geq 1 - \xi.$$

This means that $g(z) \in C_1$. Since z is arbitrary point from $z \in C$ we obtain $g(C) \subseteq C_1$ and because $C_1 = C$, we have that $g(C) \subseteq C$.

Finally, we obtain $g(C) \cap f(C) \subseteq C$.

Since $C \subseteq K_0$ and K_0 is the minimal element of collection Υ it follows that $C = K_0$. Now we have that $\delta_C(t_0) \geq 1 - \xi > \delta_{K_0}(t_0)$. This is a contradiction with $C = K_0$, i.e. the assumption that K_0 contains at least two different points is wrong, which means that K_0 contains only one point which is a common fixed point of the mapping g and f. This completes the proof. □

Now we shall give the probabilistic version of the main result in [11].

Theorem 4.5 *Let $(X, \mathcal{F}, T)$ be a strictly convex Menger PM-space with a convex structure $S(x, y, \theta)$ satisfying* (3) *and let $K \subseteq X$ be a nonempty, convex and compact subset of X. Let f be a non-expansive self-mapping on K. Then f has at least one fixed point on K.*

Putting in the Theorem 4.4 that $g = f$ we have that mapping f is a self-mapping on K and in this case, from conditions (7) and (8) we obtain that mapping f is non-expansive on K, and it is clear that Theorem 4.4 reduces to Theorem 4.5.

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