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Geometric Properties of a Certain General Family of Integral Operators

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Abstract. In this paper, we give a set of sufficient conditions for the univalence, starlikeness and convexity of a certain newly-defined general family of integral operators in the open unit disk. Relevant connections of the results presented here with those that were obtained in earlier works as well as several interesting corollaries and consequences of the main results are also presented.

1. Introduction, Definitions and Preliminaries

Let $\mathcal{H}(\mathbb{U})$ denote the class of functions which are *analytic* in the *open* unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Let $\mathcal{A}$ be the class of all functions $f \in \mathcal{H}(\mathbb{U})$, which are normalized by

$$f(0) = 0 \qquad \text{and} \qquad f'(0) = 1$$

and have the following form:

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots \qquad (z \in \mathbb{U}). \tag{1.1}$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are also *univalent* in $\mathbb{U}$. Robertson [19] studied the classes $\mathcal{S}^*(\lambda)$ and $\mathcal{K}(\lambda)$ of starlike and convex functions of order λ in $\mathbb{U}$, which are defined by

$$\mathcal{S}^*(\lambda) = \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > \lambda \quad (z \in \mathbb{U};\ \lambda < 1) \right\} \tag{1.2}$$

and

$$\mathcal{K}(\lambda) = \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \lambda \quad (z \in \mathbb{U};\ \lambda < 1) \right\}, \tag{1.3}$$

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respectively. In the case when $0 \leqq \lambda < 1$, a function in each of the classes $\mathcal{S}^*(\lambda)$ and $\mathcal{K}(\lambda)$ is univalent in $\mathbb{U}$. If $\lambda < 0$, then a function in the classes $\mathcal{S}^*(\lambda)$ and $\mathcal{K}(\lambda)$ may fail to be univalent in $\mathbb{U}$ (see, for details, [6] and [22]). In particular, we have

$$\mathcal{S}^*(0) =: \mathcal{S}^* \qquad \text{and} \qquad \mathcal{K}(0) =: \mathcal{K}. \tag{1.4}$$

Recently, Frasin and Jahangiri [8] studied a subclass of normalized analytic functions $f \in \mathcal{A}$,denoted by $\mathcal{B}(\mu, \nu)$ $(\mu \geqq 0;\ 0 \leqq \nu < 1)$, which satisfy the following condition:

$$\left|\left(\frac{z}{f(z)}\right)^{\mu} f'(z) - 1\right| < 1 - \nu \tag{1.5}$$

$(z \in \mathbb{U};\ \mu \geqq 0;\ 0 \leqq \nu < 1)$.

Clearly, we have

$$\mathcal{B}(1, \nu) = \mathcal{S}^*(\nu) \qquad (0 \leqq \nu < 1)$$

and, as observed earlier by Ozaki and Nunokawa [11], we can readily find that

$$\mathcal{B}(2, 0) = \mathcal{S}.$$

Moreover, the function class $\mathcal{B}(\nu)$ given by

$$\mathcal{B}(2, \nu) =: \mathcal{B}(\nu) \qquad (0 \leqq \nu < 1)$$

is the subclass of $\mathcal{A}$ which was studied by Frasin and Darus [7]. Further generalizations of the class $\mathcal{B}(\mu, \nu)$ were studied by Prajapat *et al.* (see [15] and [17]).

In recent years, many authors have determined various sets of sufficient conditions for univalence of many different families of integral operators (see, for details, [3], [4] and [5]; see also the references cited in each of these works). In the present paper, we derive some sufficient conditions for the univalence, starlikeness and convexity of a general *three-parameter* family of integral operators

$$I_{\alpha,\beta,\gamma} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$$

defined by

$$I_{\alpha,\beta,\gamma}(f, g)(z) = \left(\beta \int_0^z t^{\beta-1} \left[f'(t)\, e^{\gamma g(t)}\right]^{\alpha} dt\right)^{1/\beta} \tag{1.6}$$

$(f, g \in \mathcal{A};\ z \in \mathbb{U};\ \alpha, \beta, \gamma \in \mathbb{C};\ \beta \neq 0)$,

where the parameters $\alpha, \beta, \gamma \in \mathbb{C}$ $(\beta \neq 0)$ are so constrained that the integral in (1.6) exists. The integral operator $I_{\alpha,\beta,\gamma}$ generalizes several previously studied families of integral operators. For instance, we have

(i) $I_{\alpha,\beta,0}(f, g)(z) = \mathcal{H}_{\alpha,\beta}(z)$ (see [12]);
(ii) $I_{\beta,\beta,1}(z, g)(z) = Q_{\beta}(z)$ (see [13]);
(iii) $I_{\alpha,1,1}(f, g)(z) = I_{\alpha}(f, g)(z)$ (see [24]);
(iv) $I_{\alpha-1,\beta,1}(f, g)(z) = G_1(f, g)(z)$ (see [23]);
(v) $I_{\frac{\alpha-1}{M^2}, \frac{\beta-1}{M^2}, 1}(z, g)(z) = I_{\alpha,\beta,M}(z)$ (see [20]);
(vi) $I_{\alpha,1,0}(f, g)(z) = f_{\alpha}(z)$ (see [9] and [14]).

We further observe each of the following interesting special cases of the integral operator $I_{\alpha,\beta,\gamma}(f, g)$:

$$I_{1,1,\gamma}\left(z + \frac{\gamma}{2}z^2, z\right) = \int_0^z (1 + \gamma t)e^{\gamma t} dt = ze^{\gamma z} \qquad (z \in \mathbb{U}). \tag{1.7}$$

$$\begin{aligned} I_{1,\beta,-1}(z,z) &= [\beta\cdot\gamma(\beta,z)]^{1/\beta} \\ &= \left(\beta\int_0^z t^{\beta-1}\, e^{-t}dt\right)^{1/\beta} \\ &= \left(z-\frac{\beta}{\beta+1}\, z^2+\frac{\beta}{2!(\beta+2)}\, z^3-\frac{\beta}{3!(\beta+3)}\, z^4+\cdots\right)^{1/\beta} \qquad (z\in\mathbb{U}), \end{aligned} \tag{1.8}$$

where $\gamma(\kappa,z)$ $\big(\Re(\kappa)>0\big)$ denotes the incomplete gamma function (see, for example, [1]).

$$\begin{aligned} I_{\eta-1,\beta,0}\left(z-\frac{z^2}{2},z\right) &= \Big[\beta\cdot B_z(\beta,\eta)\Big]^{1/\beta} \\ &= \left(\beta\int_0^z t^{\beta-1}(1-t)^{\eta-1}dt\right)^{1/\beta} \\ &= z\Big[(1-z)^{\eta}\ {}_2F_1(1,\beta+\eta;\beta+1;z)\Big]^{1/\beta} \qquad (z\in\mathbb{U}), \end{aligned} \tag{1.9}$$

where $B_z(\kappa,\omega)$ $\big(\Re(\kappa)>0\big)$ denotes the incomplete Beta function (see, for example, [1]) and ${}_2F_1$ denotes the Gauss hypergeometric function (see, for example, [1] and [21]).

2. Univalence Properties of the Integral Operator $I_{\alpha,\beta,\gamma}(f,g)$

In order to investigate the conditions for univalence of the integral operator $I_{\alpha,\beta,\gamma}(f,g)$, we shall need each of the following lemmas.

Lemma 1 (see [12] and [16]). *Let $\alpha\in\mathbb{C}$ with $\Re(\alpha)>0$ and let the function $h\in\mathcal{A}$ satisfy the following condition*:

$$\left(\frac{1-|z|^{2\Re(\alpha)}}{\Re(\alpha)}\right)\left|\frac{zh''(z)}{h'(z)}\right|\leqq 1 \qquad (z\in\mathbb{U}). \tag{2.1}$$

Then, for any complex number β such that

$$\Re(\beta)\geqq\Re(\alpha),$$

the function $F_\beta(z)$ given by

$$\begin{aligned} F_\beta(z) &:= \left(\beta\int_0^z t^{\beta-1}h'(t)dt\right)^{1/\beta} \\ &= z+\frac{2a_2}{\beta+1}z^2+\left(\frac{3a_2}{\beta+1}+\frac{2\beta(1-\beta)a_2^2}{(\beta+1)^2}\right)z^3+\cdots \end{aligned} \tag{2.2}$$

is analytic and univalent in $\mathbb{U}$.

Lemma 2 (see [2]). *Let $h\in\mathcal{A}$ and $\beta\in\mathbb{C}$ with $\Re(\beta)>0$. Suppose also that, for some $\theta\in[0,2\pi]$, the following inequality holds true*:

$$\Re\left(e^{i\theta}\ \frac{zh''(z)}{h'(z)}\right)\leqq\begin{cases}\dfrac{1}{2}\Re(\beta) & \big(0<\Re(\beta)<1\big)\\[2ex] \dfrac{1}{4} & \big(\Re(\beta)\geqq 1\big).\end{cases} \tag{2.3}$$

Then the function $F_\beta(z)$ *defined by* (2.2) *is analytic and univalent in* $\mathbb{U}$ *for all* $\theta \in [0, 2\pi]$.

Theorem 1. *Let* $f \in \mathcal{A}$ *and* $g \in \mathcal{B}(\mu, \nu)$ $(\mu \geqq 0; 0 \leqq \nu < 1)$. *Suppose also that* M_1 *and* M_2 *are positive real numbers such that*

$$|g(z)| < M_1 \qquad \text{and} \qquad \left|\frac{f''(z)}{f'(z)}\right| \leqq M_2 \qquad (z \in \mathbb{U};\ M_1, M_2 > 0).$$

If

$$|\alpha| \leqq \frac{9}{2\sqrt{3}\left[M_2 + |\gamma|(2-\nu)M_1^\mu\right]},$$

then the function $I_{\alpha,\beta,\gamma}(f, g)(z)$ *is in the class* $\mathcal{S}$.

Proof. We consider the function $J(z)$ given by

$$J(z) := \int_0^z \left[f'(t)\, e^{\gamma g(t)}\right]^\alpha dt, \tag{2.4}$$

which is regular in $\mathbb{U}$. By differentiating (2.4) two times with respect to z, we get

$$\begin{aligned}(1-|z|^2)\left|\frac{zJ''(z)}{J'(z)}\right| &= (1-|z|^2)\left|\alpha z\left(\frac{f''(z)}{f'(z)} + \gamma g'(z)\right)\right| \\ &\leqq (1-|z|^2)|\alpha|\cdot|z|\left(\left|\frac{f''(z)}{f'(z)}\right| + |\gamma|\cdot|g'(z)|\right) \\ &\leqq (1-|z|^2)|\alpha|\cdot|z|\big(M_2 + |\gamma|\cdot|g'(z)|\big) \\ &\leqq (1-|z|^2)|\alpha|\big(M_2|z| + |\gamma|\cdot|zg'(z)|\big) \\ &\leqq (1-|z|^2)|\alpha|\cdot\left[M_2|z| + |\gamma|\cdot\left|g'(z)\left(\frac{z}{g(z)}\right)^\mu\right|\frac{|g(z)|^\mu}{|z|^{\mu-1}}\right].\end{aligned} \tag{2.5}$$

Since

$$g(0) = 0 \qquad \text{and} \qquad |g(z)| < M_1 \qquad (z \in \mathbb{U}),$$

by using the Schwarz Lemma, we have

$$|g(z)| \leqq M_1|z| \qquad (z \in \mathbb{U}).$$

Also, since (by hypothesis) $g \in \mathcal{B}(\mu, \nu)$, we find from (1.6) and (2.5) that

$$\begin{aligned}(1-|z|^2)\left|\frac{zJ''(z)}{J'(z)}\right| &\leqq (1-|z|^2)|\alpha|\cdot\left[M_2|z| + |\gamma|\cdot\left(\left|g'(z)\left(\frac{z}{g(z)}\right)^\mu - 1\right| + 1\right)M_1^\mu|z|\right] \\ &\leqq |z|(1-|z|^2)|\alpha|\cdot\left[M_2 + |\gamma|\cdot(2-\nu)M_1^\mu\right] \\ &\leqq \frac{2\sqrt{3}}{9}|\alpha|\cdot\left[M_2 + |\gamma|\cdot(2-\nu)M_1^\mu\right] \\ &\leqq 1 \qquad (z \in \mathbb{U}).\end{aligned} \tag{2.6}$$

Thus, by applying Lemma 1 for $\mathfrak{R}(\alpha) = 1$, it follows from (2.6) that the function $I_{\alpha,\beta,\gamma}(f, g)(z)$ is in the class $\mathcal{S}$. □

Remark 1. Various interesting corollaries and consequences of Theorem 1 can be deduced by suitably specializing the parameters α, β and γ, the parameters μ and ν, and the functions $f(z)$ and $g(z)$, in Theorem 1. In particular, if we set $\mu = 2$ and $\nu = 0$ in Theorem 1, we get Corollary 1 which, in its further special case when $\beta = \gamma = 1$, would provide an *improved* form of a known result due to Ularu and Breaz [24, p. 659, Theorem 1.1].

Corollary 1. *Let $f \in \mathcal{A}$ and $g \in \mathcal{S}$. Suppose also that M_1 and M_2 are positive real numbers such that*

$$|g(z)| < M_1 \qquad \text{and} \qquad \left|\frac{f''(z)}{f'(z)}\right| \leqq M_2 \qquad (z \in \mathbb{U};\ M_1, M_2 > 0).$$

If

$$|\alpha| \leqq \frac{9}{2\sqrt{3}(M_2 + 2|\gamma|M_1^2)},$$

then the function $I_{\alpha,\beta,\gamma}(f,g)(z)$ is in the class $\mathcal{S}$.

Remark 2. Theorem 1 and Corollary 1 can also be specialized to derive the corresponding univalence properties of the integral operator $I_{1,\beta,\frac{1}{M}}(z,g)$ defined by

$$I_{1,\beta,\frac{1}{M}}(z,g)(z) = \left(\beta \int_0^z t^{\beta-1} e^{\frac{g(t)}{M}}\, dt\right)^{1/\beta} \qquad (z \in \mathbb{U};\ g \in \mathcal{S}), \tag{2.7}$$

which was studied by Şendruţiu and Oros [20]. The details involved are fairly straightforward and are, therefore, being omitted here.

Theorem 2. *Let*

$$f \in \mathcal{A}, \quad \left|\frac{zf''(z)}{f'(z)}\right| < 1 \quad (z \in \mathbb{U}) \qquad \text{and} \qquad g \in \mathcal{B}(\mu, \nu).$$

Suppose also that M is a positive real number such that

$$|g(z)| < M \qquad (z \in \mathbb{U};\ M > 0).$$

If

$$|\alpha| \leqq \frac{27N^2}{2\left[(3N^2+1)^{3/2} + 9N^2 - 1\right]} \qquad \left(N := |\gamma|(2-\nu)M^\mu\right),$$

then the function $I_{\alpha,\beta,\gamma}(f,g)(z)$ is in the class $\mathcal{S}$.

Proof. Consider the function $J(z)$ defined by (2.4). Then, just as in the proof of Theorem 1, it is easily seen that

$$\begin{aligned}(1-|z|^2)\left|\frac{zJ''(z)}{J'(z)}\right| &= (1-|z|^2)\left|\alpha\left(\frac{zf''(z)}{f'(z)} + \gamma z g'(z)\right)\right| \\ &\leqq (1-|z|^2)|\alpha| \cdot \left[\left|\frac{zf''(z)}{f'(z)}\right| + |\gamma| \cdot \left|g'(z)\left(\frac{z}{g(z)}\right)^\mu\right| \frac{|g(z)|^\mu}{|z|^{\mu-1}}\right].\end{aligned} \tag{2.8}$$

Now, from the hypothesis and the Schwarz Lemma, we have

$$|g(z)| \leqq M|z| \qquad (z \in \mathbb{U}).$$

Therefore, we find from (2.8) that

$$\begin{aligned}(1-|z|^2)\left|\frac{zJ''(z)}{J'(z)}\right| &\leqq (1-|z|^2)|\alpha| \cdot \left[1 + |\gamma|\left(\left|g'(z)\left(\frac{z}{g(z)}\right)^\mu - 1\right| + 1\right)M^\mu|z|\right] \\ &\leqq (1-|z|^2)|\alpha| \cdot \left(1 + |\gamma| \cdot (2-\nu)M^\mu|z|\right) \qquad (z \in \mathbb{U}).\end{aligned} \tag{2.9}$$

We next consider a function $G : [0,1) \to \mathbb{R}$ given by

$$G(|z|) = (1-|z|^2)\Big[1+|\gamma|\cdot(2-\nu)M^{\mu}|z|\Big],$$

for which we have

$$G(|z|) \leqq \frac{2\left[(3N^2+1)^{3/2}+9N^2-1\right]}{27N^2} \qquad \Big(|z|\in[0,1);\ N=|\gamma|(2-\nu)M^{\mu}\Big).$$

Thus, by using the hypothesis and the equation (2.9), we conclude that

$$(1-|z|^2)\left|\frac{zJ''(z)}{J'(z)}\right| \leqq 1 \qquad (z\in\mathbb{U}). \tag{2.10}$$

Finally, by applying Lemma 1, it follows from (2.10) that the function $I_{\alpha,\beta,\gamma}(f,g)(z)$ is in the class $\mathcal{S}$. □

For $\mu = 2$ and $\nu = 0$ (or, alternatively, for $\mu = 1$ and $\nu = 0$), Theorem 2 yields the following corollary.

Corollary 2. *Let*

$$f\in\mathcal{A},\quad \left|\frac{zf''(z)}{f'(z)}\right|<1 \quad (z\in\mathbb{U}) \qquad \textit{and} \qquad g\in\mathcal{S}$$

or, alternatively, let

$$f\in\mathcal{A},\quad \left|\frac{zf''(z)}{f'(z)}\right|<1 \quad (z\in\mathbb{U}) \qquad \textit{and} \qquad g\in\mathcal{S}^*.$$

Suppose also that M is a positive real number such that

$$|g(z)|<M \qquad (z\in\mathbb{U};\ M>0).$$

If

$$|\alpha| \leqq \frac{54|\gamma|^2M^4}{\left(12|\gamma|^2M^4+1\right)^{3/2}+36|\gamma|^2M^4-1},$$

then the function $I_{\alpha,\beta,\gamma}(f,g)(z)$ *is in class* $\mathcal{S}$.

Theorem 3. *Let* $\alpha>0$ *and* $\beta\in\mathbb{C}$ *with* $\mathfrak{R}(\beta)>0$. *If the functions* $f,g\in\mathcal{A}$ *satisfy the following inequality*:

$$\mathfrak{R}\left(e^{i\theta}\left[\frac{zf''(z)}{f'(z)}+\gamma z g(z)\right]\right) \leqq \begin{cases} \dfrac{1}{2\alpha}\mathfrak{R}(\beta) & \big(0<\mathfrak{R}(\beta)<1\big) \\[2ex] \dfrac{1}{4\alpha} & \big(\mathfrak{R}(\beta)\geqq 1\big) \end{cases} \tag{2.11}$$

for all $z\in\mathbb{U}$ *and* $\theta\in[0,2\pi]$. *Then the function* $I_{\alpha,\beta,\gamma}(f,g)(z)$ *is in the class* $\mathcal{S}$.

Proof. For the function $J(z)$ given by (2.4), we find (as in the case of Theorem 1) that

$$\mathfrak{R}\left(e^{i\theta}\frac{zJ''(z)}{J'(z)}\right) = \alpha\,\mathfrak{R}\left(e^{i\theta}\left[\frac{zf''(z)}{f'(z)}+\gamma z g'(z)\right]\right) \qquad \Big(z\in\mathbb{U};\ \theta\in[0,2\pi]\Big).$$

The assertion of Theorem 3 about the univalence of the function $I_{\alpha,\beta,\gamma}(f,g)(z)$ now follows from Lemma 2. □

Remark 3. In its special case when $\gamma = \beta = 1$, Theorem 3 can at once be rewritten in a somewhat simpler form.

Theorem 4. *Let*

$$f \in \mathcal{A}, \qquad \left|\frac{zf''(z)}{f'(z)}\right| < M_1 \quad (z \in \mathbb{U};\ M_1 > 0) \qquad \text{and} \qquad g \in \mathcal{B}(\mu, \gamma).$$

Suppose also that M_2 *is a positive real number such that*

$$|zg'(z)| < M_2 \qquad (z \in \mathbb{U};\ M_2 > 0).$$

If

$$\frac{|\alpha|}{\Re(\alpha)} \leqq \frac{1}{M_1 + |\gamma| M_2},$$

then the function $I_{\alpha,\beta,\gamma}(f, g)(z)$ *is in the class* $\mathcal{S}$.

Proof. For the function $J(z)$ defined by (2.4), we have

$$\begin{aligned}
\left(\frac{1-|z|^{2\Re(\alpha)}}{\Re(\alpha)}\right)\left|\frac{zJ''(z)}{J'(z)}\right| &= \left(\frac{1-|z|^{2\Re(\alpha)}}{\Re(\alpha)}\right)\left|\alpha\left(\frac{zf''(z)}{f'(z)} + \gamma z g'(z)\right)\right| \\
&\leqq \left(\frac{1-|z|^{2\Re(\alpha)}}{\Re(\alpha)}\right)|\alpha| \cdot \left(\left|\frac{zf''(z)}{f'(z)}\right| + |\gamma| \cdot |zg'(z)|\right) \\
&\leqq \left(\frac{1-|z|^{2\Re(\alpha)}}{\Re(\alpha)}\right)|\alpha| \cdot \left(M_1 + |\gamma| M_2\right) \\
&\leqq 1 - |z|^{2\Re(\alpha)} \\
&\leqq 1 \qquad (z \in \mathbb{U}).
\end{aligned} \tag{2.12}$$

Applying Lemma 1 once again, it follows from (2.12) that the function $I_{\alpha,\beta,\gamma}(f, g)(z)$ is in class $\mathcal{S}$. □

Remark 4. Theorem 4 can easily be applied to the integral operator $\mathcal{I}_{\delta,M}h$ defined by

$$\mathcal{I}_{\delta,M}h(z) := I_{1,\frac{\delta}{M},1}\big(\log z, \log h(z)\big)(z) := \left[\frac{\delta}{M}\int_0^z t^{\frac{\delta}{M}-1}\left(\frac{h(t)}{t}\right) dt\right]^{M/\delta},$$

which was studied by Oros *et al.* [10].

3. Starlikeness Properties of the Integral Operator $I_{\alpha,\beta,\gamma}(f, g)$

We shall need the following lemma to investigate the starlikeness properties of the integral operator $I_{\alpha,\beta,\gamma}(f, g)$.

Lemma 3 (see [18]). *Let the functions* $\Theta(z)$ *and* $\Phi(z)$ *be analytic in* $\mathbb{U}$ *with*

$$\Theta(0) = \Phi(0) = 0$$

and let σ *be a real number. Suppose also that the function* $\Psi(z)$ *maps* $\mathbb{U}$ *onto a region which is starlike with respect to the origin. Then the inequality*:

$$\Re\left(\frac{\Theta'(z)}{\Phi'(z)}\right) > \sigma \qquad (z \in \mathbb{U})$$

implies that

$$\Re\left(\frac{\Theta(z)}{\Phi(z)}\right) > \sigma \qquad (z \in \mathbb{U}).$$

Theorem 5. *Let*

$$f \in \mathcal{A}, \qquad \left|\frac{zf''(z)}{f'(z)}\right| < 1 \quad (z \in \mathbb{U}) \qquad \textit{and} \qquad g \in \mathcal{B}(\mu, \nu).$$

Suppose also that M is a positive real number such that

$$|g(z)| < M \qquad (z \in \mathbb{U};\ M > 0).$$

If

$$|\alpha| \leqq \frac{|\beta|}{1 + |\gamma|(2-\nu)M^{\mu}},$$

then the function $I_{\alpha,\beta,\gamma}(f,g)(z)$ is in the class $\mathcal{S}^*$.

Proof. For the function $F(z)$ given by

$$F(z) = I_{\alpha,\beta,\gamma}(f,g)(z),$$

we have

$$\frac{zF'(z)}{F(z)} = \frac{z^{\beta}\left[f'(z)e^{\gamma g(z)}\right]^{\alpha}}{\beta\displaystyle\int_0^z t^{\beta-1}\left[f'(t)\,e^{\gamma g(t)}\right]^{\alpha} dt}. \tag{3.1}$$

By setting

$$\Theta(z) = zF'(z) \qquad \text{and} \qquad \Phi(z) = F(z),$$

we find from the equation (3.1) that

$$\frac{\Theta'(z)}{\Phi'(z)} = 1 + \frac{zF''(z)}{F'(z)} = 1 + \frac{\alpha}{\beta}\left(\frac{zf''(z)}{f'(z)} + \gamma z g'(z)\right). \tag{3.2}$$

Now, by using the hypothesis and the Schwarz Lemma in this last equation (3.2), we get

$$\begin{aligned}\left|\frac{\Theta'(z)}{\Phi'(z)} - 1\right| &\leqq \left|\frac{\alpha}{\beta}\right| \cdot \left[\left|\frac{zf''(z)}{f'(z)}\right| + |\gamma| \cdot \left|g'(z)\left(\frac{z}{g(z)}\right)^{\mu}\right| \frac{|g(z)|^{\mu}}{|z|^{\mu-1}}\right] \\ &\leqq \left|\frac{\alpha}{\beta}\right| \cdot \left[1 + \left(\left|g'(z)\left(\frac{z}{g(z)}\right)^{\mu} - 1\right| + 1\right) M^{\mu}|\gamma| \cdot |z|\right] \\ &\leqq \left|\frac{\alpha}{\beta}\right| \cdot \left[1 + |\gamma|(2-\nu)M^{\mu}\right] \\ &\leqq 1 \qquad (z \in \mathbb{U}),\end{aligned} \tag{3.3}$$

which implies that

$$\Re\left(\frac{\Theta'(z)}{\Phi'(z)}\right) > 0 \qquad (z \in \mathbb{U}).$$

Thus, by applying Lemma 3, we conclude that

$$\Re\left(\frac{\Theta(z)}{\Phi(z)}\right) = \Re\left(\frac{zF'(z)}{F(z)}\right) > 0 \qquad (z \in \mathbb{U}),$$

that is, that $I_{\alpha,\beta,\gamma}(f,g)(z) \in \mathcal{S}^*$. This evidently completes the proof of Theorem 3. □

Upon setting $\mu = 2$ and $\nu = 0$ (or, alternatively, for $\mu = 1$ and $\nu = 0$), Theorem 5 yields the following corollary.

Corollary 3. *Let*

$$f \in \mathcal{A}, \qquad \left|\frac{zf''(z)}{f'(z)}\right| < 1 \quad (z \in \mathbb{U}) \qquad \textit{and} \qquad g \in \mathcal{S}$$

or, alternatively, let

$$f \in \mathcal{A}, \qquad \left|\frac{zf''(z)}{f'(z)}\right| < 1 \quad (z \in \mathbb{U}) \qquad \textit{and} \qquad g \in \mathcal{S}^*.$$

Suppose also that M is a positive real number such that

$$|g(z)| < M \qquad (z \in \mathbb{U};\ M > 0).$$

If

$$|\alpha| \leqq \frac{|\beta|}{1 + 2|\gamma| M^2},$$

then the function $I_{\alpha,\beta,\gamma}(f, g)(z)$ *is in the class* $\mathcal{S}^*$.

4. Convexity Properties of the Integral Operator $I_{\alpha,\beta,\gamma}(f, g)$

In this section, we investigate the convexity properties of the integral operator $I_{\alpha,\beta,\gamma}(f, g)$.

Theorem 6. *Let* $f \in \mathcal{A}$ *and* $g \in \mathcal{B}(\mu, \nu)$. *Suppose also that M and N are positive real numbers such that*

$$\left|\frac{f''(z)}{f'(z)}\right| < M \qquad \textit{and} \qquad |g(z)| < N \qquad (z \in \mathbb{U};\ M > 0;\ N \geqq 1).$$

Then the function $I_{\alpha,\beta,\gamma}(f, g)(z)$ *is in the class* $\mathcal{K}(\rho)$ *for*

$$\rho = 1 - |\alpha| \cdot \left[M + |\gamma|(2-\nu)N^{\mu}\right] \qquad \textit{and} \qquad 0 < |\alpha| \cdot \left[M + |\gamma|(2-\nu)N^{\mu}\right] \leqq 1. \tag{4.1}$$

Proof. For the function $J(z)$ defined by (2.4), it is seen that

$$\begin{aligned}
\left|\frac{zJ''(z)}{J'(z)}\right| &= \left|\alpha\left(\frac{zf''(z)}{f'(z)} + \gamma z g'(z)\right)\right| \\
&\leqq |\alpha| \cdot \left[\left|\frac{f''(z)}{f'(z)}\right| + |\gamma| \cdot \left|g'(z)\left(\frac{z}{g(z)}\right)^{\mu}\right| \frac{|g(z)|^{\mu}}{|z|^{\mu-1}}\right] \qquad (z \in \mathbb{U}).
\end{aligned} \tag{4.2}$$

Now, by using the hypothesis and the Schwarz Lemma in (4.2), we get

$$\begin{aligned}
\left|\frac{zJ''(z)}{J'(z)}\right| &\leqq |\alpha| \cdot |z| \cdot \left[M + |\gamma|\left(\left|g'(z)\left(\frac{z}{g(z)}\right)^{\mu} - 1\right| + 1\right)N^{\mu}\right] \\
&\leqq |\alpha| \cdot \left[M + |\gamma|(2-\nu)N^{\mu}\right] \\
&= 1 - \rho \qquad (z \in \mathbb{U}),
\end{aligned} \tag{4.3}$$

where the parameter ρ is given by (and the parameter α is constrained as in) the equation (4.1). The inequality in (4.3) implies that the function $I_{\alpha,\beta,\gamma}(f, g)(z)$ is in the class $\mathcal{K}(\rho)$ for ρ given by (4.1). $\square$

Remark 5. For $\beta = \gamma = 1$, Theorem 6 would provide an *improved* form of a known result due to Ularu and Breaz [24, p. 660, Theorem 2.2].

Remark 6. Just as we observed in Remark 1 in connection with Theorem 1, many other interesting corollaries and consequences of Theorems 2 to 6 can also be deduced by suitably specializing the parameters α, β and γ, the parameters μ and ν, and the functions $f(z)$ and $g(z)$, in Theorems 2 to 6. The details involved are being left as an exercise for the interested reader.

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