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Stability and Boundedness to Certain Differential Equations of Fourth Order with Multiple Delays

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Abstract. In this paper, we give sufficient conditions to guarantee the asymptotic stability and boundedness of solutions to a kind of fourth-order functional differential equations with multiple delays. By using the Lyapunov-Krasovskii functional approach, we establish two new results on the stability and boundedness of solutions, which include and improve some related results in the literature.

1. Introduction

In 1956, Cartwright [6] investigated the asymptotic stability of zero solution of various linear and nonlinear differential equations of fourth order without delay. In [6], she considered the following differential equations of fourth order:

$$x^{(4)} + a_1 x''' + a_2 x'' + a_3 x' + a_4 x = 0,$$
$$x^{(4)} + a_1 x''' + a_2 x'' + a_3 x' + f(x) = 0$$

and

$$x^{(4)} + a_1 x''' + a_2 x'' + \psi(x) x' + f(x) = 0.$$

She first constructed a Lyapunov function for the linear equation with constant coefficients, such that its time derivative is negative semi-definite, as a sum of four linear squares, all with positive coefficients (given explicitly) if the characteristic roots have negative real parts. Then, she studied the asymptotic stability of zero solution of these equations by the Lyapunov 's direct method. The work of Cartwright [6] made very important scientific contributions to the qualitative theory of differential equations of higher order, and this paper may be accepted as a starting basic point for the later researches done on the topic. Later, many researchers investigated the stability and boundedness of solutions of various nonlinear differential equations of fourth order without delay, see, the book of Reissig et al. [24] as a survey and the papers of Abou-El-Ela& Sadek [1], Adesina&Ogundare [3], Burganskaja [5], Cartwright [6],Chukwu [7], Ezeilo [8-10], Ezeilo&Tejumola [11], Harrow [12], Hu [13], Kaufman & Harrow [15], Lalli & Skrapek [16-18], Lin et

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al. [19], Ogundare&Okecha [20], Ogurcov [21, 22], Skidmore [27], Skrapek&Lalli [28], Tejumola [29], Tunç [31-34, 36,37], Wu&Xiong [44] and the references cited.

However, to the best of our information, the stability and boundedness to nonlinear differential equations of fourth order with delay have only been discussed by a few authors, see, Bereketoglu[4], Abou-El-Ela et al. [2], Kang&Si [14], Okoronkwo [23], Sadek [25], Sinha [26], Tejumola [30], Zhu [45] and Tunç [35, 38–43], and the same topic for the differential equations of fourth order with multiple delays have not been investigated in the literature yet. The possible reason for this case is probably the difficulty of construction or definition of suitable Lyapunov functionals for higher order functional differential equations. This case remains as an open problem in the literature.

The purpose of this work is to address this problem for a kind of nonlinear differential equations of fourth order with multiple delays.

In this paper, we consider the fourth order nonlinear multi-delay differential equation of the form

$$\begin{aligned} &x^{(4)} + \phi(x'')x''' + \sum_{i=1}^{n} h_i(x''(t-r_i)) + \sum_{i=1}^{n} g_i(x'(t-r_i)) + \sum_{i=1}^{n} f_i(x(t-r_i)) \\ &= p(t,x,x',x'',x'''), \end{aligned} \tag{1}$$

where ϕ, h_i, g_i, f_i and p depend only on the variables displayed explicitly and r_i are positive constant delays. It is assumed as basic that the functions ϕ, h_i, g_i, f_i and p are continuous in their respective arguments and satisfy a Lipchitz condition in $x(t-r_i), x'(t-r_i), x'', x''(t-r_i)$ and x'''; $h_i(0) = g_i(0) = f_i(0) = 0$ and the derivatives $\frac{dg_i}{dx'} = g_i'(x')$, $\frac{dh_i}{dx''} = h_i'(x'')$ and $\frac{df_i}{dx} = f_i'(x)$ exist and are also continuous.

We write Eq.(1) in the system form,

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= u \\ u' &= -\phi(z)u - \sum_{i=1}^{n} h_i(z) - \sum_{i=1}^{n} g_i(y) - \sum_{i=1}^{n} f_i(x) + \sum_{i=1}^{n} \int_{t-r_i}^{t} h_i'(z(s))u(s)ds \\ &\quad + \sum_{i=1}^{n} \int_{t-r_i}^{t} g_i'(y(s))z(s)ds + \sum_{i=1}^{n} \int_{t-r_i}^{t} f_i'(x(s))y(s)ds + p(t,x,y,z,u). \end{aligned} \tag{2}$$

We use the functional Lyapunov approach to investigate the stability and boundedness of solutions of Eq. (1). This work is the first attempt and an improvement on the topic in the literature, and it contributes to the earlier relative works done on the nonlinear differential equations of fourth order with and without delay.

2. Preliminaries

We also consider the general autonomous delay differential system

$$\dot{x} = f(x_t), \quad x_t(\theta) = x(t+\theta), \quad -r \le \theta \le 0, \ \ t \ge 0. \tag{3}$$

Lemma 2.1. *(See Sinha [27].) Suppose $f(0) = 0$. Let V be a continuous functional defined on $C_H = C$ with $V(0) = 0$, and let $u(s)$ be a function, non-negative and continuous for $0 \le s < \infty$, $u(s) \to \infty$ as $s \to \infty$ with $u(0) = 0$. If for all $\phi \in C$, $u\left(\left|\phi(0)\right|\right) \le V(\phi)$, $V(\phi) \ge 0$, $\dot{V}(\phi) \le 0$, then the solution $x = 0$ of Eq.(3) is stable.*

If we define $Z = \left\{\phi \in C_H : \dot{V}(\phi) = 0\right\}$, then the solution $x = 0$ of Eq.(3) is asymptotically stable, provided that the largest invariant set in Z is $Q = \{0\}$.

3. Main Results

For convenience, we shall introduce the notations:

$$\phi_1(z) = \begin{cases} \frac{1}{z}\int_0^z \phi(\tau)d\tau, & z \neq 0 \\ \phi(0) & z = 0 \end{cases}$$

and

$$G(y) = \begin{cases} \sum\limits_{i=1}^{n} \frac{g_i(y)}{y}, & y \neq 0 \\ \sum\limits_{i=1}^{n} g_i'(0), & y = 0 \end{cases}.$$

Let $p(t, x, y, z, u) = 0$.
Our first main result is the following theorem.

Theorem 3.1. *In addition to the basic assumptions imposed on ϕ, h_i, g_i ve f_i , we assume that there are positive constants $a, b, c, d, \delta, \varepsilon$ and k_i, c_i, m_i $(i = 1, 2, ..., n)$ such that:*

i) $abc - c\sum\limits_{i=1}^{n} k_i - ad\phi(z) \geq \delta > 0$, $g_i'(y) \leq k_i$ *for all y and z;*

ii) $0 < d - \frac{a\delta}{4c} < \sum\limits_{i=1}^{n} f_i'(x) \leq \sum\limits_{i=1}^{n} c_i \leq d$, $0 < f_i'(x) \leq c_i$ *for all x;*

iii) $0 \leq G(y) - c < \frac{\delta}{8c}\sqrt{\frac{d}{2ac}}$ *for all y;*

iv) $0 \leq \sum\limits_{i=1}^{n} \frac{h_i(z)}{z} - b \leq \frac{c^3\varepsilon}{2d^2}$ *for all z $(z \neq 0)$ and* $h_i'(z) \leq m_i$, $\sum\limits_{i=1}^{n} h_i'(z) \leq \sum\limits_{i=1}^{n} m_i \leq b$ *for all z; in which* $0 < \varepsilon \leq \frac{\delta}{2acD}$, $D = ab + \frac{bc}{d}$;

v) $\phi(z) \geq a$, $\phi_1(z) - \phi(z) < \frac{\delta}{2a^2c}$ *for all z.*

If

$$r = \max_{1\leq i\leq n} r_i < \min\left\{\frac{\varepsilon c}{ab\beta + \beta d + \beta b + 2\lambda}, \frac{\delta}{2ac(ab + d + b + 2\mu)}, \frac{2a\varepsilon}{\alpha(ab + b + d) + 2\rho}\right\}$$

with $\alpha = \varepsilon + \frac{1}{a}$, $\beta = \varepsilon + \frac{d}{c}$, $\lambda = \frac{d}{2}(\alpha + \beta + 1) > 0$, $\mu = \frac{ab}{2}(\alpha + \beta + 1) > 0$, $\rho = \frac{b}{2}(\alpha + \beta + 1) > 0$,then the zero solution of the system (2) is asymtotically stable.

Remark 3.2. *Conditions (i) and (v) imply that*

$$\phi(z) < \frac{bc}{d}, \quad \sum_{i=1}^{n} k_i < ab, \ a\varepsilon \leq 1 \tag{4}$$

Let $p(t, x, y, z, u) \neq 0$.

Our second main result is the following theorem.

Theorem 3.3. *Let all the conditions of Theorem 3.1 and the assumption*

$$\left|p(t,x,y,z,u)\right| \leq q(t)$$

hold, where $\max q(t) < \infty$ *and* $q \in L^1(0,\infty), L^1(0,\infty)$ *is space of integrable Lebesgue functions.If*

$$r = \max_{1\leq i\leq n} r_i < \min\left\{\frac{\varepsilon c}{ab\beta + \beta d + \beta b + 2\lambda}, \frac{\delta}{2ac(ab+d+b+2\mu)}, \frac{2a\varepsilon}{\alpha(ab+b+d)+2\rho}\right\}$$

with $\alpha = \varepsilon + \frac{1}{a}, \beta = \varepsilon + \frac{d}{c}, \lambda = \frac{d}{2}(\alpha+\beta+1) > 0, \mu = \frac{ab}{2}(\alpha+\beta+1) > 0, \rho = \frac{b}{2}(\alpha+\beta+1) > 0,$

then there exists a finite positive constant K such that the solution $x(t)$ *of Eq. (1) defined by the initial function*

$$x(t) = \psi(t),\ x'(t) = \psi'(t),\ x''(t) = \psi''(t),\ x'''(t) = \psi'''(t), \text{where } t \in [t_0 - r, t_0],$$

satisfies

$$|x(t)| \leq K,\ |x'(t)| \leq K,\ |x''(t)| \leq K,\ |x'''(t)| \leq K$$

for all $t \geq t_0$*, where* $\psi \in C^3([t_0 - r, t_0], R)$.

Proof of Theorem 3.1 To prove the theorem, we define a Lyapunov functional,

$$\begin{aligned}
2V(x,y,z,u) &= 2\beta\int_0^x \sum_{i=1}^n f_i(\zeta)d\zeta + b\beta y^2 - \alpha d y^2 + 2\int_0^y \sum_{i=1}^n g_i(\eta)d\eta \\
&+ 2\alpha\int_0^z \sum_{i=1}^n h_i(\tau)d\tau + 2\int_0^z \phi(\tau)\tau d\tau - \beta z^2 + \alpha u^2 \\
&+ 2y\sum_{i=1}^n f_i(x) + 2\alpha z\sum_{i=1}^n f_i(x) + 2\alpha z\sum_{i=1}^n g_i(y) \\
&+ 2\beta y\int_0^z \phi(\tau)d\tau + 2\beta yu + 2zu + 2\lambda\int_{-r}^0\int_{t+s}^t y^2(\theta)d\theta ds \\
&+ 2\mu\int_{-r}^0\int_{t+s}^t z^2(\theta)d\theta ds + 2\rho\int_{-r}^0\int_{t+s}^t u^2(\theta)d\theta ds,
\end{aligned} \tag{5}$$

where $\alpha = \varepsilon + \frac{1}{a}, \beta = \varepsilon + \frac{d}{c}$ and λ, μ, ρ are positive constants, which will be determined later in the proof. It is clear that $V(0,0,0,0) = 0$. We can rearrange V in the following form:

$$2V = V_1 + V_2 + V_3 + V_4 + V_5, \tag{6}$$

where

$$
\begin{aligned}
V_1 &= 2\beta\int_0^x \sum_{i=1}^n f_i(\zeta)d\zeta - \left(\frac{1}{c}\right)\left[\sum_{i=1}^n f_i(x)\right]^2,\\
V_2 &= \left[b\beta - \alpha d - \beta^2\phi_1(z)\right]y^2 + 2\int_0^y \sum_{i=1}^n g_i(\eta)d\eta - cy^2,\\
V_3 &= \left[2\alpha\int_0^z \sum_{i=1}^n h_i(\tau)d\tau - (\beta+\alpha^2 c)z^2\right] + \left[2\int_0^z \phi(\tau)\tau d\tau - \phi_1(z)z^2\right],\\
V_4 &= \left[\alpha - \frac{1}{\phi_1(z)}\right]u^2 + 2\alpha yz\left[G(y) - c\right],\\
V_5 &= \frac{1}{c}\left[\sum_{i=1}^n f_i(x) + cy + \alpha cz\right]^2 + \frac{1}{\phi_1(z)}\left[u + \phi_1(z)z + \beta\phi_1(z)y\right]^2\\
&\quad + 2\lambda\int_{-r}^0\int_{t+s}^t y^2(\theta)d\theta ds + 2\mu\int_{-r}^0\int_{t+s}^t z^2(\theta)d\theta ds + 2\rho\int_{-r}^0\int_{t+s}^t u^2(\theta)d\theta ds.
\end{aligned}
$$

Subject to the assumptions of the theorem, it follows that

$$
\begin{aligned}
V_1 &= 2\beta\int_0^x \sum_{i=1}^n f_i(\zeta)d\zeta - \left(\frac{1}{c}\right)\left[\sum_{i=1}^n f_i(x)\right]^2\\
&= 2\left(\varepsilon + \frac{d}{c}\right)\int_0^x \sum_{i=1}^n f_i(\zeta)d\zeta - \left(\frac{1}{c}\right)\left[\sum_{i=1}^n f_i(x)\right]^2\\
&= 2\int_0^x \left[\left(\varepsilon + \frac{d}{c}\right) - \left(\frac{1}{c}\right)\sum_{i=1}^n f_i'(\zeta)\right]\sum_{i=1}^n f_i(\zeta)d\zeta \qquad (7)\\
&\geq 2\int_0^x \left[\left(\varepsilon + \frac{d}{c}\right) - \frac{d}{c}\right]\sum_{i=1}^n f_i(\zeta)d\zeta\\
&\geq 2\varepsilon\int_0^x \sum_{i=1}^n f_i(\zeta)d\zeta \geq \varepsilon\left(d - \frac{a\delta}{4c}\right)x^2.
\end{aligned}
$$

Since $G(y) = \sum_{i=1}^n \frac{g_i(y)}{y} = \sum_{i=1}^n g_i'(\sigma_i y)$, $(0 < \sigma_i < 1)$, from condition (iii) and (4), we have $c < ab$. Using the first mean value theorem for integral,we have

$$
\phi_1(z) = \frac{1}{z}\int_0^z \phi(s)ds = \phi(\gamma z), \quad 0 \leq \gamma \leq 1. \qquad (8)
$$

From conditions (i) and (iii), and the estimates (4) and (8), we have

$$
\begin{aligned}
V_2 &= \left[b\beta - \alpha d - \beta^2\phi_1(z)\right]y^2 + 2\int_0^y \sum_{i=1}^n g_i(\eta)d\eta - cy^2\\
&\geq \left[b\beta - \alpha d - \beta^2\phi(\theta_1 z)\right]y^2 \qquad (9)\\
&\geq \left(\frac{\delta d}{2ac^2}\right)y^2 \quad 0 \leq \theta_1 \leq 1.
\end{aligned}
$$

Since $\phi_1'(z) = -\left(\frac{1}{z^2}\right)\int_0^z \phi(\tau)d\tau + \frac{\phi(z)}{z}$, we know

$$\int_0^z \phi_1(\tau)\tau d\tau = z^2\phi_1(z) - \int_0^z \phi(\tau)\tau d\tau; \quad z^2\phi_1(z) = \int_0^z \phi_1(\tau)\tau d\tau + \int_0^z \phi(\tau)\tau d\tau. \tag{10}$$

From conditions (iii)-(v) and the estimate (10), we get

$$\begin{aligned} V_3 &= \left[2\alpha\int_0^z \sum_{i=1}^n h_i(\tau)d\tau - (\beta+\alpha^2c)z^2\right] + \left[2\int_0^z \phi(\tau)\tau d\tau - \phi_1(z)z^2\right] \\ &\geq \left[\alpha b - \beta - \alpha^2 c\right]z^2 + \int_0^z \left[\phi(\tau) - \phi_1(\tau)\right]\tau d\tau \geq \left(\frac{\delta}{2a^2c}\right)z^2, \end{aligned} \tag{11}$$

$$\left[\alpha - \frac{1}{\phi_1(z)}\right]u^2 = \left[\varepsilon + \frac{1}{a} - \frac{1}{\phi_1(z)}\right]u^2 \geq \left[\varepsilon + \frac{1}{a} - \frac{1}{a}\right]u^2 = \varepsilon u^2,$$

$$\left|2\alpha yz\left[G(y)-c\right]\right| \leq \frac{4}{a}\left|yz\left[G(y)-c\right]\right| \leq \frac{\delta}{2ac}\sqrt{\frac{d}{2ac}}yz \leq \frac{\delta d}{4ac^2}y^2 + \frac{\delta}{8a^2c}z^2,$$

$$V_4 \geq \varepsilon u^2 - \frac{\delta d}{4ac^2}y^2 - \frac{\delta}{8a^2c}z^2. \tag{12}$$

From the above analysis, we get

$$2V \geq \varepsilon\left(d - \frac{a\delta}{4c}\right)x^2 + \left(\frac{\delta d}{4ac^2}\right)y^2 + \varepsilon u^2 + \frac{\delta}{8a^2c}z^2 + V_5. \tag{13}$$

Furthermore, we can easily check

$$\begin{aligned} \frac{dV}{dt} &= -\left[d - \sum_{i=1}^n f_i'(x)\right]\left(y + \frac{\alpha}{2}z\right)^2 - \left[\sum_{i=1}^n \frac{h_i(z)}{z} - b\right]\left(z + \frac{\beta}{2}y\right)^2 \\ &- \left[\beta G(y) - d\right]y^2 + \frac{\beta^2}{4}\left[\sum_{i=1}^n \frac{h_i(z)}{z} - b\right]y^2 \\ &- \left[b - \alpha\sum_{i=1}^n g_i'(y) - \beta\phi_1(z)\right]z^2 \\ &+ \frac{\alpha^2}{4}\left[d - \sum_{i=1}^n f_i'(x)\right]z^2 - \left[\alpha\phi(z) - 1\right]u^2 \\ &+ (\alpha u + \beta y + z)\sum_{i=1}^n \int_{t-r_i}^t h_i'(z(s))u(s)ds + \rho r u^2 - \rho\int_{t-r}^t u^2(s)ds \\ &+ (\alpha u + \beta y + z)\sum_{i=1}^n \int_{t-r_i}^t g_i'(y(s))z(s)ds + \mu r z^2 - \mu\int_{t-r}^t z^2(s)ds \\ &+ (\alpha u + \beta y + z)\sum_{i=1}^n \int_{t-r_i}^t f_i'(x(s))y(s)ds + \lambda r y^2 - \lambda\int_{t-r}^t y^2(s)ds. \end{aligned} \tag{14}$$

By conditions (i), (iii)-(v), we have

$$\beta G(y) - d \geq c\varepsilon, \varepsilon \leq \frac{\delta}{2acD} < \frac{d}{c}, \frac{\beta^2}{4}\left[\sum_{i=1}^{n}\frac{h_i(z)}{z} - b\right] \leq \frac{c\varepsilon}{2}.$$

From conditions (i),(iii),(4) and (8), we get

$$\begin{aligned} & b - \alpha\sum_{i=1}^{n} g_i'(y) - \beta\phi_1(z) \\ & > \frac{1}{ac}\left[abc - c\sum_{i=1}^{n} g_i'(y) - ad\phi(\gamma z)\right] - ab\varepsilon - \varepsilon\phi(\gamma z) \\ & \geq \frac{\delta}{2ac}. \end{aligned} \tag{15}$$

By conditions (ii) and (v), we have

$$\frac{\alpha^2}{4}\left[d - \sum_{i=1}^{n} f_i'(x)\right] < \frac{\delta}{4ac}, \ \alpha\phi(z) - 1 \geq a\varepsilon.$$

In view of the above discussion, we get

$$\begin{aligned} \frac{dV}{dt} & \leq -\frac{c\varepsilon}{2}y^2 - \frac{\delta}{4ac}z^2 - a\varepsilon u^2 + (\alpha u + \beta y + z)\sum_{i=1}^{n}\int_{t-r_i}^{t} h_i'(z(s))u(s)ds \\ & + (\alpha u + \beta y + z)\sum_{i=1}^{n}\int_{t-r_i}^{t} f_i'(x(s))y(s)ds \\ & + (\alpha u + \beta y + z)\sum_{i=1}^{n}\int_{t-r_i}^{t} g_i'(y(s))z(s)ds \\ & + \rho r u^2 - \rho\int_{t-r}^{t} u^2(s)ds + \lambda r y^2 - \lambda\int_{t-r}^{t} y^2(s)ds + \mu r z^2 - \mu\int_{t-r}^{t} z^2(s)ds. \end{aligned} \tag{16}$$

Since $\sum_{i=1}^{n} f_i'(x) \leq \sum_{i=1}^{n} c_i \leq d$, $\sum_{i=1}^{n} g_i'(y) \leq \sum_{i=1}^{n} k_i \leq ab$, $\sum_{i=1}^{n} h_i'(z) \leq \sum_{i=1}^{n} m_i \leq b$ and $2mn \leq m^2 + n^2$, then we obtain

$$\begin{aligned} \frac{dV}{dt} & \leq -\frac{1}{2}\left[c\varepsilon - (ab\beta + \beta d + \beta b + 2\lambda)\, r\right] y^2 - \frac{1}{2}\left[\frac{\delta}{2ac} - (ab + d + b + 2\mu)\, r\right] z^2 \\ & - \left[a\varepsilon - \left(\frac{\alpha}{2}ab + \frac{\alpha}{2}b + \frac{\alpha}{2}d + \rho\right) r\right] u^2 \\ & + \left[\frac{d}{2}(\alpha + \beta + 1) - \lambda\right]\int_{t-r}^{t} y^2(s)ds \\ & + \left[\frac{ab}{2}(\alpha + \beta + 1) - \mu\right]\int_{t-r}^{t} z^2(s)ds + \left[\frac{b}{2}(\alpha + \beta + 1) - \rho\right]\int_{t-r}^{t} u^2(s)ds. \end{aligned} \tag{17}$$

Take $\lambda = \frac{d}{2}(\alpha + \beta + 1) > 0$, $\mu = \frac{ab}{2}(\alpha + \beta + 1) > 0$ and $\rho = \frac{b}{2}(\alpha + \beta + 1) > 0$ so that

$$\begin{aligned}\frac{dV}{dt} &\leq -\frac{1}{2}\left[c\varepsilon - (ab\beta + \beta d + \beta b + 2\lambda) r\right] y^2 \\ &- \frac{1}{2}\left[\frac{\delta}{2ac} - (ab + d + b + 2\mu) r\right] z^2 \\ &- \left[a\varepsilon - \left(\frac{\alpha}{2}ab + \frac{\alpha}{2}b + \frac{\alpha}{2}d + \rho\right) r\right] u^2.\end{aligned} \tag{18}$$

Therefore, if

$$r = \max_{1\leq i\leq n} r_i < \min\left\{\frac{\varepsilon c}{ab\beta + \beta d + \beta b + 2\lambda}, \frac{\delta}{2ac(ab + d + b + 2\mu)}, \frac{2a\varepsilon}{\alpha(ab + b + d) + 2\rho}\right\},$$

then we have

$$\frac{dV}{dt} \leq -\sigma\left(y^2 + z^2 + u^2\right) \text{for some } \sigma > 0. \tag{19}$$

Using $\frac{dV}{dt} = 0$ and system (2), we can easily obtain $x = y = z = u = 0$. Hence, all the conditions in Lemma 2.1. are satisfied, and so the zero solution of Eq.(1) is asymptotically stable. The proof of the theorem is now complete.

Proof of Theorem 3.3 From (13), we have

$$V \geq D_1\left(x^2 + y^2 + z^2 + u^2\right), \tag{20}$$

where $D_1 = \frac{1}{2}\min\left\{\varepsilon\left(d - \frac{a\delta}{4c}\right), \frac{\delta d}{4ac^2}, \frac{\delta}{8a^2c}, \varepsilon u^2\right\}$. Taking the time derivative of (5) with respect to t along the trajectory of (2), we obtain

$$\begin{aligned}\frac{dV}{dt} &\leq -\sigma\left(y^2 + z^2 + u^2\right) + \left|\alpha u + \beta y + z\right| \left|p(t, x(t), y(t), z(t), u(t))\right| \\ &\leq \left|\alpha u + \beta y + z\right| \left|p(t, x(t), y(t), z(t), u(t))\right| \\ &\leq D_2\left(\left|y\right| + |z| + |u|\right) q(t),\end{aligned}$$

where $D_2 = \max\{\alpha, \beta, 1\}$.

Making use of the estimate $\left|y\right| < 1 + y^2$, it is clear that

$$\frac{dV}{dt} \leq D_2\left(3 + y^2 + z^2 + u^2\right) q(t).$$

By (20), we also have

$$\left(y^2 + z^2 + u^2\right) \leq \left(x^2 + y^2 + z^2 + u^2\right) \leq D_1^{-1} V(x, y, z, u).$$

Hence

$$\begin{aligned}\frac{dV}{dt} &\leq D_2\left(3 + D_1^{-1} V(x, y, z, u)\right) q(t) \\ &= 3D_2 q(t) + D_2 D_1^{-1} V(x, y, z, u) q(t).\end{aligned}$$

Now, integrating the last inequality from 0 to t, using the assumption $q \in L^1(0,\infty)$ and Gronwall-Reid-Bellman inequality, we obtain

$$\begin{aligned} V(x,y,z,u) &\leq V(x_0,y_0,z_0,u_0) + 3D_2A + D_2D_1^{-1}\int_0^t V(x_s,y_s,z_s,u_s)q(s)ds \\ &\leq (V(x_0,y_0,z_0,u_0) + 3D_2A)\exp\left(D_2D_1^{-1}\int_0^t q(s)ds\right) \\ &\leq (V(x_0,y_0,z_0,u_0) + 3D_2A)\exp\left(D_2D_1^{-1}A\right) = K_1 < \infty, \end{aligned} \tag{21}$$

where $K_1 > 0$ is constant, $(V(x_0,y_0,z_0,u_0) + 3D_2A)\exp\left(D_2D_1^{-1}A\right) = K_1 < \infty$ and $A = \int_0^t q(s)ds$.

Now, the inequalities (20) and (21) together yields that

$$x^2(t) + y^2(t) + z^2(t) + u^2(t) \leq D_1^{-1}V(x_t,y_t,z_t,u_t) \leq K,$$

where $K^2 = K_1D_1^{-1}$. Thus, we can conclude that

$$|x(t)| \leq K,\ \left|y(t)\right| \leq K,\ |z(t)| \leq K,\ |u(t)| \leq K,$$

for all $t \geq t_0$. The proof of Theorem 3.3. is completed.

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