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Some Notes on Matrix Mappings and their Hausdorff Measure of Noncompactness

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Abstract. We consider the sequence spaces $s_\alpha^0(\tilde{B})$, $s_\alpha^{(c)}(\tilde{B})$ and $s_\alpha(\tilde{B})$ with their topological properties, and give the characterizations of the classes of matrix transformations from them into any of the spaces ℓ_1, ℓ_∞, c_0 and c. We also establish some estimates for the norms of bounded linear operators defined by those matrix transformations. Moreover, the Hausdorff measure of noncompactness is applied to give necessary and sufficient conditions for a linear operator on the sets $s_\alpha^0(\tilde{B})$, $s_\alpha^{(c)}(\tilde{B})$ and $s_\alpha(\tilde{B})$ to be compact. We also close a gap in the proof of the characterizations by various authors of matrix transformations on matrix domains.

1. Introduction, Definitions and Notations

We start by recalling the most important notations and definitions needed in this paper.

Let ω denote the set of all complex sequences $x = (x_k)_{k=0}^\infty$. We write ℓ_∞, c, c_0 and ϕ for the sets of all bounded, convergent, null and finite sequences, respectively; also bs, cs and ℓ_1 denote the sets of all bounded, convergent and absolutely convergent series. As usual, e and $e^{(n)}$ $(n = 0, 1, \dots)$ are the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$.

A subspace X of ω is said to be a BK space if it is a Banach space with continuous coordinates $P_n : X \to \mathbb{C}$ for $n = 0, 1, \dots$, where $P_n(x) = x_n$ for all $x = (x_k)_{k=0}^\infty \in X$. A BK space $X \supset \phi$ is said to have AK if every sequence $x = (x_k)_{k=0}^\infty \in X$ has a unique representation $x = \lim_{m\to\infty} x^{[m]}$, where $x^{[m]} = \sum_{n=0}^m x_n e^{(n)}$ is the m section of the sequence x. Let X be a normed space. Then $S_X = \{x \in X : \|x\| = 1\}$ and $\bar{B}_X = \{x \in X : |x\| \leq 1\}$ denote the unit sphere and closed unit ball in X.

If x and y are sequences and X and Y are subsets of ω, then we write $x \cdot y = (x_k y_k)_{k=0}^\infty$, $x^{-1} * Y = \{a \in \omega : a \cdot x \in Y\}$ and $M(X, Y) = \bigcap_{x\in X} x^{-1} * Y = \{a : \omega : a \cdot x \in Y \text{ for all } x \in X\}$ for the multiplier space of X and Y; in particular, we use the notations $x^\alpha = x^{-1} * \ell_1$, $x^\beta = x^{-1} * cs$ and $x^\gamma = x^{-1} * bs$, and $X^\alpha = M(X, \ell_1)$, $X^\beta = M(X, cs)$ and $X^\gamma = M(X, bs)$ for the α–, β– and γ–duals of X.

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Given any infinite matrix $A = (a_{nk})_{n,k=0}^{\infty}$ of complex numbers and any sequence x, we write $A_n = (a_{nk})_{k=0}^{\infty}$ for the sequence in the n^{th} row of A, $A_n x = \sum_{k=0}^{\infty} a_{nk}x_k$ $(n = 0, 1, \dots)$ and $Ax = (A_n x)_{n=0}^{\infty}$, provided $A_n \in x^{\beta}$ for all n. If X and Y are subsets of ω, then $X_A = \{x \in \omega : Ax \in X\}$ denotes the matrix domain of A in X and (X, Y) is the class of all infinite matrices that map X into Y; so $A \in (X, Y)$ if and only if $X \subset Y_A$.

Given $a \in \omega$, we write

$$\|a\|_X^* = \sup_{x \in S_X} \left| \sum_{k=0}^{\infty} a_k x_k \right|$$

provided the expression on the write hand side is defined and finite which is the case whenever X is a BK space and $a \in X^{\beta}$ ([15, Theorem 7.2.9]).

We write $\mathcal{U} = \{u \in \omega : u_k \neq 0 \text{ for all } k\}$ and $\mathcal{U}^+ = \{u \in \omega : u_k > 0 \text{ for all } k\}$; if $u \in \mathcal{U}$ then we write $1/u = (1/u_k)_{k=0}^{\infty}$.

A sequence $(b_n)_{n=0}^{\infty}$ in a linear metric space X is called a Schauder basis if for each $x \in X$ there exists a unique sequence $(\lambda_n)_{n=0}^{\infty}$ of scalars such that $x = \sum_{n=0}^{\infty} \lambda_n b_n$.

An infinite matrix $T = (t_{nk})_{n,k=0}^{\infty}$ is said to be a triangle if $t_{nk} = 0$ $(k > n)$ and $t_{nn} \neq 0$ for all n. Throughout, T denotes an arbitrary triangle and S its inverse. The well-known triangles $\Sigma = (\sigma_{nk})$, $\Delta = (\Delta_{nk})$, and $B(r, s) = \{b_{nk}(r, s)\}$ for $r, s \in \mathbb{R} \setminus \{0\}$ are defined by $\sigma_{nk} = 1$, $(0 \leq k \leq n)$, $\Delta_{nn} = 1$, $\Delta_{n-1,n} = -1$, and $b_{nn}(r, s) = r$, $b_{n-1,n}(r, s) = s$ and $\Delta_{n,k} = b_{n,k}(r, s) = 0$ otherwise.

The domain of the matrix $B(r, s)$ in the classical spaces ℓ_∞, c_0 and c has recently been studied by Kirişçi and Başar in [6]. The characterizations of compact matrix operators between some of those spaces were given by Djolović in [2]. Recently difference sequence spaces have extensively been studied, for instance in [2], [7], [6], [1], [11] and [10].

In the present study, we consider the spaces $s_\alpha^0(\tilde{B})$, $s_\alpha^{(c)}(\tilde{B})$ and $s_\alpha(\tilde{B})$ and their topological properties, Schauder bases and determine dual spaces. We also give necessary and sufficient conditions for a matrix operator to map X into Y where X is any of the sequence spaces $s_\alpha^0(\tilde{B})$, $s_\alpha^{(c)}(\tilde{B})$ and $s_\alpha(\tilde{B})$, and Y is any of the sequence spaces ℓ_1, ℓ_∞, c_0 and c. Furthermore we establish some estimates for the norms of bounded linear operators defined by the matrix transformations mentioned before. Moreover, the Hausdorff measure of noncompactness is applied to give necessary and sufficient conditions for a matrix operator to be compact.

We also observe that the characterizations of the classes (X_T, Y) given in [1] and several other papers seem to have a gap in the proof. If $z \in Z = X_T$ then $x = Tz \in X$ and

$$\sum_{k=0}^{m} a_{nk} z_k = \sum_{k=0}^{m} c_{nk}^{(m)} x_k \text{ for } m = 0, 1, \dots,$$

where, for each m,

$$c_{nk}^{(m)} = \begin{cases} \sum\limits_{j=k}^{m} a_{nj} s_{jk} & (0 \leq k \leq n) \\ 0 & (k > n) \end{cases} \quad (n = 0, 1, \dots).$$

No argument is given why the identity

$$\lim_{m \to \infty} \sum_{k=0}^{m} a_{nk} z_k = \sum_{k=0}^{\infty} \left(\lim_{m \to \infty} c_{nk}^{(m)} \right) x_k$$

should hold which the proof is based on. We will use the results of [12] to close this gap.

2. Topological Properties of the Sets $s_\alpha^0(\tilde{B})$, $s_\alpha^{(c)}(\tilde{B})$ and $s_\alpha(\tilde{B})$

The spaces $s_\alpha^0 = (1/\alpha)^{-1} * c_0$, $s_\alpha^{(c)} = (1/\alpha)^{-1} * c$ and $s_\alpha = (1/\alpha)^{-1} * \ell_\infty$, where $\alpha = (\alpha_n)_{n=0}^{\infty} \in \mathcal{U}^+$ were introduced by de Malafosse and Malkowsky in [8, 9]. Here we define the following sets for $\alpha \in \mathcal{U}^+$. Let

$\tilde{r} = (r_k)_{k=0}^{\infty}, \tilde{s} = (s_k)_{k=0}^{\infty} \in \mathcal{U}$ and $B(\tilde{r}, \tilde{s}) = (b_{nk}(\tilde{r}, \tilde{s}))_{n,k=0}^{\infty}$ be the double sequential band matrix defined by

$$b_{nk}(\tilde{r}, \tilde{s}) = \begin{cases} r_k & (k = n) \\ s_k & (k = n-1) \\ 0 & \text{otherwise} \end{cases} \quad (n = 0, 1, \dots).$$

Then we write $s_\alpha^0(\tilde{B}) = s_\alpha^0(B(\tilde{r}, \tilde{s})) = (s_\alpha^0)_{B(\tilde{r},\tilde{s})}$, $s_\alpha^{(c)}(\tilde{B}) = s_\alpha^{(c)}(B(\tilde{r}, \tilde{s})) = (s_\alpha^{(c)})_{B(\tilde{r},\tilde{s})}$ and $s_\alpha(\tilde{B}) = s_\alpha(B(\tilde{r}, \tilde{s})) = (s_\alpha)_{B(\tilde{r},\tilde{s})}$. It is clear that Δ can be obtained as a special case of $B(\tilde{r}, \tilde{s})$ for $\tilde{r} = e$ and $\tilde{s} = -e$ and it is also trivial that $B(\tilde{r}, \tilde{s})$ reduces to $B(r, s)$ in the special case $\tilde{r} = re$ and $\tilde{s} = se$. So, the results related to the domain of the matrix $B(\tilde{r}, \tilde{s})$ are more general and more comprehensive than the corresponding ones of the domains of the matrices Δ and $B(r, s)$.

Defining the diagonal matrix $D = (d_{nk})$ by $d_{nn} = 1/\alpha_n$ for $n = 0, 1, \dots$ and putting $\tilde{T} = DB(\tilde{r}, \tilde{s})$, we see that our spaces are the matrix domains of the triangle $\tilde{T} = DB(\tilde{r}, \tilde{s})$ in the sets c_0, c and ℓ_∞.

Remark 2.1. *The inverse $\tilde{S} = (\tilde{s}_{nk})$ of $B(\tilde{r}, \tilde{s}, \alpha)$ where*

$$b_{nk}(\tilde{r}, \tilde{s}, \alpha) = \begin{cases} r_n/\alpha_n & (k = n) \\ s_{n-1}/\alpha_n & (k = n-1) \\ 0 & \textit{(otherwise)} \end{cases} \quad (n = 0, 1, \dots)$$

is given by

$$\tilde{s}_{nk} = \begin{cases} \dfrac{(-1)^{n-k}\alpha_k}{r_n} \displaystyle\prod_{j=k}^{n-1} \frac{s_j}{r_j} & (0 \le k \le n) \\ 0 & (k > n) \end{cases} \quad (n = 0, 1, \dots) \tag{1}$$

Putting $\alpha_n = e$ or $r'_n = r_n/\alpha_n$ and $s'_n = s_n/\alpha_{n+1}$ $(n = 0, 1, \dots)$, $B(\tilde{r}, \tilde{s}, \alpha)$ reduces to $B(\tilde{r}, \tilde{s})$ studied in [1].

Now we define the sequence $\tau(x) = (\tau_n(x))$ as the $< \tilde{T} = DB(\tilde{r}, \tilde{s})$–transform of a sequence $x = (x_k)$, that is,

$$\tau_n(x) = \tilde{T}_n x = \frac{r_n x_n + s_{n-1} x_{n-1}}{\alpha_n} \text{ for all } n \in \mathbb{N}. \tag{2}$$

Since c_0, c and ℓ_∞ are BK spaces and $\tilde{T}$ is a triangle, the following result is an immediate consequence of [15, Theorem 4.3.12].

Theorem 2.1. *Let X be any of the spaces $s_\alpha^0(\tilde{B})$, $s_\alpha^{(c)}(\tilde{B})$ and $s_\alpha(\tilde{B})$. Then X is a BK space with the norm*

$$\|x\| = \sup \|\tau(x)\|_\infty = \sup_n \left\| \frac{r_n x_n + s_{n-1} x_{n-1}}{\alpha_n} \right\|.$$

We need the next known results for the study of Schauder bases for our spaces.

Remark 2.2. ([5, Remark 2.4]) *The matrix domain X_T of a linear metric sequence space X has a basis if and only if X has a basis.*

Lemma 2.3. ([5, Corollary 2.5]) *Let X be a BK space with AK and the sequences $c^{(n)}$ $(n = 0, 1, \dots)$ and $c^{(-1)}$ be defined by*

$$c_k^{(n)} = \begin{cases} 0 & (0 \le k < n) \\ s_{kn} & (k \ge n) \end{cases} \quad \textit{and} \quad c_k^{(-1)} = \sum_{n=0}^{k} s_{kn} \ (k = 0, 1, \dots).$$

(a) *Every sequence* $z = (z_n)_{n=0}^{\infty} \in Z = X_T$ *has a unique representation* $z = \sum_{n=0}^{\infty} (T_n z) c^{(n)}$.

(b) *Every sequence* $w = (w_n)_{n=0}^{\infty} \in W = (X \oplus e)_T$ *has a unique representation*

$$w = \xi c^{(-1)} + \sum_{n=0}^{\infty} (T_n w - \xi) c^{(n)},$$

where ξ *is the uniquely determined complex number such that* $Tw - \xi e \in X$.

We obtain as an immediate consequence of (1), Lemma 2.3 and Remark 2.2.

Proposition 2.4. *Let us define the sequences* $c^{(n)}$ *for* $n = 0, 1, \dots$ *and* $c^{(-1)}$ *by*

$$c_k^{(n)} = \begin{cases} 0 & (0 \le k < n) \\ \dfrac{(-1)^{k-n}\alpha_n}{r_k} \prod\limits_{j=n}^{k-1} \dfrac{s_j}{r_j} & (k \ge n) \end{cases}$$

and

$$c_k^{(-1)} = \sum_{n=0}^{k} \frac{(-1)^{k-n}\alpha_n}{r_k} \prod_{j=n}^{k-1} \frac{s_j}{r_j} \text{ for } k = 0, 1, \dots.$$

(a) *Then* $(c^{(n)})_{n=0}^{\infty}$ *is a Schauder basis for* $s_\alpha^0(\tilde{B})$, *and every sequence* $x = (x_n)_{n=0}^{\infty} \in s_\alpha^0(\tilde{B})$ *has a unique representation*

$$x = \sum_{n=0}^{\infty} \tau_n(x) \cdot c^{(n)} \text{ with } \tau_n(x) \text{ from (2) for all } n.$$

(b) *Then* $(c^{(n)})_{n=-1}^{\infty}$ *is a Schauder basis for* $s_\alpha^{(c)}(\tilde{B})$, *and every sequence* $x = (x_n)_{n=0}^{\infty} \in s_\alpha^{(c)}(\tilde{B})$ *has a unique representation*

$$x = \xi c^{(-1)} + \sum_{n=0}^{\infty} (\tau_n(x) - \xi) c^{(n)}, \text{ where } \xi = \lim_{n\to\infty} \tau_n(x).$$

(c) *The space* $s_\alpha(\tilde{B})$ *has no Schauder basis.*

3. The Alpha-, Beta- and Gamma-Duals of the Spaces $s_\alpha^0(\tilde{B})$, $s_\alpha^{(c)}(\tilde{B})$ and $s_\alpha(\tilde{B})$

In this subsection, we determine the α–, β– and γ–duals of the sets $s_\alpha^0(\tilde{B})$, $s_\alpha^{(c)}(\tilde{B})$ and $s_\alpha(\tilde{B})$.

We start with an almost trivial result which underlies the determination of the α–, β– and γ–duals of matrix domains of triangles.

Proposition 3.1. *Let* X *and* Y *be arbitrary subsets of* ω. *Then we have* $a \in M(X_T, Y)$ *if and only if* $C \in (X, Y)$ *where* C *is the matrix with the rows* $C_n = a_n S_n$ *for* $n = 0, 1, \dots$.

Proof. We write $Z = X_T$ and observe that $z \in Z$ if and only if $x = Tz \in X$. Since $a_n z_n = a_n S_n x = C_n x$ for all n, we have $a \cdot z \in Y$ for all $z \in Z$ if and only if $Cx \in Y$ for all $x \in X$, that is, $C \in (X, Y)$. □

The next result is an immediate consequence of Proposition 3.1 and the well–known characterizations of the classes (X, Y) for $X = c_0, c, \ell_\infty$ and and $Y = \ell_1, cs, bs$.

Corollary 3.2. *We have*

(a) $((c_0)_T)^\alpha = C_T^\alpha = ((\ell_\infty)_T)^\alpha$, *and* $a \in ((c_0)_T)^\alpha$ *if and only if*

$$\sup_{\substack{K \subset \mathbb{N} \\ K \text{ finite}}} \sum_{n=0}^{\infty} \left| \sum_{k \in K} c_{nk} \right| < \infty; \tag{3}$$

(b) *(i)* $a \in ((c_0)_T)^\beta$ *if and only if*

$$\sup_m \sum_{k=0}^{m} \left| \sum_{n=k}^{m} c_{nk} \right|, \infty \tag{4}$$

and

$$\sum_{n=k}^{\infty} c_{nk} = \gamma_k \text{ exists for each } k; \tag{5}$$

(ii) $a \in c_T^\beta$ *if and only if the conditions in (4) and (5) hold and*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} c_{nk} = \gamma \text{ exists;} \tag{6}$$

(iii) $a \in ((\ell_\infty)_T)^\beta$ *if and only if*

$$\lim_{n \to \infty} \sum_{k=0}^{n} \left| \sum_{j=n}^{\infty} c_{jk} \right| = 0; \tag{7}$$

(c) $((c_0)_T)^\gamma = C_T^\gamma = ((\ell_\infty)_T)^\gamma$, *and* $a \in ((c_0)_T)^\gamma$ *if and only if the condition in (4) holds.*

Proof. We apply Proposition 3.1 in each case.

(a) By [15, Example 8.4.9A], we have $(c_0, \ell_1) = (c, \ell_1) = (\ell_\infty, \ell_1)$, and $C \in (c_0, \ell_1)$ by [15, Example 8.4.3B] if and only if (3) is satisfied.

(c) By [15, Example 8.4.6A], we have $(c_0, bs) = (c, bs) = (\ell_\infty, bs)$, and $C \in (c_0, bs)$ if and only if (4) is satisfied.

(b) (i) By [15, Example 8.4.6A], we have (c_0, cs), if and only if $C \in c_0, bs)$ and (5) is satisfied.

(ii) By [15, Example 8.4.6A], we have (c, cs), if and only if $C \in c_0, cs)$ and (6) is satisfied.

(iii) By [15, Example 8.5.8], we have $C \in (\ell_\infty, cs)$ if and only if (7) is satisfied.

□

Remark 3.3. *(a) Since obviously* ([12, Theorem 3.8 (a)]) $C \in (X, cs)$ *if and only if* $D = \Sigma \cdot C \in (X, c)$, *we obtain from [15, Theorem 1.7.18 (iii)]* $C \in (X, cs)$ *if and only if*

$$\delta_k = \lim_{n \to \infty} d_{nk} = \sum_{j=k}^{\infty} c_{jk} \text{ exists for each } k \tag{8}$$

and

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} |d_{nk}| = \lim_{n \to \infty} \sum_{k=0}^{\infty} \left| \sum_{j=k}^{n} c_{jk} \right| = \sum_{k=0}^{\infty} |\delta_k| = \sum_{k=0}^{\infty} \left| \sum_{j=k}^{\infty} c_{jk} \right|. \tag{9}$$

(b) If we replace the matrix C by the matrix $\tilde{C}$ with the rows $C_n = a_n\tilde{S}_n$ $(n = 0, 1, \dots)$, where the entries of the matrix $\tilde{S}$ are given by (1), Corollary 3.2 yields the α–, β– and γ–duals of the sets $s_\alpha^0(\tilde{B})$, $s_\alpha^{(c)}(\tilde{B})$ and $s_\alpha(\tilde{B})$.
(c) In particular, if we put $\alpha_k = 1$ for $k = 0, 1, \dots$, then Corollary 3.2 (b) and (c) and (8) and (9) yield the results in [1, Corollary 3.4 (vi), (v), (iv), (i)].

We are going to use the results of [12] to close the gap in the proofs mentioned at the end of our introduction.

Lemma 3.4. ([13, Theorem 3.2, Remark 3.3 (a), (b)]) *Let T be an arbitrary triangle, S be its inverse and $R = S^t$, the transpose of S.*

(a) Let X be a BK space with AK or $X = \ell_\infty$. Then $a \in (X_T)^\beta$ if and only if $a \in (X^\beta)_R$ and $W \in (X, c_0)$ where the triangle W is defined by

$$w_{nk} = \begin{cases} 0 & (k > n) \\ \sum_{j=n}^{\infty} a_j s_{jk} & (0 \le k \le n) \end{cases} \quad \text{for } n = 0, 1, 2, \dots. \tag{10}$$

Moreover, if $a \in (X_T)^\beta$ then

$$\sum_{k=0}^{\infty} a_k z_k = \sum_{k=0}^{\infty} (R_k a)(T_k z) \text{ for all } z \in X_T. \tag{11}$$

(b) We have $a \in (c_T)^\beta$ if and only if $a \in (\ell_1)_R$ and $W \in (c, c)$. Moreover, if $a \in (c_T)^\beta$ then we have

$$\sum_{k=0}^{\infty} a_k z_k = \sum_{k=0}^{\infty} (R_k a)(T_k z) - \eta\gamma \text{ for all } z \in c_T$$

$$\text{where } \eta = \lim_k T_k z \text{ and } \gamma = \lim_n \sum_{k=0}^{n} w_{nk}. \tag{12}$$

Remark 3.5. *(a) If X is a BK space with AK then the condition $W \in (X, c_0)$ in Lemma 3.4 (a) can be replaced by*

$$W \in (X, \ell_\infty). \tag{13}$$

(b) The condition $W \in (c, c)$ in Lemma 3.4 (b) can be replaced by the conditions

$$W(c_0, \ell_\infty) \tag{14}$$

and

$$\lim_{n\to\infty} W_n e = \gamma \text{ exists.} \tag{15}$$

Proof. We have by the definition of the matrix W

$$\lim_{n\to\infty} W_n e^{(k)} = \lim_{n\to\infty} \sum_{j=n}^{\infty} a_j s_{jk} = 0 \text{ for each } k. \tag{16}$$

(a) Obviously $W \in (X, c_0)$ implies (13), and conversely, since X has AK and c_0 is a closed subspace of ℓ_∞, (13) and (16) imply $W \in (X, c_0)$ by [15, 8.3.6].

(b) Obviously $W \in (c, c)$ implies (14) and $We \in c$, that is, (15). Conversely, since c_0 has AK and c is a closed subspace of ℓ_∞, (14) and (16) imply $W \in (c_0, c)$ by [15, 8.3.6], and then $W \in (c_0, c)$ and (15) imply $W \in (c, c)$ by [15, 8.3.7]. □

Now we give an alternative representation of the β–duals of the sets $s_\alpha^0(\tilde{B})$, $s_\alpha^{(c)}(\tilde{B})$ and $s_\alpha(\tilde{B})$.

Theorem 3.1. *We have*

(a) $a \in (s_\alpha^0(\tilde{B}))^\beta$ *if and only if*

$$\sum_{k=0}^{\infty}\left|\sum_{j=k}^{\infty}\frac{(-1)^{j-k}\alpha_k}{r_j}\prod_{l=k}^{j-1}\frac{s_l}{r_l}a_j\right| < \infty \tag{17}$$

and

$$\sup_n\left(\sum_{k=0}^{n}\left|\sum_{j=n}^{\infty}\frac{(-1)^{j-k}\alpha_k}{r_j}\prod_{l=k}^{j-1}\frac{s_l}{r_l}a_j\right|\right) < \infty; \tag{18}$$

moreover, if $a \in (s_\alpha^0(\tilde{B}))^\beta$ *then*

$$\sum_{k=0}^{\infty}a_kx_k = \sum_{k=0}^{\infty}\left(\sum_{j=k}^{\infty}\frac{(-1)^{j-k}\alpha_k}{r_j}\prod_{l=k}^{j-1}\frac{s_l}{r_l}a_j\right)\cdot\tau_k(x) \text{ for all } x \in s_\alpha^0(\tilde{B})); \tag{19}$$

(b) $a \in (s_\alpha^{(c)}(\tilde{B}))^\beta$ *if and only if (17), (18) and*

$$\lim_{n\to\infty}\sum_{k=0}^{n}\left(\sum_{j=n}^{\infty}\frac{(-1)^{j-k}\alpha_k}{r_j}\prod_{l=k}^{j-1}\frac{s_l}{r_l}a_j\right) = \zeta; \tag{20}$$

moreover, if $a \in (s_\alpha^{(c)}(\tilde{B}))^\beta$, *then*

$$\sum_{k=0}^{\infty}a_kx_k = \sum_{k=0}^{\infty}\left(\sum_{j=k}^{\infty}\frac{(-1)^{j-k}\alpha_k}{r_j}\prod_{l=k}^{j-1}\frac{s_l}{r_l}a_j\right)\cdot\tau_k(x) - \eta\zeta \text{ for all } x \in s_\alpha^{(c)}(\tilde{B}), \tag{21}$$

where $\eta = \lim_{k\to\infty}\tau_k(x)$;

(c) $a \in (s_\alpha(\tilde{B}))^\beta$ *if and only if (17) and*

$$\lim_{n\to\infty}\sum_{k=0}^{n}\left|\sum_{j=k}^{\infty}\frac{(-1)^{j-k}\alpha_k}{r_j}\prod_{l=k}^{j-1}\frac{s_l}{r_l}a_j\right| = 0; \tag{22}$$

moreover, if $a \in (s_\alpha(\tilde{B}))^\beta$ *then (19) holds for all* $x \in s_\alpha(\tilde{B})$.

Proof. We apply Lemma 3.4 and Remark 3.5.
First we note that if $\tilde{R}$ denotes the transpose of the matrix $\tilde{S}$ of (1) then

$$\tilde{a}_k = \tilde{R}_k a = \sum_{j=k}^{\infty}\tilde{s}_{jk}a_j = \sum_{j=k}^{\infty}\frac{(-1)^{j-k}\alpha_k}{r_j}\prod_{l=k}^{j-1}\frac{s_l}{r_l}a_j \text{ for } k = 0, 1, \dots; \tag{23}$$

and the triangle W is given by

$$w_{nk} = \sum_{j=n}^{\infty}\tilde{s}_{jk}a_j = \sum_{j=n}^{\infty}\frac{(-1)^{j-k}\alpha_k}{r_j}\prod_{l=k}^{j-1}\frac{s_l}{r_l}a_j \text{ for } 0 \le k \le n \text{ and } n = 0, 1, \dots. \tag{24}$$

Since $c_0^\beta = c^\beta = \ell_\infty^\beta$, the condition $\tilde{R}a \in \ell_1$ of Lemma 3.4 yields (17) in each part.
By Remark 3.5 (a) and Lemma 3.4, we have to add the condition $W \in (c_0, \ell_\infty)$ which, by [15, Theorem 1.3.3], is equivalent to $\sup_n \sum_{k=0}^\infty |w_{mk}| < \infty$, and this is the condition in (18).
By Remark 3.5 (b) and Lemma 3.4, we have to add the condition in (15) to (18), and (18) is the condition in (20).
By Lemma 3.4, we have to add the condition $W \in (\ell_\infty, c_0)$ which, by [14, **(21)** (21.1)], is equivalent to $\lim_{n\to\infty} \sum_{k=0}^\infty |w_{nk}| = 0$, and this is the condition in (22).
Finally, (23) in Parts (a) and (c) and (24) come from (11) and (12), respectively. □

4. Matrix Mappings and their Operator Norms

In this section, we characterize various classes of matrix mappings on the spaces $s_\alpha^0(\tilde{B})$, $s_\alpha^{(c)}(\tilde{B})$ and $s_\alpha(\tilde{B})$, and establish identities and inequalities for their operator norms.

The following results are needed to characterize certain classes of matrix mappings on the spaces $s_\alpha^0(\tilde{B})$ and $s_\alpha^{(c)}(\tilde{B})$, and to determine the norms of bounded linear operators on our spaces.

Lemma 4.1. *Let X and Y be BK spaces.*

(a) Then we have $(X, Y) \subset \mathcal{B}(X, Y)$, that is, every $A \in (X, Y)$ defines an operator $L_A \in \mathcal{B}(X, Y)$, where $L_A(x) = Ax$ for all $x \in X$ ([12, Theorem 1.23]).

(b) If X has AK then we have $\mathcal{B}(X, Y) \subset (X, Y)$, that is, every operator $L \in \mathcal{B}(X, Y)$ is given by a matrix $A \in (X, Y)$ such that $L(x) = Ax$ for all $x \in X$ ([5, Theorem 1.9]).

Lemma 4.2. *Let Y be an arbitrary subset of ω.*

(a) Let X be a BK space with AK or $X = \ell_\infty$, and $R = S^t$. Then $A \in (X_T, Y)$ if and only if $\hat{A} \in (X, Y)$ and $W^{(A_n)} \in (X, c_0)$ for all $n = 0, 1, \ldots$. Here $\hat{A}$ is the matrix with rows $\hat{A}_n = RA_n$ for $n = 0, 1, \ldots$, and the triangles $W^{(A_n)}$ $(n = 0, 1, \ldots)$ are defined as in (29) with a_j replaced by a_{nj}. Moreover, if $A \in (X, Y)$ then we have $Az = \hat{A}(Tz)$ for all $z \in Z = X_T$ ([13, Theorem 3.4, Remark 3.5 (a)]).

(b) We have $A \in (c_T, Y)$ if and only if $\hat{A} \in (c_0, Y)$ and $W^{(A_n)} \in (c, c)$ for all $n = 0, 1, \ldots$ and $\hat{A}e - (\gamma_n)_{n=0}^\infty \in Y$, where $\gamma_n = \lim_{m\to\infty} \sum_{k=0}^m w_{mk}^{(A_n)}$ for $n = 0, 1, \ldots$. Moreover, if $A \in (c_T, Y)$ then we have $Az = \hat{A}(Tz) - \eta(\gamma_n)_{n=0}^\infty$ for all $z \in c_T$, where $\eta = \lim_{k\to\infty} T_k z$ ([13, Remark 3.5 (b)]).

Remark 4.3. *(a) If X is a BK space with AK then, by Remark 3.5 (a), the condition $W^{(A_n)} \in (X, c_0)$ in Lemma 4.2 can be replaced by*

$$W^{(A^n)} \in (X, \ell_\infty). \tag{25}$$

(b) By Remark 3.5 (b), the condition $W^{(A_n)} \in (c, c)$ can be replaced by

$$W^{(A^n)} \in (c_0, \ell_\infty). \tag{26}$$

and

$$\lim_{m\to\infty} W_m^{(A_n)} e = \gamma_n \text{ existst for each } n. \tag{27}$$

Lemma 4.4. ([3, Theorem 2.8]) *Let $X = c_0$ or $X = \ell_\infty$.*

(a) Let $Y = c_0, c, \ell_\infty$. If $A \in (X_T, Y)$ then, putting

$$\|A\|_{(X_T,\infty)} = \sup_n \left\|\hat{A}_n\right\|_1 = \sup_n \sum_{k=0}^\infty |\hat{a}_{nk}|,$$

we have $\|L_A\| = \|A\|_{(X_T,\infty)}$.

(b) *Let $Y = \ell_1$. If $A \in (X_T, \ell_1)$, then, putting*

$$\|A\|_{(X_T,1)} = \sup_{\substack{N \subset \mathbb{N} \\ N \text{ finite}}} \left(\sum_{k=0}^{\infty} \left| \sum_{n \in N} \hat{a}_{nk} \right| \right),$$

we have $\|A\|_{(X_T,1)} \leq \|L_A\| \leq 4 \cdot \|A\|_{(X_T,1)}$.

Lemma 4.5. ([3, Theorem 2.9])

(a) *Let $A \in (c_T, Y)$, where Yis any of the spaces c_0 ,c or ℓ_∞. Then we have*

$$\|L_A\| = \|A\|_{(c_T,\infty)} = \sup_n \left(\sum_{k=0}^{\infty} |\hat{a}_{nk}| + |\gamma_n| \right).$$

(b) *Let $A \in (c_T, \ell_1)$. Then we have*

$$\|A\|_{(c_T,1)} = \sup_{\substack{N \subset \mathbb{N} \\ N \text{ finite}}} \left(\sum_{k=0}^{\infty} \left| \sum_{n \in N} \hat{a}_{nk} \right| + \left| \sum_{n \in N} \gamma_n \right| \right) \leq \|L_A\| \leq 4 \cdot \|A\|_{(c_T,1)}.$$

Now we characterize of the classes of matrix mappings from the spaces $s_\alpha^0(\tilde{B})$, $s_\alpha^{(c)}(\tilde{B})$ and $s_\alpha(\tilde{B})$ into ℓ_∞, c, c_0 and ℓ_1.

Given a matrix $A = (a_{nk})_{n,k=0}^{\infty}$, we define the matrices $\tilde{A} = (\tilde{a}_{nk})_{n,k=0}^{\infty}$ and $W^{(A_n)} = (w_{mk}^{(A_n)})_{m,k=0}^{\infty}$ by

$$\tilde{a}_{nk} = \sum_{j=k}^{\infty} \frac{(-1)^{j-k} a_{nj} \alpha_k}{r_j} \prod_{l=k}^{j-1} \frac{s_l}{r_l} \text{ for all } n, k \in \mathbb{N}_0, \tag{28}$$

and

$$w_{mk}^{(A_n)} = \begin{cases} \sum\limits_{j=m}^{\infty} (-1)^{j-k} \dfrac{a_{nj}\alpha_k}{r_j} \prod\limits_{l=k}^{j-1} \dfrac{s_l}{r_l} & (0 \leq k \leq m) \\ 0 & (k > m) \end{cases} \quad (n = 0, 1, \dots). \tag{29}$$

Theorem 4.1. *Let $X = s_\alpha^0(\tilde{B}), s_\alpha^{(c)}(\tilde{B}), s_\alpha(\tilde{B})$ and $Y = \ell_\infty, c_0, c, \ell_1$. Then the necessary and sufficient conditions for $A \in (X, Y)$ can be read from the following table:*

From / *To*	$s_\alpha(\tilde{B})$	$s_\alpha^0(\tilde{B})$	$s_\alpha^{(c)}(\tilde{B})$
ℓ_∞	**1.**	**2.**	**3.**
c_0	**4.**	**5.**	**6.**
c	**7.**	**8.**	**9.**
ℓ_1	**10.**	**11.**	**12.**

where

1. $(1.1)^*$ *and* $(1.2)^*$ *where*
$(1.1)^*$ $\|A\|_{(s_\alpha(\tilde{B}),\infty)} = \sup_n \sum_{k=0}^{\infty} |\tilde{a}_{nk}| < \infty$ *and*
$(1.2)^*$ $\lim_{m\to\infty} \sum_{k=0}^{m} |w_{mk}^{(A_n)}| = 0$ *for all* n;

2. $(1.1)^*$ *and* $(2.1)^*$ *where*
$(2.1)^*$ $\|W^{(A_n)}\|_{(\ell_\infty,\ell_\infty)} = \sup_m \sum_{k=0}^{m} |w_{mk}^{(A_n)}| < \infty$ *for all* n;

3. $(1.1)^*$, $(2.1)^*$, $(3.1)^*$ *and* $(3.2)^*$ *where*
$(3.1)^*$ $\lim_{m\to\infty} \sum_{k=0}^{m} w_{mk}^{(A_n)} = \gamma_n$ *exists for each* n,
$(3.2)^*$ $\sup_n |\sum_{k=0}^{\infty} \tilde{a}_{nk} - \gamma_n| = 0$;

4. $(1.2)^*$ *and* $(4.1)^*$ *where*
$(4.1)^*$ $\lim_{m\to\infty} \sum_{k=0}^{m} |\tilde{a}_{nk}| = 0$;

5. $(1.1)^*$, $(2.1)^*$ *and* $(5.1)^*$ *where*
$(5.1)^*$ $\lim_{n\to\infty} \tilde{a}_{nk} = 0$ *for each* k;

6. $(1.1)^*$, $(2.1)^*$, $(3.1)^*$, $(5.1)^*$ *and* $(6.1)^*$ *where*
$(6.1)^*$ $\lim_{n\to\infty}(\sum_{k=0}^{\infty} \tilde{a}_{nk} - \gamma_n) = 0$;

7. $(1.2)^*$, $(7.1)^*$, $(7.2)^*$ *and* $(7.3)^*$ *where*
$(7.1)^*$ $\lim\limits_{n\to\infty} \tilde{a}_{nk} = \tilde{\alpha}_k$ *exists for each* n,
$(7.2)^*$ $\sum_{k=0}^{\infty} |\tilde{a}_{nk}|, \quad \sum_{k=0}^{\infty} |\tilde{\alpha}_k| < \infty$ *for all* n,
$(7.3)^*$ $\lim_{n\to\infty}(\sum\limits_{k=0}^{\infty} \tilde{a}_{nk} - \tilde{\alpha}_k) = 0$;

8. $(1.1)^*$, $(2.1)^*$ *and* $(7.1)^*$;

9. $(1.1)^*$, $(2.1)^*$, $(3.1)^*$, $(7.1)^*$ *and* $(9.1)^*$ *where*
$(9.1)^*$ $\lim_{n\to\infty}(\sum_{k=0}^{\infty} \tilde{a}_{nk} - \gamma_n) = \delta$ *exists*;

10. $(1.2)^*$ *and* $(10.1)^*$ *where*
$(10.1)^*$ $\sup_{K\subset\mathbb{N}\ \textit{finite}} \sum_{n=0}^{\infty} |\sum_{k\in K} \tilde{a}_{nk}| < \infty$;

11. $(2.1)^*$ *and* $(10.1)^*$;

12. $(2.1)^*$, $(3.1)^*$, $(10.1)^*$ *and* $(12.1)^*$ *where*
$(12.1)^*$ $\sum_{n=0}^{\infty} |\sum_{k=0}^{\infty} \tilde{a}_{nk} - \gamma_n| < \infty$.

Proof. Let Y be one of the spaces $\ell_\infty, c_0, c, \ell_\infty$, and $\tilde{A}$ and $W^{(A_n)}$ for $n = 0, 1, \dots$ be the matrices defined in (28) and (29)

1., **4.**, **7.**, **10.** Lemma 4.2 (a) yields $A \in (s_\alpha(\tilde{B}), Y)$ if and only if $\tilde{A} \in (\ell_\infty, Y)$ and $W^{(A_n)} \in (\ell_\infty, c_0)$ for all n. First $\tilde{A} \in (\ell_\infty, Y)$ yields $(1.1)^*$ in **1.** by [15, Theorem 1.3.3], $(4.1)^*$ in **4.** by [14, bf 21. (21.1)], $(7.1)^*$, $(7.2)^*$ and $(7.3)^*$ in **7.** by [15, Theorem 1.7.18 (ii)], and $(10.1)^*$ in **10.** by [15, 8.4.9A]. Also $W^{(A_n)} \in (\ell_\infty, c_0)$ for all n yields $(1.2)^*$ in **1.**, **4.**, **7.**, **10.** by [14, bf 21. (21.1)].

2., **5.**, **8.**, **11.** Lemma 4.2 (a) and Remark 4.3 (a) yield $A \in (s_\alpha^{(0)}(\tilde{B}), Y)$ if and only if $\tilde{A} \in (c_0, Y)$ and $W^{(A_n)} \in (c_0, \ell_\infty)$ for all n. First $\tilde{A} \in (c_0, Y)$ yields $(1.1)^*$ in **2.** by [15, Theorem 1.3.3], $(1.1)^*$ and $(5.1)^*$ in **5.** by [15, 8.4.5A], $(1.1)^*$ and $(7.1)^*$ in **8.** by [15, 8.4.5A] and $(10.1)^*$ in **11.** by [15, 8.4.3B]. Also $W^{(A_n)} \in (c_0, \ell_\infty)$ for all n yields $(2.1)^*$ in **2.**, **5.**, **8.**, **11.** by [15, Theorem 1.3.3].

3., **6.**, **9.**, **12.** Lemma 4.2 (a) and Remark 4.3 (a) yield $A \in (s_\alpha^{(0)}(\tilde{B}), Y)$ if and only if $\tilde{A} \in (c_0, Y)$, $W^{(A_n)} \in (c_0, \ell_\infty)$ for all n, (27) holds for each n, and $\tilde{A} - (\gamma_n)_{n=0}^{\infty} \in Y$. This means, we have to add the last two conditions to those for $A \in (s_\alpha^{(0)}(\tilde{B}), Y)$, that is, $(3.1)^*$ and $(3.2)^*$ in **3.** to those in **2.**, $(3.1)^*$ and $(6.1)^*$ in **6.** to those in **5.**, $(3.1)^*$ and $(9.1)^*$ in **9.** to those in **8.** and $(3.1)^*$ and $(12.1)^*$ in **12.** to those in **11.**. □

Remark 4.6. *We obtain identities or estimates for the norms of the matrix operators between the spaces in Theorem 4.1 from Lemmas 4.4 and 4.5 with the matrices* $\tilde{A}$ *and* $W^{(A_n)}$ *defined in (28) and (29).*

5. The Hausdorff Measure of Noncompactness

We recall that if X and Y are infinite–dimensional complex Banach spaces then a linear operator $L : X \to Y$ is said to be compact if the domain of L is all of X, and, for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a convergent subsequence. We denote the class of such operators by $\mathcal{C}(X, Y)$.

We also recall the definition of the Hausdorff measure of noncompactness of bounded subsets of a metric space, and the Hausdorff measure of noncompactness of operators between Banach spaces.

Definition 5.1. *Let (X, d) be a metric space, $B(x_0, \delta) = \{x \in X : d(x, x_0) < \delta\}$ denote the open ball of radius $\delta > 0$ and center in $x_0 \in X$, and $\mathcal{M}_X$ be the collection of bounded sets in X. The Hausdorff measure of noncompactness of $Q \in \mathcal{M}_X$ is*

$$\chi(Q) = \inf\{\epsilon > 0 : Q \subset \bigcup_{k=1}^{n} B(x_k, \delta_k),\ x_k \in X,\ \delta_k < \epsilon,\ 1 \le k \le n,\ n \in \mathbb{N}\}.$$

Let X and Y be Banach spaces and χ_1 and χ_2 be measures of noncompactness on X and Y. Then the operator $L : X \to Y$ is called (χ_1, χ_2)–bounded if $L(Q) \in \mathcal{M}_Y$ for every $Q \in \mathcal{M}_X$ and there exists a positive constant C such that

$$\chi_2(L(Q)) \le C\chi_1(Q) \text{ for every } Q \in \mathcal{M}_X. \tag{30}$$

If an operator L is (χ_1, χ_2)–bounded then the number

$$\|L\|_{(\chi_1,\chi_2)} = \inf\{C \ge 0 : (30) \text{ holds for all } Q \in \mathcal{M}_X\}$$

is called the (χ_1, χ_2)–measure of noncompactness of L. In particular, if $\chi_1 = \chi_2 = \chi$, then we write $\|L\|_\chi$ instead of $|L\|_{(\chi,\chi)}$.

Lemma 5.1. ([12, Theorem 2.25 and Corollary 2.26]) *Let X and Y are Banach spaces and $L \in \mathcal{B}(X, Y)$. Then we have*

$$\|L\|_\chi = \chi\left(L(\bar{B}_X)\right) = \chi\left(L(S_X)\right), \tag{31}$$

$$L \in \mathcal{C}(X, Y) \text{ if and only if } \|L\|_\chi = 0. \tag{32}$$

Lemma 5.2 (Goldenštein, Goh'berg, Markus). ([12, Theorem 2.23]) *Let X be a Banach space with a Schauder basis $(b_n)_{n=0}^\infty$, $Q \in \mathcal{M}_X$, $\mathcal{P}_n : X \to X$ be the projector onto the linear span of $\{b_0, b_1, \dots b_n\}$, I be the identity map on X and $\mathcal{R}_n = I - \mathcal{P}_n$ $(n = 0, 1, \dots)$. Then we have*

$$\frac{1}{a} \cdot \limsup_{n\to\infty}\left(\sup_{x\in Q} \|\mathcal{R}_n(x)\|\right) \le \chi(Q) \le \limsup_{n\to\infty}\left(\sup_{x\in Q} \|\mathcal{R}_n(x)\|\right), \tag{33}$$

where $a = \limsup_{n\to\infty} \|\mathcal{R}_n\|$.

Lemma 5.3. ([12, Theorem 2.8]) *Let Q be a bounded subset of the normed space X, where X is ℓ_p for $1 \le p < \infty$ or c_0. If $\mathcal{P}_n : X \to X$ is the operator defined by $\mathcal{P}_n(x) = x^{[n]}$ for $x = (x_k)_{k=0}^\infty \in X$, then we have $\chi(Q) = \lim_{n\to\infty}(\sup_{x\in Q} \|\mathcal{R}_n(x)\|)$.*

The final results of this section give the estimates of the Hausdorff measure of noncompactness of L_A when $A \in (X_T, c)$ for $X = c_0, \ell_\infty, c$.

Lemma 5.4. ([4, Corollary 5.13]) *If $A \in ((c_0)_T, c)$ or $A \in ((\ell_\infty)_T, c)$ then we have*

$$\frac{1}{2} \cdot \lim_{r\to\infty}\left(\sup_{n\ge r} \left\|\hat{A}_n - \hat{\alpha}\right\|_1\right) \le \|L_A\|_\chi \le \lim_{r\to\infty}\left(\sup_{n\ge r} \left\|\hat{A}_n - \hat{\alpha}\right\|_1\right), \tag{34}$$

where $\hat{\alpha} = (\alpha_k)_{k=0}^\infty$ with $\hat{\alpha}_k = \lim_{n\to\infty} \hat{a}_{nk}$ for $k = 0, 1, \dots$.

Lemma 5.5. ([4, Theorem 5.14]) *If $A \in (c_T, c)$ then we have*

$$\frac{1}{2}\cdot\lim_{r\to\infty}\left(\sup_{n\geq r}\left(\left|\beta-\gamma_n-\sum_{k=0}^{\infty}\hat{\alpha}_k\right|+\sum_{k=0}^{\infty}|\hat{a}_{nk}-\hat{\alpha}_k|\right)\right)\leq$$

$$\leq\|L_A\|_\chi\leq\lim_{r\to\infty}\left(\sup_{n\geq r}\left(\left|\beta-\gamma_n-\sum_{k=0}^{\infty}\hat{\alpha}_k\right|+\sum_{k=0}^{\infty}|\hat{a}_{nk}-\hat{\alpha}_k|\right)\right),$$

where $\gamma_n = \lim_{m\to\infty} w_{mk}^{(A_n)}$ for $n = 0,1,\ldots$, $\beta = \lim_{n\to\infty}(\sum_{k=0}^{\infty}\hat{a}_{nk}-\gamma_n)$ and $\hat{\alpha} = (\hat{\alpha}_k)_{k=0}^{\infty}$ with $\hat{\alpha}_k = \lim_{n\to\infty}\hat{a}_{nk}$ for $k = 0,1,\ldots$.

6. Compact Operators on the Sets

In this section, we establish necessary and sufficient conditions for a matrix operator to be a compact operator from $s_\alpha^0(\tilde{B})$, $s_\alpha^{(c)}(\tilde{B})$ and $s_\alpha(\tilde{B})$ into Y, where $Y = \ell_\infty, c_0, c, \ell_1$. This is achieved by applying the results of the previous section on the Hausdorff measure of noncompactness.

The following notations are needed to establish estimates and identities for the Hausdorff measure of noncompactness of matrix operators and characterize the classes of compact operators.

Let $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix and $r \in \mathbb{N}$. Then $A^{[r]}$ denotes the matrix with the rows $A_n^{[r]} = 0$ for $0 \leq n \leq r$ and $A_n^{[r]} = A_n$ for $n \geq r+1$. Also we write $\tilde{A}$ and $W^{(A_n)}$ $n \in \mathbb{N}$ for the matrices defined in (28) and (29), and $\tilde{\alpha} = (\tilde{\alpha}_k)_{k=0}^{\infty}$ and $\gamma = (\gamma_n)_{n=0}^{\infty}$ for the sequences with $\tilde{\alpha}_k = \lim_{n\to\infty}\tilde{a}_{nk}$ for $k = 0,1,\ldots$ and

$$\gamma_n = \lim_{m\to\infty}\sum_{k=0}^{m} w_{mk}^{(A_n)} = \lim_{m}\sum_{k=0}^{m}\sum_{j=m}^{\infty}(-1)^{j-k}\frac{a_{nj}}{r_j}\prod_{l=k}^{j-1}\frac{s_l}{r_l};$$

finally, let $\beta = \lim_{n\to\infty}(\sum_{k=0}^{\infty}\tilde{a}_{nk}-\gamma_n)$.

We also write $\tilde{C} = (\tilde{c}_{nk})_{n,k=0}^{\infty}$ for the matrix with $\tilde{c}_{nk} = \tilde{a}_{nk} - \tilde{\alpha}_k$ for all $n, k \in \mathbb{N}_0$ and $\delta = (\delta_n)_{n=0}^{\infty}$ for the sequence with

$$\delta_n = \sum_{k=0}^{\infty}\tilde{\alpha}_k - \gamma_n + \beta \; (n = 0,1,\ldots).$$

Theorem 6.1. *If $A \in (X, Y)$ where $X = s_\alpha^0(\tilde{B}), s_\alpha^{(c)}(\tilde{B}), s_\alpha(\tilde{B})$ and $Y = c_0, c, \ell_\infty, \ell_1$ then the identities or estimates for L_A can be read from the following table:*

From / *To*	$s_\alpha(\tilde{B})$	$s_\alpha^0(\tilde{B})$	$s_\alpha^{(c)}(\tilde{B})$
ℓ_∞	**1.**	**1.**	**2.**
c_0	**3.**	**3.**	**4.**
c	**5.**	**5.**	**6.**
ℓ_1	**7.**	**7.**	**8.**

where

1. $(1.1)^{**}$ $0 \leq \|L_A\|_\chi \leq \lim_{r\to\infty}\left(\sup_{n\geq r} \sum_{k=0}^{\infty} |\tilde{a}_{nk}|\right)$;

2. $(2.1)^{**}$ $0 \leq \|L_A\|_\chi \leq \lim_{r\to\infty}\left(\sup_{n\geq r} \sum_{k=0}^{\infty} |\tilde{a}_{nk}| + \left|\gamma_n\right|\right)$;

3. $(3.1)^{**}$ $\|L_A\|_\chi = \lim_{r\to\infty}\left(\sup_{n\geq r} \sum_{k=0}^{\infty} |\tilde{a}_{nk}|\right)$;

4. $(4.1)^{**}$ $\|L_A\|_\chi = \lim_{r\to\infty}\left(\sup_{n\geq r} \sum_{k=0}^{\infty} |\tilde{a}_{nk}| + \left|\gamma_n\right|\right)$;

5. $(5.1)^{**}$ $\frac{1}{2}\cdot\lim_{r\to\infty}\left(\sup_{n\geq r} \sum_{k=0}^{\infty} |\tilde{c}_{nk}|\right) \leq \|L_A\|_\chi \leq \lim_{r\to\infty}\left(\sup_{n\geq r} \sum_{k=0}^{\infty} |\tilde{c}_{nk}|\right)$;

6. $(6.1)^{**}$ $\frac{1}{2}\cdot\lim_{r\to\infty}\left(\sup_{n\geq r} \sum_{k=0}^{\infty} |\tilde{c}_{nk}| + |\delta_n|\right) \leq \|L_A\|_\chi \leq \lim_{r\to\infty}\left(\sup_{n\geq r} \sum_{k=0}^{\infty} |\tilde{c}_{nk}| + |\delta_n|\right)$;

7. $(7.1)^{**}$ $\lim_{r\to\infty} \sup_{\substack{N\subset\mathbb{N}\\ N \text{ finite}}} \left\|\sum_{n\in\mathbb{N}} \tilde{A}_n^{[r]}\right\|_1 \leq \|L_A\|_\chi \leq 4\lim_{r\to\infty} \sup_{\substack{N\subset\mathbb{N}\\ N \text{ finite}}} \left\|\sum_{n\in\mathbb{N}} \tilde{A}_n^{[r]}\right\|_1$;

8. $(8.1)^{**}$
$$\lim_{r\to\infty} \sup_{\substack{N\subset\mathbb{N}\\ N \text{ finite}}} \left(\left\|\sum_{n\in\mathbb{N}} \tilde{A}_n^{[r]}\right\|_1 + \left|\sum_{n\in N}\gamma_n\right|\right) \leq \|L_A\|_\chi$$
$$\leq 4\lim_{r\to\infty} \sup_{\substack{N\subset\mathbb{N}\\ N \text{ finite}}} \left(\left\|\sum_{n\in\mathbb{N}} \tilde{A}_n^{[r]}\right\|_1 + \left|\sum_{n\in N}\gamma_n\right|\right).$$

Proof. **1.** and **2.** We define $\mathcal{P}_r, \mathcal{R}_r : \ell_\infty \to \ell_\infty$ by $\mathcal{P}_r(x) = x^{[r]}$ for all $x \in \ell_\infty$, $R_r = I - P_r$ $(r = 0, 1, \dots)$, and write $L = L_A$ and $\bar{B} = \bar{B}_{\ell_\infty}$ for short. Then it follows from (5.1) and ([12, Theorem 2.12] that

$$\begin{aligned}0 \leq \|L\|_\chi &= \chi(L(\bar{B})) \leq \chi(\mathcal{P}_r(L(\bar{B}))) + \chi(\mathcal{R}_r(L(\bar{B})))\\ &= \chi(\mathcal{R}_r(L(\bar{B}))) \leq \sup_{x\in\bar{B}} \|\mathcal{R}_r(L(x))\|_\infty = \left\|\tilde{A}^{[r]}\right\|_{(X,\infty)}.\end{aligned}$$

If $X = s_\alpha(\tilde{B})$ or $X = s_\alpha^0(\tilde{B})$ the estimates in $(1,1)^{**}$ follow from Lemma 4.4 (a). If $X = s_\alpha^{(c)}(\tilde{B})$ then the estimates in $(2.1)^{**}$ follow from Lemma 4.5 (a).

3. – **8.** The identities or estimates in $(3.1)^{**}$–$(8.1)^{**}$ follow from [3, Corollary 3.6 (a), Theorem 3.7 (a), Corollary 3.6 (c), Theorem 3.7 (a), Corollary 3.6 (b) and Theorem 3.7 (c)]. □

Corollary 6.1. *Let X be one of the spaces $s_\alpha^0(\tilde{B})$ or $s_\alpha(\tilde{B})$. We obtain as an immediate consequence of (32) and Theorem 6.1* $(1.1)^{**}$, $(3.1)^{**}$ *and* $(5.1)^{**}$.

(a) If $A \in (X, c_0)$, then L_A is compact if and only if

$$\lim_{r\to\infty}\left(\sup_{n\geq r}\left(\sum_{k=0}^{\infty} \|\tilde{a}_{nk}\|\right)\right) = 0. \tag{35}$$

(b) If $A \in (X, c)$, then L_A is compact if and only if

$$\lim_{r\to\infty}\sup_{n\geq r}\left(\sum_{k=0}^{\infty} \|\tilde{a}_{nk} - \tilde{\alpha}_k\|\right) = 0.$$

(c) If $A \in (X, \ell_\infty)$, then L_A is compact if the condition (35) holds.

Corollary 6.2. *We obtain as an immediate consequence of (32) and Theorem 6.1* $(2.1)^{**}$, $(4.1)^{**}$ *and* $(6.1)^{**}$.

(a) If $A \in (s_\alpha^{(c)}(\tilde{B}), c_0)$, then L_A is compact if and only if

$$\lim_{r\to\infty}\left(\sup_{n\geq r}\left(\sum_{k=0}^{\infty}\|\tilde{a}_{nk}\| + \left\|\gamma_n\right\|\right)\right) = 0. \tag{36}$$

(b) If $A \in (s_\alpha^{(c)}(\tilde{B}), c)$, then L_A is compact if and only if

$$\lim_{r\to\infty}\sup_{n\geq r}\left(\sum_{k=0}\infty\|\tilde{a}_{nk} - \tilde{\alpha}_k\| + \left\|\sum_k \tilde{\alpha}_k - \gamma_n - \beta\right\|\right) = 0.$$

(c) If $A \in (s_\alpha^{(c)}(\tilde{B}), \ell_\infty)$, then L_A is compact if (36) holds.

Corollary 6.3. *For each subset N of $\mathbb{N}$ and $r \in \mathbb{N}$, let $N_r = \{n \in N : n \geq r+1\}$. We obtain as an immediate consequence of (32) and Theorem 6.1* (7.1)** *and* (8.1)**.

(a) If $A \in (s_\alpha^0(\tilde{B}), \ell_1)$ or $A \in (s_\alpha(\tilde{B}), \ell_1)$, then L_A is compact if and only if

$$\lim_{r\to\infty}\left(\sup_{\substack{N\subset\mathbb{N}\\ N \text{ finite}}}\left(\sum_{k=0}^{\infty}\left|\sum_{n\in N_r}\sum_{j=k}^{\infty}\frac{(-1)^{j-k}\alpha_k}{r_j}\prod_{m=k}^{j-1}\frac{s_m}{r_m}a_{nj}\right|\right)\right) = 0.$$

(b) If $A \in (s_\alpha^{(c)}(\tilde{B}), \ell_1)$, then L_A is compact if and only if

$$\lim_{r\to\infty}\left(\sup_{\substack{N\subset\mathbb{N}\\ N \text{ finite}}}\left(\sum_{k=0}^{\infty}\left|\sum_{n\in N_r}\sum_{j=k}^{\infty}\frac{(-1)^{j-k}\alpha_k}{r_j}\prod_{m=k}^{j-1}\frac{s_m}{r_m}a_{nj}\right| + \left|\sum_{n\in N_r}\gamma_n\right|\right)\right) = 0.$$

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