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Drazin Invertibility of Product and Difference of Idempotents in a Ring

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Abstract. In this paper, several equivalent conditions on the Drazin invertibility of product and difference of idempotents are obtained in a ring. Some results in Banach algebras are extended to the ring case.

1. Introduction

Throughout this paper, R denotes an associative ring with unity 1. Recall that an element $a \in R$ is said to be Drazin invertible if there exists $b \in R$ such that

$$ab = ba, bab = b, a^k = a^{k+1}b$$

for some positive integer k. The element b above is unique if it exists and denoted by a^D. The least such k is called the Drazin index of a, denoted by ind(a). If ind(a) = 1, then b is called the group inverse of a and is denoted by $a^\#$. By R^D we mean the set of all Drazin invertible elements of R.

Drazin inverses are closely related to some regularity of rings. It is well known that a is Drazin invertible if and only if a is strongly π-regular (i.e., $a^n \in a^{n+1}R \cap Ra^{n+1}$ for some nonnegative integer n) if and only if a^m is group invertible for some positive integer m. Hence a is Drazin invertible if and only if a^m is Drazin invertible for some (any) integer m, and in particular if and only if a^2 is Drazin invertible. In [11, 12], Koliha and Rakočević studied the invertibility of the difference and the sum of idempotents in a ring and proved that $p-q$ is invertible if and only if $1-pq$ and $p+q$ are invertible for any idempotents p and q. Many authors extended the ordinary invertibility to the Drazin invertibility. For example, Cvetković-Ilić and Deng [4] showed that $p-q$ is Drazin invertible if and only if $1-pq$ and $p+q-pq$ are Drazin invertible where p, q are idempotent operators of a Hilbert space. Further Deng and Wei [8] proved that if p, q are idempotents in $\mathscr{A} = \mathcal{B}(X)$, the ring of all bounded linear operators in a complex Banach space X, then: (1) $p-q \in \mathscr{A}^D$ if and only if $p+q \in \mathscr{A}^D$ if and only if $1-pq \in \mathscr{A}^D$; (2) $pq-qp \in \mathscr{A}^D$ if and only if $pq+qp \in \mathscr{A}^D$ if and only if $pq \in \mathscr{A}^D$ and $p-q \in \mathscr{A}^D$. These results are generalized to Banach algebras in [10]. More results on the Drazin invertibility of sum, difference and product of idempotents can be found in [3-8,10,13].

Motivated by the excellent papers [3,4,11,12], we consider the Drazin invertibility of $p-q$, pq, $pq-qp$ (commutator) and $pq+qp$ (anti-commutator), where p and q are idempotents in a ring. Some results, obtained in [3,4] in the context of Banach algebras, are extended to the ring case.

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2. Key Lemmas

In this section, we begin with some elementary and known results which play an important role in section 3.

Lemma 2.1. ([9]) (1) *Let $a \in R^D$. If $ab = ba$, then $a^D b = ba^D$,*
(2) *Let $a, b \in R^D$ and $ab = ba = 0$. Then $(a + b)^D = a^D + b^D$.*

Lemma 2.2. ([14]) *Let $a, b \in R^D$ and $ab = ba$. Then $ab \in R^D$ and $(ab)^D = b^D a^D = a^D b^D$.*

Lemma 2.3. ([2]) *Let $a, b \in R$ and $ab \in R^D$. Then $ba \in R^D$ and $(ba)^D = b((ab)^D)^2 a$.*

Lemma 2.4. ([1]) *Let $a, b \in R$. Then $1 - ab \in R^D$ if and only if $1 - ba \in R^D$.*

Lemma 2.5. *Let $a, b \in R^D$ and $p^2 = p \in R$. If $ap = pa$ and $bp = pb$, then $ap + b(1 - p) \in R^D$ and*

$$(ap + b(1 - p))^D = a^D p + b^D (1 - p).$$

Proof. Since $p^2 = p$, we have $p^D = p$. Thus, $a, b, p \in R^D$. Note that $ap = pa$ and $bp = pb$. We obtain $(ap)^D = a^D p$ and $(b(1 - p))^D = b^D (1 - p)$ by Lemma 2.2. As $apb(1 - p) = b(1 - p)ap = 0$, according to Lemma 2.1(2), it follows that $(ap + b(1 - p))^D = a^D p + b^D (1 - p)$. □

Lemma 2.6. *Let $a \in R$, $p^2 = p \in R$, $b = pa(1 - p)$ and $c = (1 - p)ap$. The following statements are equivalent:*
(1) *$b + c \in R^D$,*
(2) *$bc \in R^D$,*
(3) *$b - c \in R^D$.*

Proof. (1) ⇒ (2) Since $(b + c)^2 = (bc + cb)$ and $b + c \in R^D$, we have $bc + cb \in R^D$. Let $x = (bc + cb)^D$. As $p(bc + cb) = (bc + cb)p$, we obtain that $px = xp$ by Lemma 2.1(1). Next, we prove that $(bc)^D = pxp$.

Since $x = (bc + cb)x^2$, we get

$$pxp = p(bc + cb)x^2 p = bcx^2 p = bc(px^2 p) = bc(pxp)^2.$$

By $(bc + cb)x = x(bc + cb)$, we obtain $p(bc + cb)xp = px(bc + cb)p$. It follows that $bc(pxp) = (pxp)bc$.

Because $(bc + cb)^{n+1} x = (bc + cb)^n$ for some n, we have

$$((bc)^{n+1} + (cb)^{n+1})x = (bc)^n + (cb)^n.$$

Multiplying the equation above by p on two sides yields

$$p(bc)^{n+1} xp = p(bc)^n p,$$

i.e., $(bc)^{n+1} pxp = (bc)^n$. So, bc is Drazin invertible and $(bc)^D = pxp$.

(2) ⇒ (1) According to Lemma 2.3, $bc \in R^D$ is equivalent to $cb \in R^D$. Note that $bc \cdot cb = cb \cdot bc = 0$. We have $(b + c)^2 = (bc + cb) \in R^D$ by Lemma 2.1(2). It follows that $b + c \in R^D$.

(2) ⇔ (3) Its proof is similar to (1) ⇔ (2). □

Lemma 2.7. *Let $a \in R$ with $a - a^2 \in R^D$ or $a + a^2 \in R^D$. Then $a \in R^D$.*

Proof. We only need to prove the situation when $a - a^2 \in R^D$ with $x = (a - a^2)^D$.

By Lemma 2.1(1), it is clear $ax = xa$ since $a(a - a^2) = (a - a^2)a$.

Since $a - a^2 \in R^D$, we get $(a - a^2)^n = (a - a^2)^{n+1} x$ for some integer $n \geq 1$, that is,

$$a^n (1 - a)^n = a^{n+1} (1 - a)^{n+1} x.$$

Note that

$$a^n(1-a)^n = a^n(1+\sum_{i=1}^{n} C_n^i(-a)^i).$$

It follows that

$$a^n = a^{n+1}[(1-a)^{n+1}x + \sum_{i=1}^{n} C_n^i(-a)^{i-1}] = [(1-a)^{n+1}x + \sum_{i=1}^{n} C_n^i(-a)^{i-1}]a^{n+1}.$$

This shows $a^n \in a^{n+1}R \cap Ra^{n+1}$. Hence, $a \in R^D$. $\square$

3. Main Results

In what follows, p and q always mean two arbitrary idempotents in a ring R. We give some equivalent conditions for the Drazin invertibility of $p-q$, pq, $pq-qp$ and $pq+qp$.

Proposition 3.1. *The following statements are equivalent:*
(1) $1-pq \in R^D$, (2) $p-pq \in R^D$, (3) $p-qp \in R^D$, (4) $1-pqp \in R^D$, (5) $p-pqp \in R^D$,
(6) $1-qp \in R^D$, (7) $q-qp \in R^D$, (8) $q-pq \in R^D$, (9) $1-qpq \in R^D$, (10) $q-qpq \in R^D$.

Proof. (1) $\Leftrightarrow$ (6) is obvious by Lemma 2.4. We only need to prove that (1) $-$ (5) are equivalent.

(1) $\Leftrightarrow$ (4) $1-pq = 1-p(pq)$ is Drazin invertible if and only if $1-pqp$ is Drazin invertible by Lemma 2.4.

(4) $\Rightarrow$ (5) Since $p \in R^D$ and $p(1-pqp) = (1-pqp)p = p-pqp$, we obtain that $p-pqp$ is Drazin invertible according to Lemma 2.2.

(5) $\Rightarrow$ (4) Pose $a = p-pqp$, $b = 1$. Then a and b are Drazin invertible. Since $1-pqp = (p-pqp)+1-p$, it follows that $1-pqp = ap+b(1-p)$ is Drazin invertible in view of Lemma 2.5.

(2) $\Leftrightarrow$ (5) Since $p-pq = pp(1-q)$ and $p-pqp = p(1-q)p$, the result follows by Lemma 2.3.

(2) $\Leftrightarrow$ (3) By Lemma 2.3, $p(1-q) \in R^D \Leftrightarrow (1-q)p \in R^D$.

Finally, the other equivalences follow by interchanging p and q. $\square$

By replacing p and q with $1-p$ and $1-q$ in Proposition 3.1, respectively, we get the following result immediately.

Corollary 3.2. *The following statements are equivalent:*
(11) $p+q-pq \in R^D$, (12) $q-pq \in R^D$, (13) $q-qp \in R^D$,
(14) $p+(1-p)(q-qp) \in R^D$, (15) $(1-p)q(1-p) \in R^D$, (16) $p+q-qp \in R^D$,
(17) $p-qp \in R^D$, (18) $p-pq \in R^D$, (19) $q+(1-q)(p-pq) \in R^D$,
(20) $(1-q)p(1-q) \in R^D$.

Finally, as $p-pq$ appears in Proposition 3.1 (eq. 2) and Corollary 3.2 (eq. 18), we get

Corollary 3.3. *Statements* (1) $-$ (10) *of Proposition 3.1 are equivalent with statements* (11) $-$ (20) *of Corollary 3.2.*

In 2012, Koliha, Cvetković-Ilić and Deng [10] proved that $p-q \in \mathscr{A}^D$ if and only if $1-pq \in \mathscr{A}^D$ if and only if $p+q-pq \in \mathscr{A}^D$ where p, q are idempotents in a Banach algebra $\mathscr{A}$. It is natural to consider whether the same property can be inherited to the Drazin inverse of ring versions. The following result proves that the statement holds in the ring case.

Theorem 3.4. *The following statements are equivalent:*
(1) $p-q \in R^D$,
(2) $1-pq \in R^D$,
(3) $p+q-pq \in R^D$.

Proof. (1) $\Rightarrow$ (2) As $p(p-q)^2 = (p-q)^2p = p - pqp$, then $1 - pqp = (p-q)^2p + 1 - p$.

Let $a = (p-q)^2$ and $b = 1$. Then $ap = pa$, $bp = pb$. Since $p - q \in R^D$, we obtain that $a = (p-q)^2 \in R^D$. By Lemma 2.5, $1 - pqp = ap + b(1-p) \in R^D$. Therefore, $1 - pq$ is Drazin invertible by Lemma 2.4.

(2) $\Leftrightarrow$ (3) This is Corollary 3.3 ((2) $\Leftrightarrow$ (11)).

(3) $\Rightarrow$ (1) Let $a = 1 - pqp$, $b = 1 - (1-p)(1-q)(1-p)$. Then $ap = pa$, $bp = pb$ and $a \in R^D$ by Corollary 3.3. As $1 - (1-p)(1-p)(1-q) = p + q - pq \in R^D$, then $b \in R^D$ by Lemma 2.4. Finally, $ap + b(1-p) = (p-q)^2 \in R^D$ by Lemma 2.5, hence $p - q \in R^D$. $\square$

Cvetković-Ilić, Deng [3] considered the Drazin invertibility of product and difference of idempotents in a Banach algebra $\mathscr{A}$. Moreover, they proved that if one of pq, $1 - p - q$ and $(1-p)(1-q)$ belongs to $\mathscr{A}^D$ then they all do. We extend the result in [3] to the ring cases.

Theorem 3.5. *The following statements are equivalent:*
(1) $pq \in R^D$,
(2) $1 - p - q \in R^D$,
(3) $(1-p)(1-q) \in R^D$.

Proof. (1) $\Leftrightarrow$ (2) Let $\overline{p} = 1 - p$. Then $\overline{p} - q \in R^D$ if and only if $q - \overline{p}q \in R^D$ by Proposition 3.1 and Theorem 3.4. Since $\overline{p} - q = 1 - p - q$ and $q - \overline{p}q = pq$, (1) $\Leftrightarrow$ (2) holds.

(1) $\Leftrightarrow$ (3) Set $\overline{p} = 1 - p$. Then $\overline{p} - \overline{p}q \in R^D$ if and only if $q - \overline{p}q \in R^D$ by Proposition 3.1. Since $\overline{p} - \overline{p}q = (1-p)(1-q)$ and $q - \overline{p}q = pq$, the result follows. $\square$

Theorem 3.6. *The following statements are equivalent:*
(1) $pq - qp \in R^D$,
(2) $pq \in R^D$ *and* $p - q \in R^D$.

Proof. Pose $b = pq(1-p)$ and $c = (1-p)qp$. Then $b - c = pq - qp$.

(1) $\Rightarrow$ (2) As by hypothesis $b - c \in R^D$, then $pqp - (pqp)^2 = pq(1-p)qp = bc \in R^D$ by Lemma 2.6. It follows that $pqp \in R^D$ by Lemma 2.7, hence $pq = p(pq) \in R^D$ by Lemma 2.3.

Similarly, $(1-p)q - q(1-p) = -(pq - qp) \in R^D$ implies $q - pq = (1-p)q \in R^D$. Therefore, $p - q \in R^D$ by Proposition 3.1 and Theorem 3.4.

(2) $\Rightarrow$ (1) By Lemma 2.3, both pqp and $(p-q)^2$ are Drazin invertible. Note that $bc = pq(1-p)qp = pqp(p-q)^2 = (p-q)^2pqp$. It follows that bc is Drazin invertible by Lemma 2.2. Hence, according to Lemma 2.6, we have $pq - qp = b - c \in R^D$. $\square$

We can give an interesting result similar to Theorem 3.6.

Theorem 3.7. *The following statements are equivalent:*
(1) $pq + qp \in R^D$,
(2) $pq \in R^D$ *and* $p + q \in R^D$.

Proof. (1) $\Rightarrow$ (2) Since $pq + qp = -(p+q) + (p+q)^2 = (p+q-1) + (p+q-1)^2 \in R^D$, $p + q \in R^D$ and $p + q - 1 \in R^D$ according to Lemma 2.7. Therefore, $pq \in R^D$ by Theorem 3.5.

(2) $\Rightarrow$ (1) Since $pq + qp = (p+q)(p+q-1) = (p+q-1)(p+q)$, $pq + qp \in R^D$ by Lemma 2.2 and Theorem 3.5. $\square$

Remark 3.8. Let p, q be two idempotents in a Banach algebra. Then, $p + q$ is Drazin invertible if and only if $p - q$ is Drazin invertible [10]. Hence, $pq + qp$ is Drazin invertible is equivalent to $pq - qp$ is Drazin invertible in a Banach algebra. However, in general, this need not be true in a ring. For example, let $R = \mathbb{Z}$, $p = q = 1$. Then $p - q = 0 \in R^D$, but $p + q = 2 \notin R^D$.

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