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Existence of Fixed Points of Mappings on General Topological Spaces

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Abstract. The existence of fixed points for continuous mappings on general topological spaces via compact subsets is proved. All our results presented here are new and are generalizations, extensions and improvements of the corresponding results due to Ćirić, Jungck, Liu and many others . Further, certain results due to Ćirić are improved and extended to topological spaces which are not necessarily Hausdorff and completely regular.

1. Introduction

In fixed point theory, researchers frequently develop and solve problems on the existence of fixed points using a variety of techniques in analysis, topology and geometry. For instance [4], [6], [5], [9] and [21] have studied the existence of fixed points of mappings based on techniques from nonlinear analysis. If a topological space X is a metric space, or a linear topological space, then the fixed point theory in such spaces is very rich and easy to work in. However, if X has a topological structure only, then the fixed point theory in such spaces is comparatively less rich. A number of peoples have extended Jungck's results [13] from compact metric spaces to compact Hausdorff topological spaces and various generalized metric spaces (see for instance [1, 11, 14, 16–18, 20]).

If a topological space X is compact, then every real valued continuous mappings on X is bounded and attains its maximum and minimum. The generalization of this result is Lemma 2.4 (see section 2): every lower semi continuous from above (lsca) function on a compact topological space attains its lower bound. In this paper, we use this basic result to prove several new existence theorems on fixed points of continuous mappings on general (not necessarily Hausdorff) topological spaces. Our results are different from known, or are generalization, extension and improvement of the corresponding results due to Ćirić [5], Jungck [13], Liu [16] and Liu et al. [17].

2. Preliminaries

Definition 2.1. *Let X be a topological space.*

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(1) A function $f : X \to \mathbb{R}$ is said to be [7] lower semi-continuous from above (lsca) at a point $x_0 \in X$ if, for any net $\{x_\lambda\}_{\lambda\in\Lambda}$ converging to x_0 such that $f(x_{\lambda_1}) \le f(x_{\lambda_2})$ for any $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_2 \le \lambda_1$,

$$f(x_0) \le \lim_{\lambda\in\Lambda} f(x_\lambda).$$

(2) A function $f : X \times X \to \mathbb{R}$ is said to be lower semi-continuous from above (lsca) at a point $(x_0, y_0) \in X \times X$ if, for any net $\{(x_\lambda, y_\lambda)\}_{\lambda\in\Lambda}$ converging to (x_0, y_0) such that $f(x_{\lambda_1}, y_{\lambda_1}) \le f(x_{\lambda_2}, y_{\lambda_2})$ for any $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_2 \le \lambda_1$,

$$f(x_0, y_0) \le \lim_{\lambda\in\Lambda} f(x_\lambda, y_\lambda).$$

(3) A function $f : X \times X \to \mathbb{R}$ is said to be lsca on $X \times X$ if it is lsca at all points $(x, y) \in X \times X$.

Example 2.2. *([8]) Let $X = \mathbb{R}^+ \times \mathbb{R}^+ \cup \mathbb{R}^- \times \mathbb{R}^- \cup \{(0, 0)\}$, where $\mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of positive and negative real numbers, respectively. Define a function $f : X \to \mathbb{R}$ by*

$$f(x, y) = \begin{cases} x + y + 1, & \text{when } (x, y) \in \mathbb{R}^+ \times \mathbb{R}^+; \\ \frac{1}{2}, & \text{when } (x, y) = (0, 0); \\ -x - y + 1, & \text{when } (x, y) \in \mathbb{R}^- \times \mathbb{R}^-. \end{cases}$$

Let $\{z_n\}_{n\ge1} = \{(x_n, y_n)\}_{n\ge1}$ be a sequence in X, where $\{x_n\}_{n\ge1}$ and $\{y_n\}_{n\ge1}$ are two sequences of non-negative numbers and $x_n \to 0$, $y_n \to 0$ as $n \to \infty$. Then $\{z_n\}_{n\ge1}$ converges to $(0, 0)$ with $f(z_{n+1}) \le f(z_n)$ for $\lambda_2 = n \le n + 1 = \lambda_1$ and $f(0, 0) = \frac{1}{2} < 1 = \lim_{n\to\infty} f(z_n)$.

Similarly, if $\{z'_n\}_{n\ge1} = \{(x'_n, y'_n)\}_{n\ge1}$ is a sequence in X, where $\{x'_n\}_{n\ge1}$ and $\{y'_n\}_{n\ge1}$ are sequences of negative numbers with $x'_n \to 0$, $y'_n \to 0$ as $n \to \infty$, then $\{z'_n\}_{n\ge1}$ converges to $(0, 0)$ with $f(z'_{n+1}) \le f(z'_n)$ for $\lambda_2 = n \le n + 1 = \lambda_1$ and $f(0, 0) = \frac{1}{2} < 1 = \lim_{n\to\infty} f(z'_n)$. Thus f is lsca at $(0, 0)$.

Example 2.3. *Every lower semi-continuous function is lsca, but not conversely. One can check that the function $f : X \to \mathbb{R}$ with $X = \mathbb{R}^+ \times \mathbb{R}^+ \cup \mathbb{R}^- \times \mathbb{R}^- \cup \{(0, 0)\}$ defined below is lsca at $(0, 0)$, but is not lower semi-continuous at $(0, 0)$:*

$$f(x, y) = \begin{cases} x + y + 1, & \text{when } (x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ \cup \{(0, 0)\}; \\ x + y, & \text{when } (x, y) \in \mathbb{R}^- \times \mathbb{R}^-. \end{cases}$$

The following lemmas state some properties of lsca mappings. The first one is an analogue of Weierstrass boundedness theorem and the second one is a special case of Lemma 2 of [2].

Lemma 2.4. *([7]) Let X be a compact topological space and $f : X \longrightarrow \mathbb{R}$ be a function lsca. Then there exists $x_0 \in X$ such that $f(x_0) = \inf\{f(x) : x \in X\}$.*

Lemma 2.5. *Let X be a topological space and $f : X \times X \to X$ be a continuous function. If $g : X \to \mathbb{R}$ is a lsca function, then the composition function $F = g \circ f : X \times X \to \mathbb{R}$ is also lsca.*

Proof. Fix $(x_0, y_0) \in X\times X$ and consider a net $\{(x_\lambda, y_\lambda)\}_{\lambda\in\Lambda}$ in $X\times X$ converging to (x_0, y_0) such that, for $\lambda_2 \le \lambda_1$,

$$F(x_{\lambda_1}, y_{\lambda_1}) \le F(x_{\lambda_2}, y_{\lambda_2}).$$

Set $z_\lambda = f(x_\lambda, y_\lambda)$ and $z = f(x_0, y_0)$. Then, since f is continuous and g is lsca, $\lim_{\lambda\in\Lambda} f(x_\lambda, y_\lambda) = f(x_0, y_0) \in X$ and

$$g(z) = g(f(x_0, y_0)) \le \lim_{\lambda\in\Lambda} g(f(x_\lambda, y_\lambda)) = \lim_{\lambda\in\Lambda} g(z_\lambda)$$

with $g(z_{\lambda_1}) \le g(z_{\lambda_2})$ for $\lambda_2 \le \lambda_1$. Thus $F(x_0, y_0) \le \lim_{\lambda\in\Lambda} F(x_\lambda, y_\lambda)$ and so F is lsca. This completes the proof. $\square$

Remark 2.6. *Let X be a topological space. Let $f : X \to X$ be a continuous function and $F : X \times X \to \mathbb{R}$ be a lsca function. Then $g : X \to \mathbb{R}$ defined by $g(x) = F(x, f(x))$ is also lsca. In fact, let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net in X converging to a point $x \in X$. Since f is continuous $\lim_{\lambda\in\Lambda} f(x_\lambda) = f(x)$. Suppose that $g(x_{\lambda_1}) \le g(x_{\lambda_2})$ for $\lambda_2 \le \lambda_1$. Then, since F is lsca, we have*

$$g(x) = F(x, f(x)) \le \lim_{\lambda\in\Lambda} F(x_\lambda, f(x_\lambda)) = \lim_{\lambda\in\Lambda} g(x_\lambda)$$

and so g is lsca.

3. Existence of Fixed Points

If a topological space X is compact, then every real-valued continuous mapping on X is bounded and attains its maximum and minimum. A partial generalization of this result is Lemma 2.1: every lsca function on a compact topological space attains its lower bound. This basic result is instrumental in providing us with a point which turns out to be a fixed point of a mapping on the topological space X. Let us define

$$\mathcal{F} = \{F : X \times X \to \mathbb{R} : F \text{ is } \textit{lsca} \text{ and } F(x, y) = 0 \text{ if } x = y\}.$$

To begin with, we first prove the following existence theorem on fixed points.

Theorem 3.1. *Let X be a topological space, K be a nonempty compact subset of X and $f : K \to K$ be a continuous function. If $F \in \mathcal{F}$ and*

$$F(fx, fy) < \max\{F(x, y),\ \min\{F(x, fx), F(y, fy)\}\} + \lambda \min\{\left|F(x, fy)\right|, F(fx, y)\} \tag{1}$$

for all $x, y \in K$ with $x \neq y$ and λ an arbitrary positive real number, then f has at least one fixed point.

Proof. Define $\varphi : K \to (-\infty, \infty)$ by $\varphi(x) = F(x, f(x)),\quad x \in K$. Since f is continuous and F is lsca, so from Remark 2.6 it follows that the function φ is also lsca on K. By Lemma 2.4, there exists a point, say, $w \in K$ such that

$$\varphi(w) = F(w, f(w)) = \inf\{\varphi(x) : x \in K\}. \tag{2}$$

We prove that $f(w) = w$. Suppose, to the contrary, that $w \neq f(w)$. Then from (1),

$$\begin{aligned} F(f(w), f(f(w))) &< \max\{F(w, f(w)),\ \min\{F(w, f(w)), F(f(w), f(f(w)))\}\} \\ &\quad +\lambda \min\{\left|F(w, f(f(w)))\right|, 0\} \\ &= \max\{F(w, f(w)), \min\{F(w, f(w)), F(f(w), f(f(w)))\}\}. \end{aligned}$$

Since by (2),

$$\min\{F(w, f(w)), F(f(w), f(f(w)))\} = F(w, f(w)),$$

we have

$$F(f(w), f(f(w))) < F(w, f(w)),$$

a contradiction to (2). Thus $f(w) = w$, i.e, w is a fixed point of f. □

Corollary 3.2. *Let X be a compact topological space, K be a closed subset of X and $f : K \to K$ be a continuous function. If $F \in \mathcal{F}$ and*

$$F(fx, fy) < \max\{F(x, y),\ \min\{F(x, fx), F(y, fy)\}\} + \lambda \min\{\left|F(x, fy)\right|, F(fx, y)\}$$

for all $x, y \in K$ with $x \neq y$ and λ an arbitrary positive real number, then f has at least one fixed point.

Corollary 3.3. *Let X be a topological space, K be a nonempty compact subset of X and $f : K \to K$ be a continuous function. If $F \in \mathcal{F}$ is symmetric and*

$$F(fx, fy) < \max\{F(x,y),\ \min\{F(x,fx), F(y,fy)\}\} + \lambda \min\{\left|F(x,fy)\right|, F(fx,y)\}$$

for all $x, y \in K$ with $x \neq y$ and λ an arbitrary positive real number, then f has at least one fixed point.

Corollary 3.4. *Let (X, d) be a metric space, K be a nonempty compact subset of X and $f : K \to K$ be a continuous function. If $F \in \mathcal{F}$ and*

$$d(fx, fy) < \max\{d(x,y),\ \min\{d(x,fx), d(y,fy)\}\} + \lambda \min\{d(x,fy), d(fx,y)\}$$

for all $x, y \in K$ with $x \neq y$ and λ an arbitrary positive real number, then f has at least one fixed point.

Now, by using Theorem 3.1 we shall prove the following theorem asserting the existence of a unique fixed point.

Theorem 3.5. *Let X be a topological space, K be a nonempty compact subset of X and $f : K \to K$ be a continuous mapping. Suppose that for $F \in \mathcal{F}$ the mapping f satisfies the following condition:*

$$\begin{aligned} F(f(x), f(y)) \quad < \quad & \max\{F(x,y), \\ & [\min\{F(x,f(x)), F(y,f(y))\} + \min\{F(x,f(y)), F(f(x),,y)\}]\} \end{aligned} \tag{3}$$

for all $x, y \in K$ with $x \neq y$. Then f has a unique fixed point.

Proof. Since (3) implies (1), so by Theorem 3.1 there exists some $w \in K$ such that $f(w) = w$. Suppose that there is $v \in K$ such that $f(v) = v$ and $v \neq w$. Then from (3),

$$\begin{aligned} F(v,w) &= F(f(v), f(w)) \\ &< \max\{F(v,w), [\min\{F(v,f(v)), F(w,f(w))\} + \min\{F(v,f(w)), F(f(v),w)\}]\} \\ &= \max\{F(v,w), [\min\{F(v,v), F(w,w)\} + \min\{F(v,w), F(v,w)\}]\} \\ &= F(v,w), \end{aligned}$$

a contradiction. Thus, f has a unique fixed point. □

The following theorem generalizes Theorem 3.1.

Theorem 3.6. *Let X be a topological space, K be a compact subset of X and $f : K \to K$ be a continuous function. If $F \in \mathcal{F}$ and*

$$F(f^n x, f^n y) < \max\{F(x,y),\ \min\{F(x,f^n x), F(y,f^n y)\}\} + \lambda \min\{\left|F(x,f^n y)\right|, F(f^n x, y)\} \tag{4}$$

for all $x, y \in K$ with $x \neq y$; $n = n(x,y) \in \mathbb{N}$ and λ an arbitrary positive real number, then f has at least one periodic point.

Proof. Define $\varphi : K \to (-\infty, \infty)$ by $\varphi(x) = F(x, f^n(x)),\ \ x \in K$. Since f is continuous and F is lsca, so from Remark 2.6 it follows that the function φ is also lsca on K. By Lemma 2.4, there exists a point, say, $w \in K$ such that

$$\varphi(w) = F(w, f^n(w)) = \inf\{\varphi(x) : x \in K\}. \tag{5}$$

We prove that $f^n(w) = w$. Suppose, to the contrary, that $w \neq f^n(w)$. Then from (4),

$$\begin{aligned} F(f^n(w), f^n(f^n(w))) \quad < \quad & \max\{F(w,f^n(w)),\ \min\{F(w,f^n(w)), F(f^n(w), f^n(f^n(w)))\}\} \\ & +\lambda \min\{\left|F(w, f^n(f^n(w)))\right|, 0\} \\ = \quad & \max\{F(w,f^n(w)), \min\{F(w,f^n(w)), F(f^n(w), f^n(f(w)))\}\}. \end{aligned}$$

Since by (5),

$$\min\{F(w, f^n(w)), F(f^n(w), f^n(f^n(w)))\} = F(w, f^n(w)),$$

we have

$$F(f^n(w), f^n(f^n(w))) < F(w, f^n(w)),$$

a contradiction to (5). Thus $f^n(w) = w$ for some $n \in \mathbb{N}$ and hence w is a periodic point of f. □

Theorem 3.7. *Let X be a topological space, $f : X \to X$ be a continuous function. Let for $F \in \mathcal{F}$ the mapping f satisfy the contractive condition:*

$$F(fx, fy) < \max\{F(x, y),\ \min\{F(x, fx), F(y, fy)\}\} + \lambda \min\{\left|F(x, fy)\right|, F(fx, y)\}$$

for all $x, y \in X$ with $x \neq y$ and λ an arbitrary positive real number. If for $x_0 \in X$ and for some $K \subseteq X$

$$K = f(K) \cup \{x_0\} \Longrightarrow K \text{ is relatively compact},$$

then f has a fixed point.

Proof. Let $x_1 = f(x_0)$ and define the sequence $\{x_n\}$ in X as follows: $x_{n+1} = f(x_n)$, for $n \geq 1$. Let $A = \{x_n : n \geq 1\}$. Then

$$\begin{aligned} A &= \{x_{n+1} : n \geq 1\} \cup \{x_0\} \\ &= f(A) \cup \{x_0\}. \end{aligned}$$

Therefore, by hypothesis, A is relatively compact. Define a function $\varphi : \overline{A} \to \mathbb{R}$ by

$$\varphi(x) = F(x, f(x)).$$

Since f is continuous and F is lsca, by Remark 2.6, φ is lsca and so, by Lemma 2.4, φ has a minimum, say, at $a \in \overline{A}$. Thus by Theorem 3.1, a is a fixed point of f. □

Ćirić proved the following fixed point theorem for a Hausdorff topological space (which may not be completely regular) with

$$F \in \Psi := \{F : X \times X \to [0, \infty) : F \text{ is } \textit{continuous}, \text{ symmetric and } F(x, y) = 0 \text{ if and only if } x = y\}.$$

Theorem 3.8. *[5] Let X be a Hausdorff (not necessarily completely regular) topological space and $f : X \to X$ be a continuous self mapping and such that for some $F \in \Psi$,*

$$F(f(x), f(y)) < \max\{F(x, y), F(x, f(x)), F(y, f(y))\} + \lambda \min\{F(x, f(y)), F(y, f(x))\}, \tag{6}$$

for all $x, y \in X$, $x \neq y$; where $\lambda \geq 0$. If there exists a point $x \in X$ whose sequence of iterates $\{f^n(x)\}$ contains a convergent subsequence $\{f^{n_i}(x)\}$, then $\xi = \lim_{i\to\infty} f^{n_i}(x) \in X$ is a fixed point of f. If f satisfies (6) with $\lambda = 0$, then f has the unique fixed point.

By modifying the proof of Theorem 3.8 (which is Theorem 4 of [5]), we prove the following more general theorem in the setting of a general topological space when the real valued mapping F is continuous with $F(x, y) = 0$ when $x = y$.

Theorem 3.9. *Let X be a topological space, $f : X \to X$ be a continuous mapping and for some continuous $F : X \times X \to \mathbb{R}$ (with $F(x, y) = 0$ when $x = y$),*

$$F(f(x), f(y)) < \max\{F(x, y), \Big[\max\{F(x, f(x)), F(y, f(y))\} + \lambda \min\{\left|F(x, f(y))\right|, F(f(x), y)\}\Big]\}, \tag{7}$$

for all $x, y \in X$, $x \neq y$; where $\lambda \geq 0$. If there exists a point $x_0 \in X$ whose sequence of iterates $\{f^n(x_0)\}$ contains a convergent subsequence $\{f^{n_i}(x_0)\}$, then $a = \lim_{i\to\infty} f^{n_i}(x_0) \in X$ is a fixed point of f. If $\lambda = 0$, then f has a unique fixed point.

Proof. Let $x_0 \in X$ and define the sequence $\{x_n\}$ by $x_n = f^n(x_0)$, for $n \geq 1$. Let $\{f^{n_i}(x_0)\}$ be a subsequence of $\{x_n\}$ such that $f^{n_i}(x_0) \to a \in X$, as $i \to \infty$. We may suppose that $x_{n+1} \neq x_n$ for each n. Indeed, if we assume to the contrary, that $x_{n_0+1} = f^{n_0+1}(x_0) = f(x_{n_0}) = x_{n_0}$ for some n_0, then $x_n = x_{n_0}$ for all $n \geq n_0$ and so $x_{n_0} = f(x_{n_0}) = a = f(a)$ and the proof is done in this case.

For $i \geq 1$ put

$$p_i = n_{i+1} - n_i \geq 1, \text{ that is, } n_{i+1} = n_i + p_i.$$

Case (1). Let $p_i > 1$. Then from (7), we have

$$\begin{aligned} F(x_{n_{i+1}}, f(x_{n_{i+1}})) &= F(x_{n_i+p_i}, f(x_{n_i+p_i}) \\ &= F(f(x_{n_i+p_i-1}), f(f(x_{n_i+p_i-1}))) \\ &< \max\{F(x_{n_i+p_i-1}, f(x_{n_i+p_i-1})), [\max\{F(x_{n_i+p_i-1}, f(x_{n_i+p_i-1})), F(x_{n_i+p_i}, f(x_{n_i+p_i}))\} \\ &\quad +\lambda \min\{\left|F(x_{n_i+p_i-1}, f(x_{n_i+p_i}))\right|, F(f(x_{n_i+p_i-1}), x_{n_i+p_i})\}]\} \\ &= \max\{F(x_{n_i+p_i-1}, f(x_{n_i+p_i-1})), F(x_{n_i+p_i-1}, f(x_{n_i+p_i-1})), F(x_{n_i+p_i}, f(x_{n_i+p_i}))\} + \lambda \cdot 0 \\ &= \max\{F(x_{n_i+p_i-1}, f(x_{n_i+p_i-1})), F(x_{n_i+p_i}, f(x_{n_i+p_i}))\} \\ &= \max\{F(x_{n_i+p_i-1}, f(x_{n_i+p_i-1})), F(x_{n_{i+1}}, f(x_{n_{i+1}}))\}. \end{aligned}$$

Hence, as $F(x_{n_{i+1}}, f(x_{n_{i+1}})) = F(x_{n_i+p_i}, f(x_{n_i+p_i}) < F(x_{n_{i+1}}, f(x_{n_{i+1}}))$ is false, we get

$$F(x_{n_{i+1}}, f(x_{n_{i+1}})) = F(x_{n_i+p_i}, f(x_{n_i+p_i})) < F(x_{n_i+p_i-1}, f(x_{n_i+p_i-1})), \text{ where } p_i > 1. \tag{8}$$

Continuing this process we get

$$\begin{aligned} F(x_{n_{i+1}}, f(x_{n_{i+1}})) &= F(x_{n_i+p_i}, f(x_{n_i+p_i})) \\ &< F(x_{n_i+p_i-1}, f(x_{n_i+p_i-1})) \\ &< F(x_{n_i+p_i-2}, f(x_{n_i+p_i-2})) \\ &\quad \cdots \\ &< F(x_{n_i+1}, f(x_{n_i+1})) \\ &= F(f(x_{n_i}), f^2(x_{n_i})). \end{aligned}$$

Thus, if $p_i > 1$, then

$$F(x_{n_{i+1}}, f(x_{n_{i+1}})) < F(f(x_{n_i}), f^2(x_{n_i})).$$

Case (2). Let $p_i = 1$. Then $x_{n_{i+1}} = x_{n_i+1} = f^{n_i+1}(x_0) = f(f^{n_i}(x_0)) = f(x_{n_i})$ and so

$$F(x_{n_{i+1}}, f(x_{n_{i+1}})) = F(f(x_{n_i}), f^2(x_{n_i})).$$

Therefore, from both the cases, i.e, for each $p_i \geq 1$, we get

$$F(x_{n_{i+1}}, f(x_{n_{i+1}})) \leq F(f(x_{n_i}), f^2(x_{n_i})). \tag{9}$$

Since f is continuous and $x_{n_i} \to a$ as $i \to \infty$, it follows that $f(x_{n_i}) \to f(a)$ and $f^2(x_{n_i}) \to f^2(a)$ as $i \to \infty$. Since $F : X \times X \to \mathbb{R}$ is continuous, we get

$$F(a, f(a)) = \lim_{i\to\infty} F(x_{n_{i+1}}, f(x_{n_{i+1}})),$$

and

$$F(f(a), f^2(a)) = \lim_{i\to\infty} F(f(x_{n_i}), f^2(x_{n_i})).$$

These two equations combined with (9) imply that

$$F(a, f(a)) \leq F(f(a), f^2(a)). \tag{10}$$

Now we show that $f(a) = a$. Suppose, to the contrary, that $f(a) \neq a$. Then from (7),

$$\begin{aligned} F(f(a), f^2(a)) &= F(f(a), f(f(a))) \\ &< \max\{F(a, f(a))), [\max\{F(a, f(a)), F(f(a), f^2(a))\} \\ &\quad + \lambda \min\{\left|F(a, f^2(a))\right|, F(f(a), f(a))\}]\} \\ &= \max\{F(a, f(a)), \max\{F(a, f(a)), F(f(a), f^2(a))\}\} \\ &= F(f(a), f^2(a)), \end{aligned}$$

which is not true. Thus $f(a) = a$ and hence f has a fixed point.
If $\lambda = 0$, then it can easily be shown that f has a unique fixed point. □

Remark 3.10. *Theorem 3.9 generalizes and improves the corresponding results due to Ćirić [3–5], Edelstein [10], Jungck [12, 13] and Liu et al. [17].*

Theorem 3.11. *Let X be a topological space, $f : X \to X$ be a continuous mapping and for some $F \in \mathcal{F}$, we have*

$$F(f(x), f(y)) < \max\{F(x, y), [\max\{F(x, f(x)), F(y, f(y))\} + \min\{\left|F(x, f(y))\right|, F(f(x), y)\}]\}, \tag{11}$$

for all $x, y \in X$, $x \neq y$. If there exists a point $x_0 \in X$ whose sequence of iterates $\{f^n(x_0)\}$ contains a convergent subsequence $\{f^{n_i}(x_0)\}$, then $a = \lim_{i\to\infty} f^{n_i}(x_0) \in X$ is a unique fixed point of f.

Proof. Since (11) implies (7), so by Theorem 3.9 there exists some $a \in K$ such that $f(a) = a$. Suppose that there, also, exists $b \in K$ such that $f(b) = b$ and $a \neq b$. Then from (11),

$$\begin{aligned} F(a, b) &= F(f(a), f(b)) \\ &< \max\{F(a, b), [\max\{F(a, f(a)), F(b, f(b))\} + \min\{\left|F(a, f(b))\right|, F(f(a), b)\}]\} \\ &= \max\{F(a, b), [\max\{F(a, a), F(b, b)\} + \min\{|F(a, b)|, F(a, b)\}]\} \\ &= \max\{F(a, b), [0 + F(a, b)]\} \text{ since } |F(a, b)| \geq F(a, b), \\ &= F(a, b), \end{aligned}$$

a contradiction. Thus, f has a unique fixed point. □

Our last theorem is on the existence of coincidence and fixed points of four continuous mappings.

Theorem 3.12. *Let X be a topological space, K be a nonempty compact subset of X and T, S, G $H : K \to K$ be continuous self mappings on K such that T, S are surjective on K, T commutes with $\{G, H\}$ and S commutes with $\{G, H\}$. If for $F \in \mathcal{F}$ the four mappings T, S, G and H satisfy the following condition:*

$$\begin{aligned} F(S(x), T(y)) &> \inf\{F(t, P(t)), F(Q(t), P(t)), F(Q(t), P(Q(t))), F(Q(x), Q(y)) \\ &: t \in \{x, y\},\ P \in \{T, S\} \text{ and } Q \in \{G, H\}\} \end{aligned} \tag{12}$$

for any $x, y \in K$ with $T(x) \neq S(y)$, then at least one of the following assertions holds:
(1) T has a fixed point in K;
(2) S has a fixed point in K;
(3) T and G have a coincidence point in K;
(4) T and H have a coincidence point in K;
(5) S and G have a coincidence point in K;
(6) S and H have a coincidence point in K.

Proof. Since K is a compact and F, T, S, G and H are continuous functions, it follows from Remark 2.6 that the functions $x \longmapsto F(x, P(x))$ and $x \longmapsto F(Q(x), P(x))$ with $P \in \{T, S\}$ and $Q \in \{G, H\}$, are continuous on K, and by Lemma 2.4 there exists $a, b, p, q, r, s \in K$ such that

$$\begin{aligned}
F(a, T(a)) &= \inf\{F(x, T(x)) : x \in K\},\\
F(b, S(b)) &= \inf\{F(x, S(x)) : x \in K\},\\
F(G(p), T(p)) &= \inf\{F(G(x), T(x)) : x \in K\},\\
F(H(q), T(q)) &= \inf\{F(H(x), T(x)) : x \in K\},\\
F(G(r), S(r)) &= \inf\{F(G(x), S(x)) : x \in K\},\\
F(H(s), S(s)) &= \inf\{F(H(x), S(x)) : x \in K\}.
\end{aligned}$$

We need to consider the following cases.

Assertion (1). Suppose that

$$\begin{aligned}
F(a, T(a)) = \min\{&F(a, T(a)), F(b, S(b)), F(G(p), T(p)),\\
&F(H(q), T(q)), F(G(r), S(r)), F(H(s), S(s))\}. \qquad (13)
\end{aligned}$$

Since $S(K) = K$, there exists some $w \in K$ such that $S(w) = a$. Suppose that $T(S(w)) \neq S(w)$, that is, $T(a) \neq a$. Since S commutes with G and H, from (12) we get

$$\begin{aligned}
F(S(w), T(S(w))) &> \inf\{F(t, P(t)), F(Q(t), P(t)), F(Q(t), PQ(t)), F(Q(S(w)), Q(w))\\
&\quad : \ t \in \{S(w), w\}, \ P \in \{T, S\} \text{ and } Q \in \{G, H\}\}\\
&= \min\{\inf_{t\in\{S(w),w\}}\{F(t, P(t))\}, \inf_{t\in\{S(w),w\}}\{F(Q(t), P(t))\},\\
&\qquad \inf_{t\in\{S(w),w\}}\{F(Q(t), PQ(t))\}, \inf_{t\in\{S(w),w\}}\{F(S(Q(w)), Q(w))\}\\
&\quad : \ P \in \{T, S\} \text{ and } Q \in \{G, H\}\}\\
&\geq \min\{F(a, T(a)), F(b, S(b)), F(G(p), T(p)),\\
&\qquad F(H(q), T(q)), F(G(r), S(r)), F(H(s), S(s))\}.
\end{aligned}$$

Since

$$F(t, P(t)) = F(t, T(t)), \quad \text{or } F(t, P(t)) = F(t, S(t)),$$

then

$$\inf_{t\in\{S(w),w\}}\{F(t, P(t))\} \geq \inf_{t\in K}\{F(t, P(t))\} = \min\{F(a, T(a)), F(b, S(b)).$$

Similarly, as

$$\begin{aligned}
F(Q(t), P(t)) &= F(G(t), T(t)), \quad \text{or } F(Q(t), P(t)) = F(G(t), S(t)),\\
\text{or } F(Q(t), P(t)) &= F(H(t), T(t)), \quad \text{or } F(Q(t), P(t)) = F(H(t), S(t)),
\end{aligned}$$

then

$$\inf_{t\in\{S(w),w\}} F(Q(t), P(t)) \geq \min\{F(G(p), T(p)), F(G(r), S(r)), F(H(q), T(q)), F(H(s), S(s)).$$

Further, as

$$F(Q(t), PQ(t)) = F(Q(t), TQ(t)) \quad \text{or } F(Q(t), PQ(t)) = F(Q(t), SQ(t)),$$

then

$$\inf_{G(t)\in K, H(t)\in K; t\in\{S(w),w\}} F(Q(t), TQ(t)) \geq \inf_{x\in K} F(x, T(x)) = F(a, T(a)),$$

$$\inf_{G(t)\in K, H(t)\in K; t\in\{S(w),w\}} F(Q(t), SQ(t)) \geq \inf_{x\in K} F(x, S(x)) = F(b, S(b)).$$

Also, as

$$\begin{aligned}
F(Q(S(w)), Q(w)) &= F(S(Q(w)), Q(w)) = F(S(G(w)), G(w)), \quad \text{or} \\
F(Q(S(w)), Q(w)) &= F(S(G(w)), H(w)), \quad \text{or } F(Q(S(w)), Q(w)) = F(S(H(w)), G(w)), \ldots
\end{aligned}$$

then

$$\inf_{G(w)\in K, H(w)\in K;} F(Q(S(w)), Q(w)) \geq \inf_{x\in K} F(S(x), x)) = F(b, S(b)).$$

Therefore, we obtain

$$\begin{aligned}
F(S(w), T(S(w))) &> \inf\{F(t, P(t)), F(Q(t), P(t)), F(Q(t), PQ(t)), F(Q(S(w)), Q(w)) \\
&\geq \min\{F(a, T(a)), F(b, S(b)), F(G(p), T(p)), \\
&\quad F(H(q), T(q)), F(G(r), S(r)), F(H(s), S(s))\}.
\end{aligned}$$

Hence, by (13), we have

$$\begin{aligned}
F(S(w), T(S(w))) &= F(a, T(a)) \\
&> \min\{F(a, T(a)), F(b, S(b)), F(G(p), T(p)), \\
&\quad F(H(q), T(q)), F(G(r), S(r)), F(H(s), S(s))\} \\
&= F(a, T(a)),
\end{aligned}$$

a contradiction. Thus $T(a) = a$, i.e. a is the fixed point of T.

Assertion (5). Suppose that

$$\begin{aligned}
F(G(r), S(r)) &= \min\{F(a, T(a)), F(b, S(b)), F(G(p), T(p)), \\
&\quad F(H(q), T(q)), F(G(r), S(r)), F(H(s), S(s))\}. \qquad (14)
\end{aligned}$$

As $T(K) = K$, there exists some $z \in K$ such that $T(z) = r$. Suppose that $S(T(z)) \neq G(T(z))$, that is, $S(r) \neq G(r)$. From (12) we get

$$\begin{aligned}
F(T(G(z), S(T(z))) &> \inf\{F(t, P(t)), F(Q(t), P(t)), \\
&\quad F(Q(t), P(Q(t))), F(Q(T(z)), Q(G(z))) \\
&: \ t \in \{T(z)), G(z)\}, \ P \in \{T, S\} \text{ and } Q \in \{G, H\}\} \\
&\geq \min\{F(a, T(a)), F(b, S(b)), F(G(p), T(p)), \\
&\quad F(H(q), T(q)), F(G(r), S(r)), F(H(s), S(s))\}.
\end{aligned}$$

Hence, as $F(T(G(z)), S(T(z))) = F(G(T(z)), S(T(z))) = F(G(r), S(r))$, by (14), we have

$$F(G(r), S(r)) > F(G(r), S(r)),$$

a contradiction. Thus $G(r) = S(r)$ which means that r is the coincidence of S and G. Thus we have proved assertions (1) and (5).

Considering the remaining cases:

$$\begin{aligned}
F(b, S(b)) &= \min\{F(a, T(a)), F(b, S(b)), F(G(p), T(p)), \\
&\quad F(H(q), T(q)), F(G(r), S(r)), F(H(s), S(s))\}.
\end{aligned}$$

$$\begin{aligned}
F(G(p), T(p)) &= \min\{F(a, T(a)), F(b, S(b)), F(G(p), T(p)), \\
&\quad F(H(q), T(q)), F(G(r), S(r)), F(H(s), S(s))\}.
\end{aligned}$$

$$\begin{aligned} F(H(q),T(q)) &= \min\{F(a,T(a)),F(b,S(b)),F(G(p),T(p)),\\ &\quad F(H(q),T(q)),F(G(r),S(r)),F(H(s),S(s))\}. \end{aligned}$$

or

$$\begin{aligned} F(H(s),S(s)) &= \min\{F(a,T(a)),F(b,S(b)),F(G(p),T(p)),\\ &\quad F(H(q),T(q)),F(G(r),S(r)),F(H(s),S(s))\}. \end{aligned}$$

and proceeding on lines as in the proof of *Assertion* (1), or *Assertion* (2), we can conclude that our assertions (2), (3), (4) or (6) also hold. □

Corollary 3.13. *Let X be a topological space, K be a nonempty compact subset of X and T, S, G $H : K \to K$ be continuous self mappings on K such that S, T are surjective on K and commutes with G, H. If for $F \in \mathcal{F}$ the four mappings T, S, G and H satisfy the following condition:*

$$\begin{aligned} F(S(x),T(y)) &> \sup\{F(t,P(t)),F(Q(t),P(t)),F(Q(t),P(Q(t))),F(Q(x),Q(y))\\ &: \quad t \in \{x,y\},\ P \in \{T,S\} \quad and \quad Q \in \{G,H\}\} \end{aligned}$$

for any $x, y \in K$ with $T(x) \neq S(y)$, then at least one of the following assertions holds:
(1) T has a fixed point in K; (2) S has a fixed point in K;
(3) T and G have a coincidence point in K; (4) T and H have a coincidence point in K;
(5) S and G have a coincidence point in K; (6) S and H have a coincidence point in K.

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