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(ψ,φ) -weak Contraction on Ordered Uniform Spaces

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Abstract. In this paper, we prove a fixed point theorem for $(\psi,\varphi)-$contractive mappings on ordered uniform space.

1. Introduction

We call a pair (X,ϑ) to be a uniform space which consists of a non-empty set X together with an uniformity ϑ of wherein the latter begins with a special kind of filter on $X\times X$ whose all elements contain the diagonal $\Delta=\{(x,x):x\in X\}$. If $V\in\vartheta$ and $(x,y)\in V$, $(y,x)\in V$ then x and y are said to be $V-$close. Also a sequence $\{x_n\}$ in X, is said to be a Cauchy sequence with regard to uniformity ϑ if for any $V\in\vartheta$, there exists $N\geq 1$ such that x_n and x_m are $V-$close for $m,n\geq N$. An uniformity ϑ defines a unique topology $\tau(\vartheta)$ on X for which the neighborhoods of $x\in X$ are the sets $V(x)=\{y\in X:(x,y)\in V\}$ when V runs over ϑ.

A uniform space (X,ϑ) is said to be Hausdorff if and only if the intersection of all the $V\in\vartheta$ reduces to diagonal Δ of X i.e. $(x,y)\in V$ for all $V\in\vartheta$ implies $x=y$. Notice that Hausdorffness of the topology induced by the uniformity guarantees the uniqueness of limit of a sequence in uniform space. An element V of uniformity ϑ is said to be symmetrical if $V=V^{-1}=\{(y,x):(x,y)\in V\}$. Since each $V\in\vartheta$ contains a symmetrical $W\in\vartheta$ and if $(x,y)\in W$ then x and y are both W and $V-$close and then one may assume that each $V\in\vartheta$ is symmetrical. When topological concepts are mentioned in the context of a uniform space (X,ϑ), they are naturally interpreted with respect to the topological space $(X,\tau(\upsilon))$.

Aamri and El Moutawakil [2] proved some common fixed point theorems for some new contractive or expansive maps in uniform spaces by introducing the notions of an $E-$distance. Some other authors proved fixed point theorems using this concept ([4],[8],[10],[11],[16],[17]). In [5],[6] and [19] authors used the order relation on uniform space.

Existence of fixed points in partially ordered metric spaces was first investigated in 2004 by Ran and Reurings [18] and then by Nieto and Lopez [15]. Further results in this direction under weak contraction conditions were proved, e.g. ([3],[7],[9],[12],[14]).

In this paper, we establish a fixed point theorem satisfying $(\psi,\varphi)-$contractive condition on ordered uniform space. We also give an example.

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2. Preliminaries

Definition 2.1. ([2]) *Let (X, ϑ) be a uniform space. A function $p : X \times X \longrightarrow \mathbb{R}^+$ is said to be an $A-$ distance if for any $V \in \vartheta$, there exists $\delta > 0$, such that $p(z, x) \le \delta$ and $p(z, y) \le \delta$ for some $z \in X$ imply $(x, y) \in V$.*

Definition 2.2. ([2]) *Let (X, ϑ) be a uniform space. A function $p : X \times X \longrightarrow \mathbb{R}^+$ is said to be an $E-$ distance if*
$(p1)$ p is an $A-$ distance,
$(p2)$ $p(x, y) \le p(x, z) + p(z, y)$ for all $x, y, z \in X$.

Example 2.3. ([2]) *Let $X = [0, +\infty)$ and $p(x, y) = \max\{x, y\}$. The function p is an $A-$distance. Also, p is an $E-$distance.*

The following lemma embodies some useful properties of $E-$ distance.

Lemma 2.4. ([1], [2]) *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be arbitrary sequences in X and $\{\alpha_n\}$,$\{\beta_n\}$be sequences in $\mathbb{R}^+$ converging to 0. Then, for $x, y, z \in X$, the following holds*
(a) If $p(x_n, y) \le \alpha_n$ and $p(x_n, z) \le \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In partıcular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$.
(b) If $p(x_n, y_n) \le \alpha_n$ and $p(x_n, z) \le \beta_n$ for all $n \in \mathbb{N}$, then $\{y_n\}$ converges to z.
(c) If $p(x_n, x_m) \le \alpha_n$ for all $m > n$, then $\{x_n\}$ is a Cauchy sequence in (X, ϑ).

Let (X, ϑ) be a uniform space equipped with $E-$ distance p. A sequence in X is $p-$Cauchy if it satisfies the usual metric condition. There are several concepts of completeness in this setting.

Definition 2.5. ([1], [2]) *Let (X, ϑ) be a uniform space and p be an $E-$ distance on X.Then*
i) X is said to be $S-$complete if for every $p-$Cauchy sequence $\{x_n\}$ there exists $x \in X$ with $\lim_{n\to\infty} p(x_n, x) = 0$,
ii) X is said to be $p-$Cauchy complete if for every $p-$Cauchy sequence $\{x_n\}$ there exists $x \in X$ with $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\vartheta)$,
iii) $f : X \longrightarrow X$ is $p-$continuous if $\lim_{n\to\infty} p(x_n, x) = 0$ implies $\lim_{n\to\infty} p(fx_n, fx) = 0$,
iv) $f : X \longrightarrow X$ is $\tau(\vartheta)-$continuous if $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\vartheta)$ implies $\lim_{n\to\infty} fx_n = fx$ with respect to $\tau(\vartheta)$.

Remark 2.6. ([2]) *Let (X, ϑ) be a Hausdorff uniform space and let $\{x_n\}$ be a $p-$Cauchy sequence. Suppose that X is $S-$complete, then there exists $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = 0$. Lemma 2.4 (b) then gives $\lim_{n\to\infty} x_n = x$ with respect to the topology $\tau(\vartheta)$. Therefore $S-$completeness implies $p-$Cauchy completeness.*

We shall also state the following definition of altering distance function which is required in the sequel to establish a fixed point theorem in uniform space.

Definition 2.7. ([6]) *A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:*
(i) $\psi(0) = 0$,
(ii) ψ is continuous and monotonically nondecreasing.

3. Fixed Point Result

Theorem 3.1. *Let (X, ϑ) be a Hausdorff uniform space, "$\le$" be a partial order on X. Suppose p be an E-distance on S-complete space X. Let $T : X \to X$ be a $p-$continuous or $\tau(\vartheta)-$continuous nondecreasing mapping such that for all comparable $x, y \in X$ with*

$$\psi(p(Tx, Ty)) \le \psi(p(x, y)) - \varphi(p(x, y)), \tag{1}$$

where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are altering distance functions.
If there exists $x_0 \in X$ with $x_0 \le T(x_0)$ then T has a fixed point.

Proof. If $T(x_0) = x_0$ then the proof is finished. Suppose that $T(x_0) \neq x_0$. Since $x_0 \preceq T(x_0)$ and T is nondecreasing, we obtain by induction that

$$x_0 \preceq T(x_0) \preceq T^2(x_0) \preceq T^3(x_0) \preceq \cdots \preceq T^n(x_0) \preceq T^{n+1}(x_0) \preceq \cdots .$$

Put $x_{n+1} = Tx_n$, for all $n \geq 1$. If there exists a positive integer N such that $x_N = x_{N+1}$, then x_N is a fixed point of T. Now, we may assume that $x_n \neq x_{n+1}$, for all $n \geq 0$.

From (1), we have for all $n \geq 0$,

$$\begin{aligned} \psi(p(x_{n+2}, x_{n+1})) &= \psi(p(Tx_{n+1}, Tx_n)) \\ &\leq \psi(p(x_{n+1}, x_n)) - \varphi(p(x_{n+1}, x_n)) \end{aligned} \tag{2}$$

Together with that ψ is nondecreasing implies that the sequence $\{p(x_{n+1}, x_n)\}$ is monotone decreasing and hence there exists an $r \geq 0$ such that

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = r .$$

Letting $n \to \infty$ in (2) and using the continuity of ψ and φ, we obtain

$$\psi(r) \leq \psi(r) - \varphi(r)$$

which is a contradiction unless $r = 0$. Hence,

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = 0.$$

Similarly, we can show $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$.

Next we show that $\{x_n\}$ is a $p-$Cauchy sequence. Assume $\{x_n\}$ is not $p-$Cauchy. Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $m(k) > n(k) > k$ such that

$$p\left(x_{n(k)}, x_{m(k)}\right) \geq \varepsilon. \tag{3}$$

Further, corresponding to $n(k)$, we can choose $m(k)$ in such a way that it is the smallest integer with $m(k) > n(k)$ and satisfying (3). Hence,

$$p\left(x_{n(k)}, x_{m(k)-1}\right) < \varepsilon.$$

Then we have

$$\varepsilon \leq p\left(x_{n(k)}, x_{m(k)}\right) \leq p\left(x_{n(k)}, x_{m(k)-1}\right) + p\left(x_{m(k)-1}, x_{m(k)}\right),$$

that is

$$\varepsilon \leq p\left(x_{n(k)}, x_{m(k)}\right) < \varepsilon + p\left(x_{n(k)-1}, x_{n(k)}\right).$$

Taking the limit as $k \to \infty$, we have

$$\lim_{k\to\infty} p\left(x_{n(k)}, x_{m(k)}\right) = \varepsilon. \tag{4}$$

From (p2),

$$p\left(x_{n(k)}, x_{m(k)}\right) \leq p\left(x_{n(k)}, x_{n(k)+1}\right) + p\left(x_{n(k)+1}, x_{m(k)+1}\right) + p\left(x_{m(k)+1}, x_{m(k)}\right)$$

and

$$p\left(x_{n(k)+1}, x_{m(k)+1}\right) \leq p\left(x_{n(k)+1}, x_{n(k)}\right) + p\left(x_{n(k)}, x_{m(k)}\right) + p\left(x_{m(k)}, x_{m(k)+1}\right).$$

Taking the limit as $k \to \infty$ we have

$$\lim_{k\to\infty} p\left(x_{n(k)+1}, x_{m(k)+1}\right) = \varepsilon. \tag{5}$$

From (1),

$$\psi\left(p\left(x_{n(k)+1}, x_{m(k)+1}\right)\right) \le \psi\left(p\left(x_{n(k)}, x_{m(k)}\right)\right) - \varphi\left(p\left(x_{n(k)}, x_{m(k)}\right)\right).$$

Letting $k \to \infty$ in the above inequality, using (4), (5) and the continuities of ψ and φ, we have

$$\psi(\varepsilon) \le \psi(\varepsilon) - \varphi(\varepsilon),$$

which is a contradiction by virtue of a property of φ.

Hence $\{x_n\}$ is a p-Cauchy sequence. Since $S-$completeness of X, there exists a $z \in X$ such that

$$\lim_{n\to\infty} p(x_n, z) = 0$$

Moreover, the $p-$continuity of T implies that $\lim_{n\to\infty} p(Tx_n, Tz) = 0$. So, by Lemma 2.4 (a), $z = Tz$. Using Remark 2.6, the proof is similar when T is $\tau(\vartheta)-$continuous . □

Example 3.2. *Let $X = [0,1]$ equipped with usual metric $d(x,y) = \left|x - y\right|$ and a partial order be defined as $x \preceq y$ whenever $y \le x$ and suppose*

$$\vartheta = \{V \subset X \times X : \Delta \subset V\}.$$

Define the function p as $p(x,y) = y$ for all x, y in X and $T : X \to X$ defined by $T(t) = \frac{t^2}{1+t}$. Consider the functions φ and ψ defined as follows

$$\varphi(t) = \frac{t}{1+t} \qquad and \qquad \psi(t) = t.$$

Definition of ϑ, $\cap_{V\in\vartheta} V = \Delta$ and this show that the uniform space (X, ϑ) is Hausdorff uniform space. And also X is $S-$complete.On the other hand, p is an $E-$distance. T is $p-$continuous and φ and ψ are continuous, monotone nondecreasing. For $x = 0.5$ and $y = 0.3$, using usual metric, (1) does not hold. However, we have that for all $x, y \in X$

$$\psi(p(Tx, Ty)) \le \psi(p(x,y)) - \varphi(p(x,y)).$$

And 0 is the fixed point of T.

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