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Some approximation properties of a certain nonlinear Bernstein operators

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Abstract. The present paper concerns with a certain sequence of nonlinear Bernstein operators $NB_n f$ of the form

$$(NB_n f)(x) = \sum_{k=0}^{n} P_{k,n}\left(x, f\left(\frac{k}{n}\right)\right) \ , \ 0 \le x \le 1 \ , \quad n \in \mathbb{N},$$

acting on bounded functions on an interval $[0,1]$, where $P_{k,n}$ satisfy some suitable assumptions. We will also investigate the pointwise convergence of this operators in some functional spaces. As a result, this study can be considered as an extension of the results dealing with the linear Bernstein Polynomials. As far as we know this kind of study is the first one on the nonlinear Bernstein approximation operators.

1. Introduction

Let f be a function defined on the interval $[0,1]$ and let $\mathbb{N} := \{1, 2, ...\}$. The classical Bernstein operators $B_n f$ applied to f are defined as

$$(B_n f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{k,n}(x) \ , \ 0 \le x \le 1 \ , \quad n \in \mathbb{N}, \tag{1}$$

where $p_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ is the Bernstein basis. These polynomials were introduced by Bernstein [9] in 1912 to give the first constructive proof of the Weierstrass approximation theorem. Some remarkable approximation properties of the polynomials (1) are presented in Lorentz [23].

The present paper concerns with pointwise convergence of certain families of nonlinear Bernstein operators $B_n f$ of the form

$$(NB_n f)(x) = \sum_{k=0}^{n} P_{k,n}\left(x, f\left(\frac{k}{n}\right)\right) \ , \ 0 \le x \le 1 \ , \quad n \in \mathbb{N}, \tag{2}$$

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acting on bounded functions f on an interval $[0,1]$, where $P_{k,n}$ satisfy some suitable assumptions. In particular, we obtain some pointwise convergence for the nonlinear sequence of Bernstein operators (2) to some point x of f, as $n \to \infty$.

We note that the approximation theory with nonlinear integral operators of convolution type was introduced by J. Musielak in [24] and widely developed in [4]. To the best of our knowledge, the approximation problem were limited to linear operators because the notion of singularity of an integral operator is closely connected with its linearity until the fundamental paper of Musielak [24]. In [24], the assumption of linearity of the singular integral operators was replaced by an assumption of a Lipschitz condition for the kernel function $K_\lambda(t,u)$ with respect to the second variable. After this approach, several mathematicians have undertaken the program of extending approximation by nonlinear operators in many ways, including pointwise and uniform convergence, Korovkin type theorems in abstract function spaces, sampling series and so on. Especially, nonlinear integral operators of type

$$(T_\lambda f)(x) = \int_a^b K_\lambda(t-x, f(t))\, dt, \quad x \in (a,b),$$

and its special cases were studied by Bardaro, Karsli and G. Vinti [6]-[7], Karsli [17]-[18] and Karsli-Ibikli [20] in some Lebesgue spaces.

Such developments delineated a theory which is nowadays referred to as the theory of approximation by nonlinear integral operators.

For further reading, we also refer the reader to [1]-[3], [5], [8] as well as the monographs [13] and [4] where other kinds of convergence results of linear and nonlinear operators in the Lebesgue spaces, Musielak-Orlicz spaces, BV-spaces and BV_φ-spaces have been considered.

An outline of the paper is as follows: The next section contains basic definitions and notations.

In Section 3, the main approximation results of this study are given. They are dealing with the rate of pointwise convergence of the nonlinear Bernstein operators $NB_n f$ to the limit $\psi o \left|f\right|$ and f of functions of bounded variation on the interval $[0,1]$. At the point x, which is a discontinuity of the first kind of f and of its derivative, we shall prove that $(NB_n f)(x)$ converge to the limit $f(x)$. Let us note that the counterpart of such kind of results for positive linear operators of functions of bounded variation in the Jordan sense were first obtained by Bojanic [10] and Cheng [14]. Some other important papers on this topic are [11], [12], [15], [20], [22] and [25].

In Section 4, we give some certain results which are necessary to prove the main result.

The final section, that is Section 5, concerns with the proof of the main results presented in Section 3.

2. Preliminaries

In this section, we assemble the main definitions and notations which will be used throughout the paper.

Let X be the set of all bounded Lebesgue measurable functions $f : [0,1] \to \mathbb{R}$.

Let Ψ be the class of all functions $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that the function ψ is continuous and concave with $\psi(0) = 0$, $\psi(u) > 0$ for $u > 0$.

We now introduce a sequence of functions. Let $\{P_{k,n}\}_{n\in\mathbb{N}}$ be a sequence functions $P_{k,n} : [0,1] \times \mathbb{R} \to \mathbb{R}$ defined by

$$P_{k,n}(t,u) = p_{k,n}(t) H_n(u) \tag{3}$$

for every $t \in [0,1], u \in \mathbb{R}$, where $H_n : \mathbb{R} \to \mathbb{R}$ is such that $H_n(0) = 0$ and $p_{k,n}(t)$ is the Bernstein basis.

Throughout the paper we assume that $\mu : \mathbb{N} \to \mathbb{R}^+$ is an increasing and continuous function such that $\lim_{n\to\infty} \mu(n) = \infty$.

First of all we assume that the following conditions hold:

a) $H_n : \mathbb{R} \to \mathbb{R}$ is such that

$$|H_n(u) - H_n(v)| \leq \psi\left(|u - v|\right), \quad \psi \in \Psi,$$

holds for every $u, v \in \mathbb{R}$, for every $n \in \mathbb{N}$. That is, H_n satisfies a $(L - \psi)$ Lipschitz condition.

b) We now set

$$K_n(x,u) := \begin{cases} \sum\limits_{k \leq nu} p_{n,k}(x) & , \quad 0 < u \leq 1 \\ \\ 0 & , \quad u = 0 \end{cases} . \tag{4}$$

and

$$B_n(x) := \int_{x - x/n^{\gamma/\beta}}^{x + (1-x)/n^{\gamma/\beta}} d_t\left(K_n(x,t)\right) \quad \text{for any fixed } x \in (0,1).$$

where $\beta > 0$, $\gamma \geq 1$.

We note that the use of the function $B_n(x)$ concerns the behavior of the approximation near to the point x. Similar approach and some particular examples can be found in [8], [21], [16], [19] and [25].

c) denoting by $r_n(u) := H_n(u) - u$, $u \in \mathbb{R}$ and $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} r_n(u) = 0$$

uniformly with respect to u.

In other words, for n sufficiently large

$$\sup_u |r_n(u)| = \sup_u |H_n(u) - u| \leq \frac{1}{\mu(n)},$$

holds.

The symbol $[a]$ will denote the greatest integer not greater than a.

3. Convergence Results

We will consider the following type nonlinear Bernstein operators,

$$\left(NB_n f\right)(x) = \sum_{k=0}^{n} P_{k,n}\left(x, f\left(\frac{k}{n}\right)\right)$$

defined for every $f \in X$ for which $NB_n f$ is well-defined, where

$$P_{k,n}(x,u) = p_{k,n}(x) H_n(u)$$

for every $x \in [0,1]$, $u \in \mathbb{R}$.

As before, we let

$$f_x(t) = \begin{cases} f(t) - f(x+) & , \ x < t \le 1 \\ 0 & , \ t = x \\ f(t) - f(x-) & , \ 0 \le t < x \end{cases} , \tag{5}$$

$e_n(x) = \begin{cases} 1 & , \ x = \frac{k'}{n} \text{ for some } k' \in IN \\ 0 & , \ x \neq \frac{k'}{n} \text{ for all } k' \in IN \end{cases}$, $e = 2.71...$and $\bigvee_0^1 \psi(|f_x|)$ is the total variation of $\psi(|f_x|)$ on $[0,1]$.

We are now ready to establish the main results of this study:

Theorem 1. *Let $\psi \in \Psi$ and $f \in X$ be such that $\psi o |f| \in BV([0,1])$. Suppose that $P_{k,n}(x,u)$ satisfies conditions $a) - c)$. Then for every $x \in (0,1)$ and for all $n > \frac{256}{25x(1-x)}$, we have*

$$\begin{aligned} & (NB_n f)(x) - \left[\psi\left(\left| \frac{f(x+) + f(x-)}{2} \right| \right) + \psi\left(\left| \frac{f(x+) - f(x-)}{2} \right| \right) \right] \\ \le \ & B_n^*(x) n^{-\gamma/\beta} \left[\bigvee_0^1 \psi(|f_x|) + \sum_{k=1}^{[n^\gamma]} \bigvee_{x-x/k^{1/\beta}}^{x+(1-x)/k^{1/\beta}} \psi(|f_x|) \right] + B_n(x) \bigvee_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} \psi(|f_x|) \\ & + \psi\left(\left| f(x) - \frac{f(x+) + f(x-)}{2} \right| \frac{e_n(x)}{\sqrt{2enx(1-x)}} \right) \end{aligned}$$

where $B_n^*(x) = B_n(x) \max\{x^{-\beta}, (1-x)^{-\beta}\}, (\beta > 0)$.

The following result is a corollary of the Theorem 1.

Corollary 1. *If we choose $f \in C[0,1]$ in Theorem 1, then we have*

$$\begin{aligned} & (NB_n f)(x) - \psi(|f(x)|) \\ \le \ & B_n^*(x) n^{-\gamma/\beta} \left[\bigvee_0^1 \psi(|f_x|) + \sum_{k=1}^{[n^\gamma]} \bigvee_{x-x/k^{1/\beta}}^{x+(1-x)/k^{1/\beta}} \psi(|f_x|) \right] + B_n(x) \bigvee_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} \psi(|f_x|) \end{aligned}$$

where $B_n^*(x) = B_n(x) \max\{x^{-\beta}, (1-x)^{-\beta}\}, (\beta > 0)$.

Theorem 2. *Let $\psi \in \Psi$ and $f \in X$ be such that $\psi o |f| \in BV([0,1])$. Suppose that $P_{k,n}(x,u)$ satisfies conditions $a) - c)$. Then for every $x \in (0,1)$ and for all $n > \frac{256}{25y(1-y)}$, we have*

$$\begin{aligned} & \left| (NB_n f)(x) - \frac{f(x+) + f(x-)}{2} \right| \le B_n^*(x) n^{-\gamma/\beta} \left[\bigvee_0^1 \psi(|f_x|) + \sum_{k=1}^{[n^\gamma]} \bigvee_{x-x/k^{1/\beta}}^{x+(1-x)/k^{1/\beta}} \psi(|f_x|) \right] \\ & + B_n(x) \bigvee_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} \psi(|f_x|) + \psi\left(\left| f(x) - \frac{f(x+) + f(x-)}{2} \right| \frac{e_n(x)}{\sqrt{2enx(1-x)}} \right) \\ & + \psi\left(\frac{|f(x+) - f(x-)|}{2} \right) + \frac{1}{\mu(n)} \end{aligned}$$

where $B_n^*(x) = B_n(x) \max\{x^{-\beta}, (1-x)^{-\beta}\}, (\beta > 0)$.

The following results are corollaries of the Theorem 2.

Corollary 2. *If we choose $f \in C[0,1]$ in Theorem 2, then we have*

$$\begin{aligned}\left|(NB_n f)(x) - f(x)\right| &\leq B_n^*(x) n^{-\gamma/\beta}\left[\bigvee_0^1 \psi\left(\left|f_x\right|\right) + \sum_{k=1}^{[n^\gamma]} \bigvee_{x-x/k^{1/\beta}}^{x+(1-x)/k^{1/\beta}} \psi\left(\left|f_x\right|\right)\right] \\ &\quad + B_n(x) \bigvee_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} \psi\left(\left|f_x\right|\right)\end{aligned}$$

where $B_n^*(x) = B_n(x)\max\{x^{-\beta}, (1-x)^{-\beta}\}$, $(\beta > 0)$.

Corollary 3. *If $f \in C[0,1]$ and $\psi(t) = t$ (that is, strongly Lipschitz condition) in Theorem 2, then we have*

$$\begin{aligned}\left|(NB_n f)(x) - f(x)\right| &\leq B_n^*(x) n^{-\gamma/\beta}\left[\bigvee_0^1 \left(\left|f_x\right|\right) + \sum_{k=1}^{[n^\gamma]} \bigvee_{x-x/k^{1/\beta}}^{x+(1-x)/k^{1/\beta}} \left(\left|f_x\right|\right)\right] \\ &\quad + B_n(x) \bigvee_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} \left(\left|f_x\right|\right)\end{aligned}$$

where $B_n^*(x) = B_n(x)\max\{x^{-\beta}, (1-x)^{-\beta}\}$, $(\beta > 0)$.

4. Auxiliary Result

In this section we give certain results, which are necessary to prove our theorems.

Lemma 1. *For $(B_n t^s)(x)$, $s = 0, 1, 2$, one has*

$$\begin{aligned}(B_n 1)(x) &= 1 \\ (B_n t)(x) &= x \\ (B_n t^2)(x) &= x^2 + \frac{x(1-x)}{n}.\end{aligned}$$

For proof of this Lemma see [23].

By direct calculation, we find the following equalities:

$$(B_n (t-x)^2)(x) = \frac{x(1-x)}{n} \quad , \quad (B_n (t-x))(x) = 0\,.$$

Lemma 2. *For all $x \in (0,1)$ and for each $n \in \mathbb{N}$, let*

$$NB_n(|t-x|^\beta; x) \leq \frac{B_n(x)}{n^{\gamma/\beta}},\ (\beta > 0) \tag{6}$$

holds, where $B_n(x)$ is as defined in Section 2. Then one has

$$\lambda_n(x,t) =: \int_0^t d_u\left(K_n(x,u)\right) \leq \frac{B_n(x)}{(x-t)^\beta n^{\gamma/\beta}}\ , \quad 0 \leq t < x, \tag{7}$$

and

$$1 - \lambda_n(x,t) = \int_t^1 d_u\left(K_n(x,u)\right) \leq \frac{B_n(x)}{(t-x)^\beta n^{\gamma/\beta}}\ , \quad x < t < 1. \tag{8}$$

Proof. We have

$$\begin{aligned}\lambda_n(x,t) &= :\int_0^t d_u\left(K_n(x,u)\right) \le \int_0^t \left(\frac{x-u}{x-t}\right)^{\beta} d_u\left(K_n(x,u)\right) \\ &\le \frac{1}{(x-t)^{\beta}}\int_0^1 |u-x|^{\beta}\, d_u\left(K_n(x,u)\right).\end{aligned}$$

According to (6), we have

$$\lambda_n(x,t) \le \frac{B_n(x)}{(x-t)^{\beta} n^{\gamma/\beta}}.$$

Proof of (8) is analogous.

Lemma 3 ([26], Theorem 1). *For all $x \in (0,1)$ and for all $n > \frac{256}{25x(1-x)}$,we have*

$$p_{n,k}(x) \le \frac{1}{\sqrt{2enx(1-x)}},$$

where $e = 2.71...$ is the Napierian constant.

5. Proof of the Theorems

Proof of Theorem 1. For any $f \in X$, it is easily verified from (5), that

$$\begin{aligned}f(t) &= \frac{f(x+)+f(x-)}{2} + f_x(t) + \frac{f(x+)-f(x-)}{2}sgn(t-x) \\ &\quad +\delta_x(t)\left[f(x) - \frac{f(x+)+f(x-)}{2}\right], \qquad (9)\end{aligned}$$

where

$$\delta_x(t) = \begin{cases} 1\ , & x = t \\ 0\ , & x \neq t. \end{cases}$$

Applying the operator (2) to (9), using (3) and from condition *a*), we have

$$\begin{aligned}(NB_n f)(x) &= \sum_{k=0}^{n} P_{k,n}\left(x, f\left(\frac{k}{n}\right)\right) = \sum_{k=0}^{n} p_{k,n}(x) H_n\left(f\left(\frac{k}{n}\right)\right) \\ &\le \sum_{k=0}^{n} p_{k,n}(x)\psi\left(\left|f\left(\frac{k}{n}\right)\right|\right) \\ &\le \psi\left(\left|\frac{f(x+)+f(x-)}{2}\right|\right)\sum_{k=0}^{n} p_{k,n}(x) + \sum_{k=0}^{n} p_{k,n}(x)\psi\left(\left|f_x\left(\frac{k}{n}\right)\right|\right) \\ &\quad + \sum_{k=0}^{n} p_{k,n}(x)\psi\left(\left|\frac{f(x+)-f(x-)}{2}\right|\left|sgn\left(\frac{k}{n}-x\right)\right|\right) \\ &\quad + \sum_{k=0}^{n} p_{k,n}(x)\psi\left(\left|f(x) - \frac{f(x+)+f(x-)}{2}\right|\delta_x\left(\frac{k}{n}\right)\right)\end{aligned}$$

Hence from (9) , we have

$$(NB_n f)(x) \le \left[\psi\left(\left|\frac{f(x+)+f(x-)}{2}\right|\right)+\psi\left(\left|\frac{f(x+)-f(x-)}{2}\right|\right)\right]\sum_{k=0}^{n} p_{k,n}(x)$$

$$+\sum_{k=0}^{n} p_{k,n}(x)\psi\left(\left|f_x\left(\frac{k}{n}\right)\right|\right)+\psi\left(\sum_{k=0}^{n} p_{k,n}(x)\left|f(x)-\frac{f(x+)+f(x-)}{2}\right|\delta_x\left(\frac{k}{n}\right)\right).$$

Note that

$$\sum_{k=0}^{n} p_{k,n}(x)\delta_x\left(\frac{k}{n}\right)=e_n(x)p_{k',n}(x)$$

where $e_n(x)=1$ if there exists a k' such that $x=k'/n$, $e_n(x)=0$ if $x\neq k'/n$ for all $k\in\{0,1,...,n\}$. Thus,

$$(NB_n f)(x) \le \left[\psi\left(\left|\frac{f(x+)+f(x-)}{2}\right|\right)+\psi\left(\left|\frac{f(x+)-f(x-)}{2}\right|\right)\right]\sum_{k=0}^{n} p_{k,n}(x)$$

$$+\sum_{k=0}^{n} p_{k,n}(x)\psi\left(\left|f_x\left(\frac{k}{n}\right)\right|\right)+\psi\left(\left|f(x)-\frac{f(x+)+f(x-)}{2}\right|e_n(x)p_{k',n}(x)\right).$$

In view of Lemma 2, we get

$$(NB_n f)(x)-\left[\psi\left(\left|\frac{f(x+)+f(x-)}{2}\right|\right)+\psi\left(\left|\frac{f(x+)-f(x-)}{2}\right|\right)\right]$$

$$\le\sum_{k=0}^{n} p_{k,n}(x)\psi\left(\left|f_x\left(\frac{k}{n}\right)\right|\right)+\psi\left(\left|f(x)-\frac{f(x+)+f(x-)}{2}\right|\frac{e_n(x)}{\sqrt{2enx(1-x)}}\right). \tag{10}$$

In order to complete the proof of Theorem 1, we need an estimation for the term

$$\sum_{k=0}^{n} p_{k,n}(x)\psi\left(\left|f_x\left(\frac{k}{n}\right)\right|\right)$$

in (10).

Let

$$K_n(x,u)=:\begin{cases}\sum_{k\le nu} p_{n,k}(x) & , \quad 0<u\le 1\\ 0 & , \quad u=0\end{cases}.$$

Now, in view of (4) we split the last integral in three parts as follows;

$$\begin{aligned}\sum_{k=0}^{n} p_{k,n}(x)\psi\left(\left|f_x\left(\frac{k}{n}\right)\right|\right) &\le \left(\int_0^{x-x/n^{\gamma/\beta}}+\int_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}}+\int_{x+(1-x)/n^{\gamma/\beta}}^{1}\right)\psi\left(\left|f_x(t)\right|\right)d_t\left(K_n(x,t)\right)\\ &=: |I_1(\lambda,x)|+|I_2(\lambda,x)|+|I_3(\lambda,x)|.\end{aligned} \tag{11}$$

First we estimate $I_2(n,x)$. Since $f_x(x)=0$ and $\psi(0)=0$, for $t\in\left[x-x/n^{\gamma/\beta},x+(1-x)/n^{\gamma/\beta}\right]$, we have

$$|I_2(n,x)|=\int_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}}\left[\psi\left(\left|f_x(t)\right|\right)-\psi\left(\left|f_x(x)\right|\right)\right]d_t\left(K_n(x,t)\right),$$

also by the condition *b*)

$$
\begin{aligned}
|I_2(n,x)| &\leq \bigvee_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} \psi\left(\left|f_x\right|\right) \int_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} d_t\left(K_n(x,t)\right) \\
&\leq B_n(x) \bigvee_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} \psi\left(\left|f_x\right|\right). \qquad (12)
\end{aligned}
$$

Next, we estimate $I_1(n,x)$. Using partial Lebesgue-Stieltjes integration, we obtain

$$
|I_1(n,x)| = \int_0^{x-x/n^{\gamma/\beta}} \psi\left(\left|f_x(t)\right|\right) d_t\left(K_n(x,t)\right)
$$

$$
= \psi\left(\left|f_x(x-x/n^{\gamma/\beta})\right|\right) K_n(x, x-x/n^{\gamma/\beta}) - \int_0^{x-x/n^{\gamma/\beta}} K_n(x,t) d_t\left(\psi\left(\left|f_x(t)\right|\right)\right).
$$

Let $y = x - x/n^{\gamma/\beta}$. By (7), it is clear that

$$
K_n(x,y) \leq B_n(x)(x-y)^{-\beta} n^{\gamma(\beta-1)/\beta}. \qquad (13)
$$

Here we note that

$$
\psi\left(\left|f_x(x-x/n^{\gamma/\beta})\right|\right) = \psi\left(\left|f_x(x-x/n^{\gamma/\beta})\right|\right) - \psi\left(\left|f_x(x)\right|\right) \leq \bigvee_{x-x/n^{\gamma/\beta}}^{x} \psi\left(\left|f_x\right|\right).
$$

Using partial integration and applying (13), we obtain

$$
|I_1(n,x)| \leq \bigvee_{x-x/n^{\gamma/\beta}}^{x} \psi\left(\left|f_x\right|\right) \left|K_n(x, x-x/n^{\gamma/\beta})\right| + \int_0^{x-x/n^{\gamma/\beta}} K_n(x,t) d_t\left(-\bigvee_t^x \psi\left(\left|f_x\right|\right)\right)
$$

$$
\begin{aligned}
&\leq \bigvee_{x-x/n^{\gamma/\beta}}^{x} \psi\left(\left|f_x\right|\right) B_n(x) x^{-\beta} n^{\gamma(\beta-1)/\beta} + B_n(x) n^{-\gamma/\beta} \int_0^{x-x/n^{\gamma/\beta}} (x-t)^{-\beta} d_t\left(-\bigvee_t^x \psi\left(\left|f_x\right|\right)\right) \\
&= \bigvee_{x-x/n^{\gamma/\beta}}^{x} \psi\left(\left|f_x\right|\right) B_n(x) x^{-\beta} n^{\gamma(\beta-1)/\beta} + B_n(x) n^{-\gamma/\beta} \left[-x^{-\beta}/n^{-\gamma} \bigvee_{x-x/n^{\gamma/\beta}}^{x} \psi\left(\left|f_x\right|\right) \right. \\
&\quad \left. + x^{-\beta} \bigvee_0^x \psi\left(\left|f_x\right|\right) + \int_0^{x-x/n^{\gamma/\beta}} \bigvee_t^x \psi\left(\left|f_x\right|\right) \frac{\beta}{(x-t)^{\beta+1}} dt \right] \\
&= B_n(x) n^{-\gamma/\beta} \left[x^{-\beta} \bigvee_0^x \psi\left(\left|f_x\right|\right) + \int_0^{x-x/n^{\gamma/\beta}} \bigvee_t^x \psi\left(\left|f_x\right|\right) \frac{\beta}{(x-t)^{\beta+1}} dt \right].
\end{aligned}
$$

Changing the variable t by $x - x/u^{1/\beta}$ in the last integral, we have

$$\int_0^{x-x/n^{\gamma/\beta}} \bigvee_t^x \psi\left(\left|f_x\right|\right) \frac{\beta}{(x-t)^{\beta+1}} dt = \frac{1}{x^\beta} \int_1^{n^\gamma} \bigvee_{x-x/u^{1/\beta}}^{x} \psi\left(\left|f_x\right|\right) du \leq \frac{1}{x^\beta} \sum_{k=1}^{[n^\gamma]} \bigvee_{x-x/k^{1/\beta}}^{x} \psi\left(\left|f_x\right|\right).$$

Consequently, we obtain

$$|I_1(n,x)| \leq B_n(x) n^{-\gamma/\beta} x^{-\beta} \left[\bigvee_0^x \psi\left(\left|f_x\right|\right) + \sum_{k=1}^{[n^\gamma]} \bigvee_{x-x/k^{1/\beta}}^{x} \psi\left(\left|f_x\right|\right) \right]. \tag{14}$$

Using the similar method, we can find

$$|I_3(n,x)| \leq B_n(x) n^{-\gamma/\beta} (1-x)^{-\beta} \left[\bigvee_x^1 \psi\left(\left|f_x\right|\right) + \sum_{k=1}^{[n^\gamma]} \bigvee_{x}^{x+(1-x)/k^{1/\beta}} \psi\left(\left|f_x\right|\right) \right]. \tag{15}$$

Combining (12), (14) and (15) in (11), we obtain

$$\begin{aligned} \sum_{k=0}^{n} p_{k,n}(x) \psi\left(\left|f_x\left(\frac{k}{n}\right)\right|\right) &\leq B_n(x) n^{-\gamma/\beta} x^{-\beta} \left[\bigvee_0^x \psi\left(\left|f_x\right|\right) + \sum_{k=1}^{[n^\gamma]} \bigvee_{x-x/k^{1/\beta}}^{x} \psi\left(\left|f_x\right|\right) \right] \\ &\quad + B_n(x) n^{-\gamma/\beta} (1-x)^{-\beta} \left[\bigvee_x^1 \psi\left(\left|f_x\right|\right) + \sum_{k=1}^{[n^\gamma]} \bigvee_{x}^{x+(1-x)/k^{1/\beta}} \psi\left(\left|f_x\right|\right) \right] + B_n(x) \bigvee_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} \psi\left(\left|f_x\right|\right) \\ &\leq B_n^*(x) n^{-\gamma/\beta} \left[\bigvee_0^1 \psi\left(\left|f_x\right|\right) + \sum_{k=1}^{[n^\gamma]} \bigvee_{x-x/k^{1/\beta}}^{x+(1-x)/k^{1/\beta}} \psi\left(\left|f_x\right|\right) \right] + B_n(x) \bigvee_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} \psi\left(\left|f_x\right|\right). \end{aligned} \tag{16}$$

Collecting (10) and (16), we get the desired result. This completes the proof of the theorem.

Proof of Theorem 2. For any $f \in X$ we have

$$\begin{aligned} \left| (NB_n f)(x) - \frac{f(x+)+f(x-)}{2} \right| &= \left| \sum_{k=0}^{n} p_{k,n}(x) H_n\left(f\left(\frac{k}{n}\right)\right) - \frac{f(x+)+f(x-)}{2} \right| \\ &\leq \sum_{k=0}^{n} p_{k,n}(x) \psi\left(\left| f\left(\frac{k}{n}\right) - \frac{f(x+)+f(x-)}{2} \right|\right) \\ &\quad + \sum_{k=0}^{n} p_{k,n}(x) \left| H_n\left(\frac{f(x+)+f(x-)}{2}\right) - \frac{f(x+)+f(x-)}{2} \right| \\ &\leq \sum_{k=0}^{n} p_{k,n}(x) \psi\left(\left|f_x\left(\frac{k}{n}\right)\right|\right) + \sum_{k=0}^{n} p_{k,n}(x) \psi\left(\left| f(x) - \frac{f(x+)+f(x-)}{2} \right| \delta_x\left(\frac{k}{n}\right)\right) \\ &\quad + \psi\left(\left|\frac{f(x+)-f(x-)}{2}\right|\right) + \sum_{k=0}^{n} p_{k,n}(x) \left| H_n\left(\frac{f(x+)+f(x-)}{2}\right) - \frac{f(x+)+f(x-)}{2} \right| \end{aligned}$$

As in the proof of the Theorem 1, one has

$$\left|(NB_n f)(x) - \frac{f(x+)+f(x-)}{2}\right| \le B_n^*(x) n^{-\gamma/\beta} \left[\bigvee_0^1 \psi\left(\left|f_x\right|\right) + \sum_{k=1}^{[n^\gamma]} \bigvee_{x-x/k^{1/\beta}}^{x+(1-x)/k^{1/\beta}} \psi\left(\left|f_x\right|\right)\right]$$

$$+B_n(x) \bigvee_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} \psi\left(\left|f_x\right|\right) + \psi\left(\left|f(x) - \frac{f(x+)+f(x-)}{2}\right| \frac{e_n(x)}{\sqrt{2enx(1-x)}}\right)$$

$$+\psi\left(\frac{\left|f(x+)-f(x-)\right|}{2}\right) + \sum_{k=0}^{n} p_{k,n}(x) \left|H_n\left(\frac{f(x+)+f(x-)}{2}\right) - \frac{f(x+)+f(x-)}{2}\right|.$$

In view of c), we have $\sup_u |H_n(u) - u| \le \frac{1}{\mu(n)}$. This completes the proof of the theorem.

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